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ELECTRIC FIELD FLUCTUATIONS IN TURBULENT PLASMAS

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ELECTRIC FIELD FLUCTUATIONS IN TURBULENT PLASMAS

by

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ABSTRACT

It is well known that the properties of stable quiescent plasma may be calculated by a method known as the "superposition of dressed particles". In this paper an analogous superposition principle is constructed for turbulent plasmas. This shows that the turbulent fluctuations may suppress the growth of the instabilities which give rise to them. The fluctuation spectrum, diffusion coefficient and dielectric constant of turbulent plasma are calculated for the guiding centre model. The relationship of this theory to that of Dupree and Weinstock is discussed.

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1. INTRODUCTION

The study of fully turbulent plasma of necessity involves departures from conventional small amplitude perturbation theory. In this paper we investigate one such unconventional approach which illustrates how the turbulent fluctuations may limit the growth of the instabilities which gave rise to them, so leading to a steady turbulent state which can be described in some detail.

In order to introduce the approximation employed for the turbulent state we first re-examine the dressed-particle picture of the fluctuations in a quiescent plasma. This interprets the fluctuations in terms of an assembly of non-interacting particles each accompanied by by a screening cloud. Our derivation follows closely that of Dawson and Nakayama (1966) and leads to the expressions for the electric field fluctuations given by Rostoker (1961) and by Hubbard (1961). By considering how this result breaks down as instability is approached one is led to a new dressed-particle picture applicable to the fully turbulent state.

In the turbulent plasma, fluctuations are again represented by an assembly of independently moving particles but the screening accompanying each charge is now dependent on the field fluctuations themselves. An important point about the calculation of this screening is that it does not start from the assumption that the fluctuations are small, only that they are random. The present calculation is thus a complete departure from, for example, quasi-linear theory.

Our results have much in common with those of Dupree (1967,1968) and of Weinstock (1968, 1969, 1970); indeed in many respects all three approaches are equivalent. However we believe that our development brings out more clearly what is involved in the theory, provides a valuable physical picture of the mechanism and in some minor respects yields a more satisfactory result. In particular, Dupree and Weinstock determine the diffusion coefficient (from a non-linear dispersion equation) independently of the level of fluctuations and of their source - in fact the spectrum of such fluctuations remains undetermined in their calculations. In the present work the source of fluctuations is introduced explicitly and both the equilibrium level of fluctuations and the diffusion are determined as part of a single integrated development. The equations of Dupree and Weinstock then appear as approximations to our formulae.

2. THE DRESSED PARTICLE MODEL

For simplicity we consider a single species, spatially homogeneous plasma governed by the equation

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f}{\partial \underset{\sim}{x}} + \frac{e}{m} (\underset{\sim}{v} \times \underset{\sim}{B}) \frac{\partial f}{\partial \underset{\sim}{v}} + \frac{e\underset{\sim}{E}}{m} \cdot \frac{\partial f}{\partial \underset{\sim}{v}} = 0 \, . \tag{1}$$

The electro-static field is determined by Poisson's equation

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = 4\pi e \int f d\underset{\sim}{v}, \tag{2}$$

the magnetic field we take to be constant and uniform.

It must be emphasised that Eq.1 is to be interpreted as the exact equation for the (singular) distribution, not as the macroscopic Vlasov equation referring to the smoothed fields and distribution function.

The dressed particle picture can be introduced by splitting the distribution function into three parts

$$f \equiv f_0 + g + h \tag{3}$$

where

(i) f_0 is the space-averaged distribution. This is a smooth function of velocity only, $f_0 = f_0(\underset{\sim}{v})$, which we further assume to depend only on v^2,

(ii) $g(\underset{\sim}{x},\underset{\sim}{v})$ is the difference between the distribution $f_0(\underset{\sim}{v})$ and that of an assembly of non-interacting particles, that is

$$g(\underset{\sim}{x},\underset{\sim}{v},\underset{\sim}{t}) = \sum_i \delta\left(\underset{\sim}{x} - \underset{\sim}{x}_i(t)\right) \delta\left(\underset{\sim}{v} - \underset{\sim}{v}_i(t)\right) - f_0(\underset{\sim}{v}) \tag{4}$$

where

$$\frac{d\underset{\sim}{v}_i}{dt} = \frac{e}{m} (\underset{\sim}{v}_i \times \underset{\sim}{B}), \tag{5}$$

and

(iii) $h(\underset{\sim}{x},\underset{\sim}{v},t)$ is all that remains of f and is to be determined by the theory.

