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NEOCLASSICAL DIFFUSION ARISING FROM MAGNETIC FIELD RIPPLES IN TOKAMAKS

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NEOCLASSICAL DIFFUSION ARISING FROM MAGNETIC FIELD RIPPLES IN TOKAMAKS

by

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ABSTRACT

A solution of the Fokker-Planck equation is obtained for the distribution of trapped particles in the magnetic field ripples of an imperfectly axisymmetric Tokamak; the resulting particle flux and ion heat flux arising from the toroidal drift of such particles are calculated, revisimg previous estimates which neglected the asymmetry of the ripple profile.

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1. INTRODUCTION

The discrete nature of the coils producing the main toroidal field in a supposedly axisymmetric torus destroys this symmetry and is responsible for an additional neoclassical diffusion [1] having essentially the same origin as superbanana diffusion in a stellarator [2, 3]. The resulting magnetic fields can be written

$$\underline{B} = \frac{B_o}{1 + \varepsilon \cos \theta} \left(0,\ \Theta(r),\ 1 - \delta(r) \cos N\phi\right) \qquad (1)$$

where r and θ are polar coordinates with respect to the magnetic axis of the torus and ϕ measures angular distance along this axis; $\varepsilon = \frac{r}{R}$ is the inverse aspect ratio, $\delta(r) \lesssim \varepsilon$ is a measure of the magnitude of the ripple, N is the number of turns in the B_ϕ winding and a rotational transform ι is defined through $\Theta(r) = \varepsilon \iota$. A component B_r to satisfy div B = 0 has been neglected since we assume $N\varepsilon < 1$. The magnetic field strength is approximately

$$B \simeq B_o(1 - \varepsilon \cos\theta - \delta(r) \cos N\phi) \qquad (2)$$

reminiscent of that in a stellarator and one might conclude that merely substituting $\delta(r)$ for ε_h, the variation of the helical field in a stellarator, in the expressions for super-banana diffusion [2, 3] would yield the results for 'ripple' diffusion in a Tokamak.

In one fundamental respect, however, ripple induced diffusion differs from the analogous superbanana diffusion in stellarators. Stringer [4] has given a detailed discussion of this difference which appears at small values of the ripple well depth δ, or at very large values of N. He shows that for values of $\alpha = \frac{\varepsilon_t}{N\delta}$ in excess of unity the ripple does not result in the formation of local magnetic mirrors and consequent trapping over the whole minor azimuth, and that in general for α of order unity, significant reduction of the mirror depth occurs at all angles. This can be seen from Eq. (2) by noting that minima in $B(\ell)$ (where ℓ is arclength along a field line) occur only if

$$\alpha \sin\theta < 1 .$$

Even when this holds each ripple is an asymmetric mirror with mirror ratios $R_{\pm}(\theta)$ where

$$(R_- - 1) < 2\delta < R_+ - 1$$

and a well depth $\Delta(\theta) \equiv (R_- - 1)$ given by

$$\Delta(\theta) = 2\delta \left[\sqrt{1 - \alpha^2 \sin^2\theta} - \alpha \left\{ \frac{\pi}{2} - \sin^{-1}(\alpha |\sin\theta|) \right\} |\sin\theta| \right] \quad (3)$$

replacing 2δ. Stringer's estimate of the reduction in diffusion and ion thermal conductivity is obtained by taking account of this reduction in well depth from 2δ to $\Delta(\theta)$, but although this effect is of greatest importance it neglects the asymmetric distortion of the ripple wells. In the next section we present the solution of the Fokker-Planck equation for the ripple-trapped distribution in a field of the form (1), and derive the consequent transport coefficients.

The analysis presented is valid in the experimentally important range of collision frequencies ν_j given by

$$\frac{\delta^{1/2} v_{Th_j} N}{R} > \frac{\nu_j}{\delta} > \frac{v_{Th_j}^2 \, \delta}{\omega_{c_j} \, r^2} \quad (4)$$

where v_{Th_j} is the thermal speed of species j and ω_{c_j} is the gyro-frequency. This means physically that the effective collision frequency $\nu_{eff} = \frac{\nu}{\delta}$ is less than the bounce time of a ripple trapped particle $\omega_b = \frac{\delta^{1/2} v_{Th} N}{R}$, but greater than the drift frequency around a complete superbanana orbit in the minor azimuth of the torus.