Now by the definition of $(f_0 + g)$ equation (1) can be written

$$\frac{dh}{dt} + \frac{e}{m}\left(\underset{\sim}{E}\cdot\frac{\partial f_o}{\partial \underset{\sim}{v}} + \underset{\sim}{E}\,\frac{\partial g}{\partial \underset{\sim}{v}} + \underset{\sim}{E}\,\frac{\partial h}{\partial \underset{\sim}{v}}\right) = 0 \tag{6}$$

where d/dt denotes the convective derivative along the orbits defined by (5). Similarly, equation (2) becomes

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{E} = 4\pi e\,\left(\int g\,d\underset{\sim}{v} + \int h\,d\underset{\sim}{v}\right) \tag{7}$$

(we have assumed the usual neutralising background which cancels the contribution from f_o).

a) Stable Plasma

Equations (6) and (7) are exact; to solve them we must introduce an approximate procedure. One approximation, applicable to stable systems only, is based on the observation that g,h,E are all zero in the fluid limit ($1/n \to 0$, $e \to 0$, with ne, e/m and thermal velocity remaining finite). In other words, g,h,E are non-zero only by virtue of the discreteness of the particles and we can therefore treat them as small, first order, quantities in an expansion in terms of a discreteness parameter, formally proportional to 1/n. The quantity f_o is zero-order in this expansion.

The unknown first order quantity h is then determined by

$$\frac{dh}{dt} + \frac{e}{m}\,\underset{\sim}{E}\cdot\frac{\partial f^o}{\partial \underset{\sim}{v}} = 0 \tag{8}$$

and

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{E} = 4\pi e\,\left(\int g\,d\underset{\sim}{v} + \int h\,d\underset{\sim}{v}\right). \tag{9}$$

Equation (8) can be solved to give

$$h(\underset{\sim}{x},\underset{\sim}{v},t) = -\frac{e}{m}\,\hat{G}_o\,\underset{\sim}{E}\cdot\frac{\partial f_o}{\partial v} \tag{10}$$

where $\hat{G}_o$ is the appropriate propagation operator of equation (8), i.e.

$$\hat{G}_o \equiv \left(\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\frac{\partial}{\partial \underset{\sim}{x}} + \frac{e}{m}\,(\underset{\sim}{v}\times\underset{\sim}{B})\cdot\frac{\partial}{\partial \underset{\sim}{v}}\right)^{-1}. \tag{11}$$

This expression is, of course, nothing more than the well known "integral along unperturbed orbits". The electric field is then given by the linear equation

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{E} + \frac{4\pi e^2}{m}\int d\underset{\sim}{v}\,\hat{G}_o\,\underset{\sim}{E}\cdot\frac{\partial f_o}{\partial \underset{\sim}{v}} = 4\pi e\sum_i \delta\,(x - x_i(t)) \tag{12}$$

and $\underset{\sim}{E}$ can therefore be expressed as the sum of independent contributions, each associated with one of the sources on the right side of (12). Introducing fourier-laplace transforms and observing that the left side of equation (12) defines the usual linear dielectric constant $\varepsilon(\underset{\sim}{k},\omega)$ one finds

$$\underset{\sim}{E}(\underset{\sim}{k},\omega) = - 4\pi e \sum_i \frac{i\, \underset{\sim}{k}\, \rho^i(\underset{\sim}{k},\omega)}{k^2 \varepsilon\ (\underset{\sim}{k},\omega)} \tag{13}$$

where $\rho^{(i)}$ is the transform of the i^{th} independent particle

$$\rho^i(\underset{\sim}{k},\omega) = \int_0^\infty dt\ \exp\left(- i\ \underset{\sim}{k}\cdot\underset{\sim}{x}_i(t) + i\omega t \right). \tag{14}$$

Thus, the electric field of the <u>interacting</u> particles is equivalent to that of an assembly of <u>independent</u> particles, but each independent particle is 'dressed' i.e. carries with it a charge cloud whose effect is to replace the actual coulomb field $1/k^2$ by the "dressed particle field" $1/k^2\varepsilon(k,\omega)$. In terms of the three components of the distribution function introduced in equation (3), g represents the bare particles and h their screening clouds. In the case of unmagnetised plasma (14) is particularly simple and the field of each dressed particle is

$$E(\underset{\sim}{x},\underset{\sim}{x}_i) = \frac{e}{2\pi^2} \int \frac{d\underset{\sim}{k}\ i\underset{\sim}{k}\ \exp(i\underset{\sim}{k}\cdot[\underset{\sim}{x} - \underset{\sim}{x}_i - \underset{\sim}{v}_i t])}{k^2\ \varepsilon(\underset{\sim}{k},\ \underset{\sim}{k}\cdot\underset{\sim}{v}_i)} \tag{15}$$

and the mean square amplitude of field fluctuations is

$$\langle E^2(\underset{\sim}{x},t)\rangle = 4\pi e^2 \int d\underset{\sim}{k}\ d\omega \cdot \frac{1}{k^3} \frac{f_o(\omega/k)}{|\varepsilon(\underset{\sim}{k}\ \omega)\ |^2}, \tag{16}$$

as given by Rostoker (1961) and Hubbard (1961).