2. SOLUTION OF THE FOKKER-PLANCK EQUATION

Collision frequencies in the range (4) may be treated by writing a kinetic equation which assumes the bounce frequency and effective collision frequency $\nu_{eff} = \nu/\delta$ are comparable [5] and then considering a subsidiary expansion in ν_{eff}/ω_b. We introduce the distribution function f_j which is the average over a gyro-period of the larmor radius correction to a Maxwellian F_j satisfying the equation [5]

$$\sigma q \,\underline{n}.\nabla f_j + \underline{v}_{d_j}.\nabla F_j = C(f_j) \tag{5}$$

where

$$F_j = \left(\frac{m_j}{2\pi T_j(r)}\right)^{3/2} n_j(r) \exp\left(-\frac{m_j \kappa}{T_j(r)}\right) \tag{6}$$

with $\kappa = v^2/2$, the kinetic energy per unit mass, and σ is the sign of the velocity q along the field. The gradient operators in Eq. (5) are taken at constant energy ε

$$\varepsilon = \frac{v^2}{2} + \frac{e_j \Phi(r)}{m_j}$$

where $\Phi(r)$ is an electrostatic potential, a function of radius only. The drift velocity $\underline{v}_{d_j}$ is given by

$$\underline{v}_{d_j} = \frac{1}{\omega_{c_j}} \underline{n} \Lambda (\mu\nabla B + q^2 \underline{n}.\nabla\underline{n}) - \frac{\nabla\Phi\Lambda B}{B^2} \tag{7}$$

where $\underline{n}$ is a unit vector along the field and μ is the magnetic moment: $\kappa = q^2/2 + \mu B$.

In order to obtain a tractable equation for f_j we must replace the exact Fokker-Planck collision operator $C(f_j)$ by a simpler model. For electron-ion collisions one may use the excellent Lorentz approximation

$$C_{ei}(f_e) = \nu_{ei}(\kappa) \frac{q}{B} \frac{\partial}{\partial\mu} q\mu \frac{\partial}{\partial\mu} f_e \tag{8}$$

while ion-electron collisions may be ignored in comparison with ion-ion collisions. The two like particle collision operators C_{ee} and C_{ii} may also be replaced by operators resembling (8), namely

$$C_{jj}(f_j) = \nu_{jj}(\kappa) \frac{q}{B} \frac{\partial}{\partial\mu} q\mu \frac{\partial}{\partial\mu} f_j + \nu_{jj}(\kappa)\, q\, F_j \frac{p_j m_j}{T_j} \tag{9}$$

with

$$P_j = \frac{\int d^3v\ \nu_{jj}(\kappa)\, q f_j}{\frac{m_j}{T_j} \int d^3v\ \nu_{jj}(\kappa)\ q^2 F_j} .$$

The justification for this form of operator is that it may be derived from the Rosenbluth-McDonald-Judd form for distributions f_j localised in velocity space [6], as in the present case. The term containing p ensures momentum conservation but may be ignored here, for with a localised distribution its contribution is small. In fact it may be included in the analysis, but does not alter the result. This term is essential in the axisymmetric case where momentum conservation plays an important role. In that case results obtained using this collision operator are identical with those obtained from a more accurate variational treatment [6]. We remark that reference 2 employs this form of collision operator. Finally we must define the form of the collision frequencies [2]:

$$\nu_{jk}(\kappa) = \frac{\sqrt{2}\pi \; ne^4 \lambda}{m_j^{1/2} \; T_j^{3/2}} \; A_{jk}(x_j) \qquad (10)$$

where $x_j = \frac{m_j \kappa}{T_j}$ and

$$A_{jk}(x_j) = (\eta_k + \eta_k' - \frac{\eta_k}{2x_k}) \; x_j^{-3/2} \qquad (11)$$

with

$$\eta_k(x_k) = \frac{2}{\sqrt{\pi}} \int_0^{x_k} e^{-t} \, t^{1/2} \, dt, \; \eta_k' = \frac{d\eta}{dx_k} .$$