b) <u>Onset of Instability</u>

The results described in the preceding section are valid only for a stable plasma, that is one for which there are no roots $\omega(k)$ of the dispersion equation

$$\varepsilon(\underset{\sim}{k},\omega) = 0 \tag{17}$$

in the upper half plane. The effect of approaching instability can be seen from (16); as one of the zero's of $\varepsilon(\underset{\sim}{k},\omega)$ approaches the real ω axis from below the integral in (16) increases and eventually diverges when the zero of $\varepsilon(k,\omega)$ reaches the real axis. When there is a zero of ε in the upper half of the ω-plane the theory breaks down completely and the fluctuations are no longer given by equation (16).

We have described the dressed particle model in detail to illustrate the important point that although the g-term, representing the bare particles, is the <u>source</u> of the fluctuations, their divergence as instability is approached is due entirely to the h-component which amplifies the fluctuations embodied in g. Indeed the fluctuation level directly due to the bare particles is quite independent of the stability or otherwise of the plasma and is

$$\langle E_g^2 \rangle = 4\pi ne^2 \int \frac{d^3 k}{k^2} . \qquad (18)$$

These observations provide the key to a modified dressed particle approximation suitable for the description of turbulent plasma. In this approximation we continue to regard the 'g-component' of the distribution as a small quantity but the 'h-component' and the electric field itself which become large in an unstable plasma, are treated on a more exact footing.

c) <u>Turbulent Plasma</u>

For a turbulent plasma, we return to the exact equations (6) and (7) and utilise the fact that $g \ll h$ to justify the neglect of g in (6). We cannot, of course, neglect g in (7) for it acts there as the source of the fluctuations which are to be amplified by h. Then we have

$$\frac{dh}{dt} + \frac{e}{m} \underset{\sim}{E} \cdot \frac{\partial h}{\partial \underset{\sim}{v}} + \frac{e}{m} \underset{\sim}{E} \cdot \frac{\partial f_o}{\partial \underset{\sim}{v}} = 0 , \qquad (19)$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = 4\pi e \left(\int g \, d\underset{\sim}{v} + \int h \, d\underset{\sim}{v} \right) . \qquad (20)$$

The first of these equations can again be formally solved to give

$$h = - \frac{e}{m} \hat{G}(\underset{\sim}{E}) \underset{\sim}{E} \cdot \frac{\partial f_o}{\partial \underset{\sim}{v}} \qquad (21)$$

where $\hat{G}(\underset{\sim}{E})$ is now the propagator <u>including</u> the effect of the electric field $\underset{\sim}{E}$, i.e.

$$\hat{G}(\underset{\sim}{E}) \equiv \left(\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{x}} + \frac{e}{m} (\underset{\sim}{v} \times \underset{\sim}{B}) \cdot \frac{\partial}{\partial \underset{\sim}{v}} + \frac{e}{m} \underset{\sim}{E} \cdot \frac{\partial}{\partial \underset{\sim}{v}} \right)^{-1} . \qquad (22)$$

Then the electric field is given by

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{E} + \int d^3 v\, G(\underset{\sim}{E})\underset{\sim}{E}\cdot\frac{\partial f_o}{\partial \underset{\sim}{v}} = 4\pi e \sum \delta(\underset{\sim}{x} - \underset{\sim}{x}_i(t)) . \tag{23}$$

The field is therefore no longer determined by a linear equation. Nevertheless, if we were to regard $G(\underset{\sim}{E})$ as known, then the solution of (23) could again be expressed as the sum of individual contributions each arising from one of the terms in the sum on the right of (23). That is, E would again be the field of independent dressed particles but the charge cloud accompanying each particle would depend on the field of all particles. Since the dressed particles are, by construction, independent, the central limit theorem suggests that the distribution of the sum of their fields, $\underset{\sim}{E}$, can be closely approximated by a normal distribution with (functional) probability

$$P(\{\underset{\sim}{E}\}) = N \exp\left\{-\frac{1}{2}\int dx \int dx' \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, \underset{\sim}{E}(\underset{\sim}{x},t)\cdot\underset{\sim}{E}(\underset{\sim}{x}',t')W(\underset{\sim}{x}-\underset{\sim}{x}', t-t')\right\}. \tag{24}$$

N is the normalisation and $W(\underset{\sim}{x},t)$ is the inverse of the covariance $Q(\underset{\sim}{x},t)$:

$$\int d\underset{\sim}{x}'' \int_{-\infty}^{\infty} dt''\, W(\underset{\sim}{x}-\underset{\sim}{x}'', t-t'')Q(\underset{\sim}{x}''-\underset{\sim}{x}', t''-t') = \delta(\underset{\sim}{x} - \underset{\sim}{x}')\delta(t - t'). \tag{25a}$$

The covariance $Q(\underset{\sim}{x},t)$ is defined by

$$Q(\underset{\sim}{x} - \underset{\sim}{x}', t - t') \equiv \langle \underset{\sim}{E}(\underset{\sim}{x},t)\cdot \underset{\sim}{E}(\underset{\sim}{x}',t')\rangle . \tag{25b}$$

Hence we are led to an approximate solution of (23) by first taking the propagator $G(\underset{\sim}{E})$ to be the average propagator $\langle G(E)\rangle$ for a normally distributed field with covariance $Q(x,t)$ and using this average propagator to express $\underset{\sim}{E}$ as the sum of independent dressed particle contributions. The auto-correlation of this field $\underset{\sim}{E}$ is then calculated and identified with the covariance $Q(\underset{\sim}{x},t)$ of the assumed normal form, so rendering the calculation self-consistent. It will be noted that this approach does not assume that the fluctuations are small, only that they are random.

3. THE GUIDING CENTRE MODEL

The essential step in the theory outlined above is the calculation of the average propagator $\langle G(E)\rangle$ and we shall illustrate this using a model in which the propagator has a simple form. In this model* the effect of the electric fields is represented by the guiding-centre velocity

$$\underset{\sim}{U} = \frac{C}{B^2} (\underset{\sim}{E} \times \underset{\sim}{B}) \tag{26}$$

and the propagator therefore satisfies

$$\left(\frac{d}{dt} + \underset{\sim}{U} \cdot \frac{\partial}{\partial \underset{\sim}{x}}\right) G = \delta\ (\underset{\sim}{x} - \underset{\sim}{x}')\ \delta\ (v-v')\ \delta\ (t-t') \tag{27}$$

where

$$\frac{d}{dt} \equiv \left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial z}\right) . \tag{28}$$

Then $G(E)$ is given explicitly by

$$G(\{\underset{\sim}{E}\}) = \Theta(t-t')\delta(v-v')\ \delta \left\{\underset{\sim}{x} - \underset{\sim}{x}' - v\hat{\underset{\sim}{z}}(t-t') - \int_{t'}^{t} ds\ \underset{\sim}{U}(\underset{\sim}{R}(s),s)\right\} \tag{29}$$

where

$$\frac{\partial \underset{\sim}{R}}{\partial s} = v\ \hat{z} + \underset{\sim}{U}(\underset{\sim}{R}(s),s)$$

$$\underset{\sim}{R}(t') = \underset{\sim}{x}' . \tag{30}$$

Using the Fourier representation of the δ-function we can write the average propagator, $\Gamma \equiv \langle G \rangle$ as

$$\Gamma = \Theta(t-t')\delta(v-v') \int \frac{d^3 \lambda}{(2\pi)^3}\ e^{i\lambda \cdot (\underset{\sim}{x} - \underset{\sim}{x}' - v\hat{\underset{\sim}{z}}(t-t'))} \left\langle \exp\left\{ - i\underset{\sim}{\lambda} \cdot \int_{t'}^{t} ds \underset{\sim}{U}(\underset{\sim}{R}(s),s)\right\}\right\rangle . \tag{31}$$

It is shown in appendix A that when $\underset{\sim}{U}$ is assumed to be normally distributed and isotropic

$$\left\langle \exp\left\{- i\underset{\sim}{\lambda} \cdot \int_{t'}^{t} ds\ \underset{\sim}{U}(\underset{\sim}{R}(s),s)\right\}\right\rangle = \exp\left\{ - \frac{\lambda_{\perp}^2}{2} I(v,t,t')\right\} \tag{32}$$

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* We shall not discuss the applicability of this model as it is mainly intended for illustration of the general argument. However it would be valid if e.g. the turbulence was of long-wavelength compared to larmor radii and the field fluctuation aligned with the magnetic field ($k_{\parallel} \ll k_{\perp}$) - a not unrealistic situation.

where

$$I(v,t,t') \equiv \frac{1}{2}\int_{t'}^{t} ds \int_{t'}^{t} ds' \left\langle \underset{\sim}{U}(\underset{\sim}{R}(s)s)\cdot\underset{\sim}{U}(\underset{\sim}{R}(s'),s')\right\rangle . \tag{33}$$

Inserting (32) into (31) and carrying out the λ-integration the average propagator Γ becomes

$$\Gamma = \frac{\Theta(t-t')\delta(v-v')\delta(z-z'-v(t-t'))}{2\pi\ I(v,t,t')} \exp\left[-\frac{(\underset{\sim}{x}_{\perp} - \underset{\sim}{x}'_{\perp})^2}{2\ I(v,t,t')}\right]. \tag{34}$$