Clearly in A_{ei}, $x_i \gg x_e$ so that

$$A_{ei}(x_e) = x_e^{-3/2} . \qquad (12)$$

Returning to the solution of the kinetic equation (5), in order to include the effect discussed by Stringer we must take account of the slow variation in θ over a ripple period by using the field line equation $\theta = \theta_o + \iota\phi$, where θ_o is a reference angle, constant for each field line. Thus, introducing the variable θ_o explicitly into the basic kinetic equation the first term becomes

$$\sigma q \, \underline{n} . \nabla \, f_j = \frac{\sigma q}{R} \frac{\partial f_j}{\partial \phi} \qquad (13)$$

while the second term is

$$\underline{v}_d . \nabla \, F_j = - \frac{B_o}{\omega_{c_j}} \left(\frac{q^2 + \mu B}{B} \right) \frac{1}{R} \, \frac{\partial F_j}{\partial r} \; \sin(\theta_o + \iota\phi) + O(\varepsilon^2) . (14)$$

We seek a solution for those particles trapped in a magnetic field ripple and for these we expand equation (5) in $\frac{\nu_{eff}}{\omega_b}$.

The term $\frac{\sigma q}{R} \frac{\partial f_j}{\partial \phi}$ may be annihilated by the operator $\sum_\sigma \int_{\phi_1}^{\phi_2} \frac{d\phi}{q}$ where $q(\phi_1) = q(\phi_2) = 0$. Unless we make the ansatz

$$f = \frac{\omega_b}{\nu_{eff}} f^{(-1)} + f^{(o)} + \frac{\nu_{eff}}{\omega_b} f^{(1)} + \ldots \tag{15}$$

we are led to a contradiction since $\sum_\sigma \int_{\phi_1}^{\phi_2} \frac{d\phi}{q} \underline{v}_{d_j} . \nabla F_j \neq 0.$

However, with the expansion (15) we have in lowest order

$$\frac{\partial f_j^{(-1)}}{\partial \phi} = 0 \tag{16}$$

and in the first order, after applying the annihilator, one obtains the following equation for $f_j^{(-1)}$

$$- \frac{\partial F_j}{\partial r} \frac{1}{\omega_{co_j} R} \int_{\phi_1}^{\phi_2} \frac{d\phi (q^2 + \mu B)}{q} \sin (\theta_o + \iota\phi) = \sum_k \nu_{jk}(\kappa) \frac{\partial}{\partial \mu} \frac{\mu}{B} J \frac{\partial f_j^{(-1)}}{\partial \mu} \tag{17}$$

where $J \equiv \int_{\phi_1}^{\phi_2} q \, d\phi = \int_{\phi_1}^{\phi_2} \sqrt{2[\kappa - \mu B_o (1 - \delta \cos N\phi - \varepsilon \cos(\theta_o + \iota\phi)]} \, d\phi .$ (18)

On the left hand side of equation (17) taking $X = N\phi$ as the variable of integration with a range of less than 2π it follows that

$$\sin (\theta_o + \iota\phi) = \sin (\theta_o + \frac{\iota}{N} X) \sim \sin \theta_o + O\left(\frac{1}{N}\right)$$

and since $\frac{\partial (\mu J)}{\partial \mu} = - \int_{\phi_1}^{\phi_2} \frac{d\phi}{q} (\mu B - q^2)$ and $\frac{q^2}{\mu B} \sim O(\delta)$

we may write equation (17) in the form

$$\frac{\partial F_j}{\partial r} \frac{1}{\omega_{co_j} R} \sin \theta_o \frac{\partial}{\partial \mu} \mu J = \frac{\partial}{\partial \mu} \frac{\mu}{B_o} J \frac{\partial f_j^{(-1)}}{\partial \mu} \left(\sum_k \nu_{jk}(\kappa) \right) + O\left(\delta, \frac{1}{N}\right). \tag{19}$$