This satisfies the differential equation

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z} - \frac{1}{2}\frac{\partial I(v,t,t')}{\partial t}\frac{\partial^2}{\partial \underset{\sim}{x}_{\perp}^{2}}\right)\Gamma = \delta(\underset{\sim}{x}-\underset{\sim}{x}')\delta(v-v')\delta(t-t') . \tag{35}$$

The meaning of (34) and (35) is clear. In the absence of fluctuations particles move along rigidly prescribed orbits, (in the model, straight lines) represented by appropriate δ-functions. In the presence of the fluctuating electric fields the orbits become "smeared-out", and are represented by the Gaussian function whose 'width' $I(v,t,s)$ increases with time.

We shall require only the long-term behaviour of the propagator Γ so that I may be replaced by

$$I(v,t,t') = 2D(v)(t-t') \tag{36}$$

where

$$D(v) = \frac{1}{2}\int_{0}^{\infty} ds \left\langle \underset{\sim}{u}(\underset{\sim}{R}(s),s) \cdot \underset{\sim}{u}(0,0)\right\rangle \tag{37}$$

and the propagator then satisfies

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z} - D(v)\frac{\partial^2}{\partial x_{\perp}^{2}}\right)\Gamma = \delta(\underset{\sim}{x}-\underset{\sim}{x}')\delta(v-v')\delta(t-t') . \tag{38}$$

In terms of this propagator the electric field is given by

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{E} + \frac{4\pi e^2}{m}\int dv \int dt' \int d\underset{\sim}{x}' \int dv'\,\Gamma(\underset{\sim}{x}-\underset{\sim}{x}', v-v', t-t')E(\underset{\sim}{x}', t')\frac{\partial f_0}{\partial v} =$$

$$4\pi e \sum_i \delta(\underset{\sim}{x}-\underset{\sim}{x}_i(t)) . \tag{39}$$

The left hand side of the equation represents a non-linear dielectric constant $\hat{\varepsilon}$. Equations (38) and (39) can be solved by Fourier-Laplace transform yielding

$$\hat{\varepsilon}(\underset{\sim}{k},\omega) = 1 + \frac{4\pi e^2}{mk^2}\int \frac{d\underset{\sim}{v}\;\underset{\sim}{k}\cdot\frac{\partial f_o}{\partial \underset{\sim}{v}}}{(\omega - k_z v + ik_\perp^2\, D(\underset{\sim}{v}))} \tag{40}$$

and

$$\underset{\sim}{E}_{k\omega} = -\,4\pi e \sum_i \frac{i\;k\;\rho^i_{k\omega}}{k^2\;\hat{\varepsilon}(\underset{\sim}{k},\omega)} \tag{41}$$

which is a non-linear analogue of equation (13).

Equation (40) defines a 'non-linear' dielectric constant describing the response of the turbulent plasma and embodies the stabilisation of the system by its own fluctuations. As the fluctuation level increases the coefficient $D(v)$ also increases. Consequently the roots of $\hat{\varepsilon}(\underset{\sim}{k},\omega) = 0$, in the ω-plane, move down until they are below the real axis and the system becomes (non-linearly) stable.

4. THE DIFFUSION COEFFICIENT

The final step in the theory is now to identify the autocorrelation of the final electric field E given by equation (41), with the covariance of the assumed gaussian form (24) in order to make the calculation self consistent. In the present model the self consistency condition is reduced to the requirement that $D(v)$ as determined from the electric field (41), be identical with $D(v)$ appearing in equation (38) defining the propagator.

Using (26) and expressing $\underset{\sim}{E}(\underset{\sim}{x},t)$ in terms of its fourier transform we rewrite (37) in the form

$$D(v) = \frac{c^2}{2VB^2}\int_0^\infty dt \int \frac{d\underset{\sim}{k}}{(2\pi)^3}\; \langle \underset{\sim}{E}_k(t)\cdot\underset{\sim}{E}_{-k}(0)\exp\,(i\underset{\sim}{k}\cdot R(t))\,\rangle \tag{42}$$

where V is the volume of the system. The factor $\exp(i\underset{\sim}{k}\cdot R(t))$ may be recognised as the fourier transform of the propagator G. In accordance with the general approach of this paper we replace G by its average Γ, then

$$D(v) = \frac{c^2}{2B^2}\int^\infty dt\int d\underset{\sim}{k}\,\frac{\langle \underset{\sim}{E}_k(t)\cdot\underset{\sim}{E}_{-k}(0)\rangle}{V}\exp\,(i\underset{\sim}{k}\cdot\underset{\sim}{v}t - k_\perp^2 D(v)t). \tag{43}$$

In order to calculate $E_k(t)$ we use (41) and observe that because of the non-linear stabilisation described above, all the roots of $\hat{\varepsilon}(k,\omega) = 0$ will be in the lower half of the ω-plane. Thus the asymptotic, steady state, value of $\underset{\sim}{E}_k(t)$ will be determined by the contributions to the inverse laplace transform arising from the zeros of $\rho_i(\underset{\sim}{k},\omega)$ which are on the real axis. Then

$$\underset{\sim}{E}_k(t) = -4\pi e i \frac{\underset{\sim}{k}}{k} \sum_i \frac{\exp(-i\underset{\sim}{k}\cdot[x_i+v_i t])}{\hat{\varepsilon}(\underset{\sim}{k},\underset{\sim}{k}\cdot\underset{\sim}{v}_i)} \, . \tag{44}$$

The next step is to multiply (43) by the corresponding expression for $\underset{\sim}{E}_{-k}(0)$ and form the ensemble average. Since the bare particle coordinates and velocities are, by construction, uncorrelated, this leads to

$$\frac{\langle \underset{\sim}{E}_k(t)\cdot\underset{\sim}{E}_{-k}(0)\rangle}{V} = \frac{(4\pi)^2 e^2}{k^2} \int_{-\infty}^{\infty} \frac{dv\ f_0(v)\ \exp\ (i\ k_z\ vt)}{\hat{\varepsilon}\ (\underset{\sim}{k},k_z v)\ \hat{\varepsilon}\ (-\underset{\sim}{k},-k_z v)} \, . \tag{45}$$

Substituting (45) in (43) and performing the time integral, we obtain

$$D(v) = \frac{(4\pi)^2 e^2 c^2}{2B^2} \int d\underset{\sim}{k} \int_{-\infty}^{\infty} du \frac{f_0(u)\ k_{\perp}^2\ D(v)}{k^2 \hat{\varepsilon}(\underset{\sim}{k},k_z u)\hat{\varepsilon}(-\underset{\sim}{k},\ -k_z u)\ \{k_z^2(v-u)^2 + k_{\perp}^4\ D^2(v)\}} \, . \tag{46}$$

This expression for the diffusion coefficient completes the theory and is one of the main results of this paper. The equilibrium fluctuation, obtained by putting $t = 0$ in (45), is

$$\frac{\langle\ |\underset{\sim}{E}_k|^2\rangle}{V} = \frac{(4\pi)^2 e^2}{k^2} \int_{-\infty}^{\infty} \frac{dv\ f_0(v)}{\hat{\varepsilon}(\underset{\sim}{k},k_z v)\hat{\varepsilon}(-\underset{\sim}{k},-k_z v)} \, . \tag{47}$$

5. DISCUSSION

Equations (46) and (47), together with the non-linear dielectric constant (40), provide a complete description of the turbulent plasma in terms of the mean distribution function $f_0(\underset{\sim}{v})$. (We do not discuss the determination of $f_0(\underset{\sim}{v})$ which depends in part on the external conditions.) The essential content of this theory is that a plasma which is unstable according to the usual linear theory, may be stabilised by the effect of its fluctuating electric field which produces a 'smearing out' of the particle orbits. The extent of this smearing is represented by the coefficient D and is determined self consistently by (46), the corresponding level of fluctuations is then given by (47).

That the dielectric constant embodies the self stabilising mechanism of the fluctuations can be made more explicit if one replaces $D(v)$ by a constant D. Then the non-linear dielectric constant is simply related to the linear one

$$\hat{\varepsilon}(\underset{\sim}{k},\omega) = \varepsilon\,(\underset{\sim}{k},\omega + ik_{\perp}^2 D) \tag{48}$$

and the non-linear growth rate $\hat{\gamma}$ is given in terms of the linear growth γ as

$$\hat{\gamma}(k) = \gamma(k) - k_{\perp}^2 D \,. \tag{49}$$

The system is therefore stabilised when the fluctuations have raised the diffusion coefficient to such a value that

$$D > D_0 \equiv \text{Max}\,(\gamma(k)/k_{\perp}^2)\ . \tag{50}$$

This quantity D_0 is an oft-quoted estimate for the diffusion coefficient of a linearly unstable plasma. In fact, however, the diffusion coefficient is not determined solely by the dielectric constant but by equation (46) and the proper interpretation of (50) is as a condition for the validity of the theory leading to eq.(46)!