The appropriate solution of this equation in the range

$\frac{\kappa}{B_{max}} < \mu < \frac{\kappa}{B_{min}}$ is

$$f_j^{(-1)} = \frac{\partial F_j}{\partial r} \frac{m_j}{e_j} \frac{1}{R} \sin \theta_o \left(\mu - \frac{\kappa}{B_{max}}\right) \frac{1}{\sum \nu_{jk}(\kappa)} \tag{20}$$

where B_{max}, the lower of the two maxima containing the well is given, with the aid of expression (3) by

$$\frac{B_{max}}{B_o} = 1 - \varepsilon \cos \theta_o + \delta\left\{\sqrt{1 - \alpha^2 \sin^2\theta_o} - \alpha|\sin\theta_o|[\pi - \sin^{-1}(\alpha|\sin\theta_o|)]\right\}. \tag{21}$$

3. DIFFUSION AND HEAT FLUX

The diffusion flux is defined by

$$\Gamma_j = \int_o^{2\pi} \frac{d\theta}{2\pi} \int_o^{2\pi} \frac{d\phi}{2\pi} \sum_\sigma \int_o^\infty 2\pi d\kappa \int_{\kappa/B_{max}}^{\kappa/B} \frac{B\, d\mu}{q}\, f_j^{(-1)}\, v_{dr_j} \tag{22}$$

and the ion heat flux by

$$Q_i = \int_o^{2\pi} \frac{d\theta}{2\pi} \int_o^{2\pi} \frac{d\phi}{2\pi} \sum_\sigma \int_o^\infty 2\pi d\kappa \int_{\kappa/B_{max}}^{\kappa/B} \frac{B\, d\mu}{q}\, f_i^{(-1)}\, m_i\, \kappa\, v_{dr_i} \tag{23}$$

where

$$v_{dr_j} = - \frac{\kappa}{\omega_{co_j} R} \sin \theta_o \tag{24}$$

and $f_j^{(-1)}$ is given by equation (20).

The μ integration can be performed to yield

$$\Gamma_j = - \frac{A}{(2\pi)^{3/2}} \left(\frac{T_j}{eBR}\right)^2 \int_o^\infty \frac{dx_j\, x_j^{5/2}\, e^{-x_j}}{\sum_k \nu_{jk}(x_j)} \left[\frac{n_j'}{n_j} + \frac{e_j\Phi'}{T_j} - \left(\frac{3}{2} - x_j\right)\frac{T_j'}{T_j}\right] \tag{25}$$

and

$$Q_i = - \frac{A}{(2\pi)^{3/2}} T_i \left(\frac{T_i}{eBR}\right)^2 \int_o^\infty \frac{dx_i\, x_i^{7/2}\, e^{-x_i}}{\sum_k \nu_{ik}(x_i)} \left[\frac{n_i'}{n_i} + \frac{e\Phi'}{T_i} - \left(\frac{3}{2} - x_i\right)\frac{T_i'}{T_i}\right] \tag{26}$$

where

$$A = - \frac{2\sqrt{2}N}{3\pi} \int_o^{2\pi} d\theta_o \sin^2\theta_o \int_a^b d\phi \left(1 - \frac{B}{B_{max}}\right)^{3/2} \tag{27}$$

and $B(a) = B(b) = B_{max}$. Earlier results follow from different approximations to this numerical factor A.

1. Limit of $\alpha \ll 1$. In this limit $(1 - B/B_{max}) \sim \delta(1 + \cos N\phi)$, the θ_o and ϕ integrations can be performed separately, and

$$\lim_{\alpha \to 0} A = \frac{64}{9} \delta^{3/2} . \tag{28}$$

This limit, $\alpha \to 0$, corresponds to the results obtained by analogy with stellarator diffusion and applies when there are deep ripples, e.g. $\delta \sim \varepsilon$.

2. Taking a model mirror with an appropriately defined well depth to take account of the decrease in well depth for finite values of α

$$B = B_o(1 - \tfrac{1}{2}\Delta(\theta_o) \cos N\phi - \varepsilon \cos \theta_o) \tag{29}$$

with $\Delta(\theta)$ given by equation (3), the ϕ integration can be performed explicitly, and

$$A = \frac{64}{9} \delta^{3/2} \frac{1}{\pi} \int_o^{2\pi} \sin^2 \theta_o \left(\frac{\Delta}{2\delta}\right)^{3/2} d\theta_o \equiv \frac{64}{9} \delta^{3/2} I(\alpha) \tag{30}$$

which is the result obtained by Stringer [4].

In general, however,

$$A = \frac{64}{9} \delta^{3/2} G(\alpha) \tag{31}$$

where

$$G(\alpha) = \frac{3\sqrt{2}}{8\pi\alpha^2} \int_o^c dX \frac{\cos X \sin^2 X}{\sqrt{\alpha^2 - \sin^2 X}} \int_X^{Y_1(X)} dY \{\cos X - \cos Y - (Y - X)\sin X\}^{3/2} \tag{32}$$

where $Y = Y_1(X)$ is a zero of the inner integrand.

$$\left.\begin{aligned} c &= \frac{\pi}{2} \quad ; \quad \alpha > 1 \\ &= \sin^{-1}(\alpha) \quad ; \quad \alpha < 1 \end{aligned}\right\} .$$

Asymptotic forms for $G(\alpha)$, for $\alpha \ll 1$ and $\alpha \gg 1$ are

$$G(\alpha) \sim 1 - 3\alpha + O(\alpha^2) \qquad ; \quad \alpha \ll 1$$

$$G(\alpha) \sim \frac{0.02}{\alpha^3} + O\left(\frac{1}{\alpha^5}\right) \qquad ; \quad \alpha \gg 1$$

and comparison with the correction factor $I(\alpha)$ obtained by Stringer [4] shows that $G(\alpha) \leqslant I(\alpha)$ for all α , and that $G(\alpha) \sim I(\alpha)/1.6$ for most of the range of α of interest. Thus the diffusion estimates of reference 4 are slightly pessimistic. In figure 1, $G(\alpha)$ and $I(\alpha)$ are plotted against α.

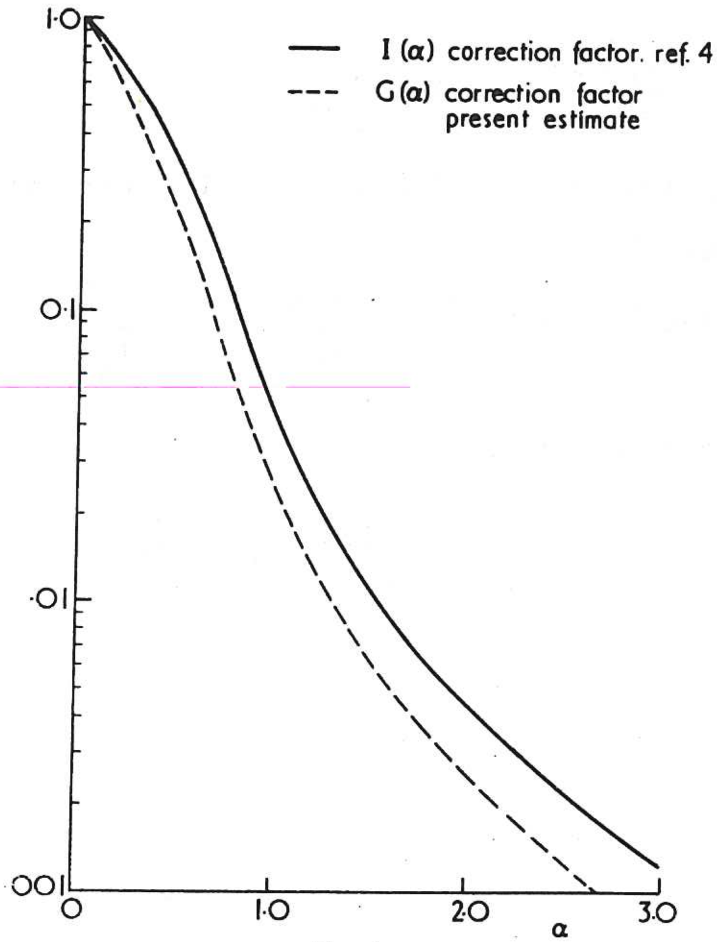

Fig. 1.