However D_0 may, in fact, be the solution of (46) in the limit $e^2 \to 0$ when such a limiting solution exists. For, defining

$$I(D) = \int d^3k \int du \frac{f(u)k_{\perp}^2}{k^2\,\hat{\varepsilon}(\underset{\sim}{k},k_z u)\hat{\varepsilon}(-\underset{\sim}{k},-k_z u)\,\{k_z^2 u^2 + k_{\perp}^4 D^2\}} \tag{51}$$

it is apparent that as $e^2 \to 0$ equation (46) can be satisfied only if $I(D) \to \infty$. This may occur when the poles in the integrand at

$$k_z u = \omega_k \pm i\,(\gamma_k - k_{\perp}^2\,D), \tag{52}$$

arising from the zeros of $\hat{\varepsilon}(\underset{\sim}{k},k_z u)$ $\hat{\varepsilon}(-\underset{\sim}{k},-k_z u)$, approach the real axis, as they do for some value of k when $D \to D_0$. If indeed $I(D) \to \infty$ then its limiting form can be found by expanding $\hat{\varepsilon}$ in the vicinity of $\hat{\varepsilon} = 0$ to give

$$|\varepsilon(\underset{\sim}{k},k_z u)|^2 \sim \left|\frac{\partial\varepsilon}{\partial\omega}\right|^2 \{(k_z u - \omega_k)^2 + (\gamma_k - k_{\perp}^2 D)^2\}\,. \tag{53}$$

The resonant factor arising in this way has a width $\sim(\gamma - k_{\perp}^2\,D)$ and a height $(\gamma - k_{\perp}^2 D)^{-2}$ so that as $I(D)$ approaches its limiting value variation of the other factors may be ignored. Then

$$I(D) \sim \int d^3k \cdot \frac{1}{(\gamma_k - k_{\perp}^2 D)}\,. \tag{54}$$

Whether $I(D)$ does diverge as $D \to D_0$ is now seen to depend on the form (but not the magnitude) of $(\gamma(k)/k_\perp^2)$ in the vicinity of its maximum, for this determines the type of singularity in (54). If the maximum of $\gamma(k)/k_\perp^2$ occurs along a line in three dimensional $\underset{\sim}{k}$ space (as would be the case when γ_k depends on $k_\perp$ through $k_\perp^2$ and the maximum is not at $k_\perp = 0$) then the singularity in (54) when $D \to \max(\gamma(k)/k_\perp^2)$ is of the form

$$\int \frac{d^3 k}{k^2}$$

and $I(D)$ necessarily diverges as $D \to D_0$ which is therefore the limiting solution of (46) as $e^2 \to 0$. Furthermore there must be a solution of (46) for finite e satisfying $D > D_0$.

In other cases the structure of $\gamma(k)$ will be such that $\max(\gamma(k)/k_\perp^2)$ occurs only at a point in k space. Then (54) need not diverge and $I(D)$ is bounded for all $D \geqslant D_0$. We cannot then be certain that (46) possesses a valid solution. If it did not this would imply that the dispersion of particle orbits by fluctuations was never sufficient to stabilise the original instability no matter how large the fluctuations! It may be noteworthy that this can apparently occur only when the instability is completely three dimensional in character.

The dielectric constant $\hat{\varepsilon}(\underset{\sim}{k},\omega)$ and the estimate (50) were derived some time ago by Dupree (1967) and subsequently by Weinstock (1970). However equations (45) - (47), which are needed to complete the description of turbulent plasma, were not contained in these earlier investigations. Instead, both Dupree and Weinstock used a single equation expressing the diffusion coefficient in terms of the fluctuations $\langle E^2(k) \rangle$. As the spectrum $\langle E^2(k) \rangle$ then remains undetermined such a description is, in principle, incomplete. The formulae of Dupree and Weinstock may be obtained by assuming that the poles (52) lie very close to the real axis and that most of the contribution to the integral (46) will therefore come from the vicinity of $(k_z u) = \omega_k^{NL}$. Then we may ignore the variation of other factors with u and write

$$D(v) = \frac{(4\pi)^2 e^2 c^2}{2B^2} \int d\underset{\sim}{k} \int_{-\infty}^{\infty} du \frac{f_0(u)\, k_\perp^2\, D(v)}{k^2 \hat{\varepsilon}(\underset{\sim}{k},k_z u)\hat{\varepsilon}(-\underset{\sim}{k},-k_z u)\ \{ (k_z v-\omega_k^{NL})^2 + k_\perp^4 D^2(v)\}} \cdot \tag{55}$$

Comparing the remaining integration over u with the right hand side of equation (47), we see that $D(v)$ can alternatively be expressed as

$$D(v) = \frac{c^2}{B^2} \int d\underset{\sim}{k} \frac{\langle |E_k|^2 \rangle}{V} \cdot \frac{k_\perp^2 \, D(v)}{\{(k_z v - \omega_k^{NL})^2 + k_\perp^4 \, D^2(v)\}} \tag{56}$$

which is similar to the expressions derived by Dupree (1967) and by Weinstock (1968).