Finally we must perform the energy integrations appearing in expressions (25) and (26); these have been evaluated numerically [7] to give

$$\Gamma_i = -\frac{64}{9}\frac{\delta^{3/2}G(\alpha)}{(2\pi)^{3/2}}\left(\frac{T_i}{eBR}\right)^2\frac{27.42}{\nu_{ii}}\left[\frac{n_i'}{n_i}+\frac{e\Phi'}{T_i}+3.37\frac{T_i'}{T_i}\right] \quad (33)$$

and

$$\Gamma_e = -\frac{64}{9}\frac{\delta^{3/2}G(\alpha)}{(2\pi)^{3/2}}\left(\frac{T_e}{eBR}\right)^2\frac{12.78}{\nu_{ei}}\left[\frac{n_e'}{n_e}-\frac{e\Phi'}{T_e}+3.45\frac{T_e'}{T_e}\right] \quad (34)$$

where

$$\nu_{jk} = \frac{\sqrt{2}\pi\ ne^4\lambda}{m_j^{1/2}\,T_j^{3/2}} \ .$$

Thus the ions diffuse more rapidly and the radial electrostatic field adjusts to reduce their diffusion rate to that of the electrons. Using quasi-neutrality, $n_i = n_e = n$ we obtain

$$\Phi' = -\frac{T_i}{e}\left[\frac{n'}{n}+3.37\frac{T_i'}{T_i}\right] . \quad (35)$$

Substituting this expression in equation (33) we find the ambipolar flux Γ

$$\Gamma = -4.34\frac{\delta^{3/2}G(\alpha)}{\nu_{ei}^{*}}\left(\frac{T_e}{eBR}\right)^2\left\{\frac{n'}{n}\left(1+\frac{T_i}{T_e}\right)+3.37\frac{T_e'}{T_e}+3.45\frac{T_i'}{T_e}\right\} \quad (36)$$

and similarly for the ion heat flux

$$Q_i = -46.5\frac{\delta^{3/2}G(\alpha)}{\nu_{ii}^{*}}\left(\frac{T_i}{eBR}\right)^2 T_i' \quad (37)$$

where

$$\nu_{jk}^{*} = \frac{4}{3}\ \frac{\sqrt{2\pi}\ ne^4\lambda}{m_j^{1/2}T_j^{3/2}} \ .$$

We note that the numerical coefficients appearing in expressions (36) and (37) differ slightly from those of Stringer [4]; the coefficients used by Stringer were obtained by approximating $A_{jj}(x)$ by $x^{-3/2}$. We refer to Stringer [4] for further discussion of these results and their importance for present and planned experiments.

REFERENCES

[1] KADOMTSEV, B.B. and POGUTSE, O.P., Nuclear Fusion 11, 67 (1971).

[2] GALEEV, A.A., SAGDEEV, R.Z., FURTH, H.P. and ROSENBLUTH, M.N., Phys. Rev. Letters 22, 511 (1969).

[3] KOVRIZHNYKH, L.M., ZhETF 56, 877 (1969) [Sov. Phys. JETP 29, 475 (1969)].

[4] STRINGER, T.E., Nuclear Fusion (to be published).

[5] FRIEMAN, E.A., Phys. Fluids 13, 490 (1970).

[6] ROSENBLUTH, M.N., HAZELTINE, R.D. and HINTON, F.L., Phys. Fluids 15, 116 (1972).

[7] The authors are grateful to R.C. Grimm for performing these numerical integrations.