This last expression also has some similarity to that arising in quasi-linear theory and it is appropriate therefore to comment on the relation of this to the present discussion. Essentially the two theories are complimentary. Quasi-linear theory describes the stabilisation of an instability by the effect of fluctuating fields on the average distribution $f_0(\underset{\sim}{v})$, essentially by assuming that the fluctuations convert $f_0(\underset{\sim}{v})$ into a stable distribution. The present theory describes the stabilisation which arises from the influence of electric field fluctuations on the particle orbits, even though the mean distribution function $f_0(v)$ may remain one which would be unstable in the conventional linear theory.

Appendix A

We need to evaluate an average of the type

$$A \equiv \langle e^{\varphi} \rangle \tag{A.1}$$

where

$$\varphi \equiv - i\underset{\sim}{\lambda} \cdot \int_{t'}^{t} \underset{\sim}{u}(\underset{\sim}{R}(s),s)ds \tag{A.2}$$

and the probability of u is to be given by a normal functional distribution.

If $\underset{\sim}{R}(s)$ is a <u>fixed</u> path, independent of u, (for example the unperturbed orbit, $\underset{\sim}{R}(s) = \underset{\sim}{v}s$), the average is easily performed, for φ is then a linear functional of $\underset{\sim}{u}(r,s)$. In this case φ is a gaussian random function when $\underset{\sim}{u}(r,s)$ is a gaussian random function. However, in reality $\underset{\sim}{R}(s)$ is itself a functional of $\underset{\sim}{u}(\underset{\sim}{r},s)$ and so a strict application of this argument is not possible. We shall therefore simply assume that $\underset{\sim}{u}(\underset{\sim}{R}(s),s)$ is itself a gaussian random function of time, recognising that although this is not entirely satisfactory it is no less plausible than the assumption that $\underset{\sim}{u}(\underset{\sim}{r},s)$ is gaussian. (We are not aware of any general theory concerning the relationship between the statistical properties of Eulerian and Lagrangian variables.)

Subject to the above assumption we may determine A by a method due to Edwards (1964). Writing $\underset{\sim}{u}(s)$ for $\underset{\sim}{u}(\underset{\sim}{R}(s),s)$, we may indicate the averaging procedure explicitly in the form of a functional integral, thus:-

$$A = N \int \delta\underset{\sim}{u}(s) \exp\left\{ - i\underset{\sim}{\lambda} \cdot \int_{t'}^{t} \underset{\sim}{u}(s)ds - \frac{1}{2}\int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds' \; \underset{\sim}{u}(s)\cdot\underset{\sim}{u}(s')W(s,s') \right\} \tag{A.3}$$

N indicates the normalisation constant. $W(s,s')$ is the inverse of the covariance function $Q(s,s')$:-

$$\int_{-\infty}^{\infty} ds' \; W(s,s')Q(s,s'') = \delta(s - s'') \tag{A.4}$$

and $Q(s,s')$ is defined as

$$Q(s,s') = \langle \underset{\sim}{u}(s)\cdot\underset{\sim}{u}(s') \rangle \; . \tag{A.5}$$

We introduce a change of variable:-

$$\underset{\sim}{u}(s) = \underset{\sim}{v}(s) - i\underset{\sim}{\lambda} \int_{t'}^{t} ds' Q(s,s') . \tag{A.6}$$

Then in terms of $\underset{\sim}{v}(s)$, A is

$$A = N \int \delta\underset{\sim}{v}(s) \exp\left\{ -\frac{1}{2}\int_{\infty}^{\infty} ds \int_{\infty}^{\infty} ds' \underset{\sim}{v}(s)\cdot\underset{\sim}{v}(s') W(s,s') - \frac{\lambda_{\perp}^{2}}{2}\int_{t'}^{t} ds \int_{t'}^{t} ds'\; Q(s,s')\right\} . \tag{A.7}$$

The functional integral now cancels the normalisation, leaving

$$A = \exp\left\{ -\frac{\lambda_{\perp}^{2}}{2} \int_{t'}^{t} ds \int_{t'}^{t} ds'\; Q(s,s')\right\} . \tag{A.8}$$

Using the definition (A.5) of $Q(s,s')$ and returning to the original notation, we may alternatively write (A.8) as

$$A = \exp\left\{ \frac{\lambda_{\perp}^{2}}{4} \int_{t'}^{t} ds \int_{t'}^{t} ds' \left\langle \underset{\sim}{u}(\underset{\sim}{R}(s),s)\cdot\underset{\sim}{u}(\underset{\sim}{R}(s'),s')\right\rangle \right\} \tag{A.9}$$

which is the result needed to establish equations (32) and (33).

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