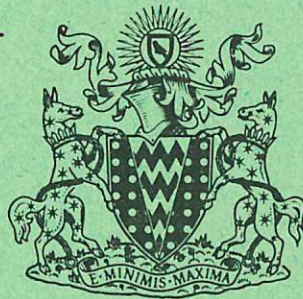
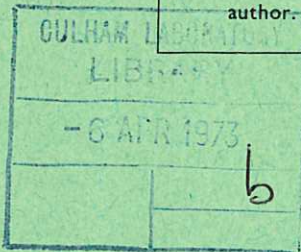


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Preprint

ANALYSIS OF THE FORMATION
OF A VAPOUR LAYER AT A
LIQUID/HOT-WALL INTERFACE

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1972

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ANALYSIS OF THE FORMATION OF A VAPOUR LAYER AT A LIQUID/HOT-WALL INTERFACE

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(submitted for publication in International Journal
of Heat & Mass Transfer)

ABSTRACT

A set of coupled nonlinear partial differential equations (PDE) describing a liquid in contact with a hot wall is derived. As an alternative to obtaining an approximate solution by linearization it is shown how the nonlinear PDEs may be reduced to nonlinear ordinary differential equations (ODE) under a variety of approximations. In the case of the acoustic approximation the important parameter is the rate of heat transfer, r_1 , across the interface. For large values of this parameter vaporization is initially suppressed due to the rapid pressure rise. The system then relaxes by volume expansion and vaporization occurs. The temperature, pressure, thickness and time of formation of the vapour layer are all calculated. For exceptionally large values of r_1 , vaporization is completely suppressed until supercritical temperatures are reached. Although large pressures are obtained their duration is short; however, if the system is contained, large, long-lived pressures occur for quite moderate heat-transfer rates. It is suggested that the large pressures which occur in a contained system could account for a substantial proportion of the shredding that occurs in a thermal interaction. That is, much of the area increase is an effect rather than the cause of a thermal interaction.

(Risley Intergroup Requisition No.F214 -
originally produced as paper FREC/P(72)12).

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Abingdon, Berkshire.

December 1972

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NOMENCLATURE

A,	cross-sectional area;
A ₁ ,	constant;
A ₂ ,	constant;
A ₃ ,	constant;
A ₄ ,	constant;
A ₅ ,	constant;
A _i ,	Airy function;
A _i '	derivative of Airy function;
B ₁ ,	constant;
B ₂ ,	constant;
B ₃ ,	constant;
B ₄ ,	constant;
B ₅ ,	constant;
B _i ,	Airy function;
B _i '	derivative of Airy function;
C ₁ ,	constant;
C ₂ ,	constant;
C ₃ ,	constant;
D,	length of constraining column of liquid;
D ₂ ,	constant;
D ₄ ,	constant;
D _a (x)	Dawson's integral;
D _{ag} (x;a,b,c),	$= \exp(-ax - be^{-cx}) \int_0^x \exp(az + be^{-cz}) dz;$
E ₄ ,	constant;
F(t),	function of time;
F ₅ ,	constant;
M,	mass of slab;
M _i ,	mass of constraining column of liquid;
N,	positive integer;
P,	pressure of liquid;
P ₀ ,	initial pressure of liquid;
P _c ,	critical pressure;
P _{max} ,	maximum pressure;

$Q,$ = $P - P_0$;
 $R,$ heating rate;
 $T,$ temperature of liquid;
 $T_0,$ initial temperature of liquid;
 $T_c,$ critical temperature;
 $T_w,$ temperature of hot wall;
 $V,$ volume of slab;
 $V_0,$ initial volume of slab;
 $a,$ constant;
 $b,$ constant;
 $c,$ specific heat of liquid; constant;
 $\text{erfc}(x),$ complementary error function;
 $f(t),$ function of time;
 $k,$ thermal conductivity of liquid;
 $m,$ mass of slab/unit area;
 $n,$ summation index;
 $p,$ constant;
 $r_1,$ heat flux;
 $r_2,$ = $-dR/dT$;
 $s,$ sonic velocity;
 $t,$ time;
 $t',$ dummy variable for integration;
 $t_1,$ time;
 $t_2,$ time;
 $u,$ velocity of liquid; variable;
 $x,$ space dimension; variable;
 $x_0,$ = 2σ ;
 $z,$ variable;
 $z_0,$ constant;

Greek symbols:

$\alpha,$ variable;
 $\gamma,$ coefficient of expansion of liquid;
 $\gamma(n,x),$ = $\int_0^x e^{-t} t^{n-1} dt$ the incomplete gamma function;
 $\kappa,$ coefficient of compressibility of liquid;

$\pi,$ = 3.14159... ;
 $\rho,$ density of liquid;
 $\rho_0,$ initial density of liquid;
 $\sigma,$ = $\sqrt{k/\rho_0 c}$;
 $\Omega,$ integral of Q.

1. INTRODUCTION

It is well known that when a hot molten metal comes into contact with a cold vaporizable liquid an explosion of considerable violence may occur. This type of explosion is called a thermal interaction or a fuel-coolant interaction, the hot liquid playing the role of the fuel whilst the vaporizable liquid is the coolant, and it is known to be the result of some physical rather than chemical process.

Explosions can also occur in liquefied hydrocarbon/water systems [1]. Witte, Cox and Bouvier [2] quote many examples of such interactions. In addition, these authors as well as Brauer, Green and Mesler [3] and Groenveld [4] give resumés of some theoretical models of thermal interactions. Numerous references may also be found in those papers.

There has been much discussion on the importance of vapour production in thermal interactions. For example, the violent explosion is sometimes attributed to an extremely rapid rate of vapour production. On the other hand a vapour layer between a hot surface and a cold vaporizable liquid will inhibit heat transfer. It is, therefore, important to investigate the conditions under which vapour production will occur and this is the subject of this paper. To be specific, we deal with the problem of a cold vaporizable liquid adjacent to a hot wall. (The word 'cold' implies only that the temperature of the vaporizable liquid is initially less than that of the wall. An extremely important case where a thermal interaction may occur is in certain hypothetical accidents in the sodium cooled fast reactor. (There the 'cold' liquid is liquid sodium.)). The situation under investigation is depicted in Fig.1. The initial conditions could occur in several ways. For example, the wall could have the same temperature as the liquid and then be heated very rapidly; or the hot wall might initially be far from the liquid and then be brought up extremely quickly. We assume that heat transfer essentially takes place in only one direction. By solving the coupled heat transfer and pressure equations we are able to predict the temperature and pressure within the cold liquid. By comparing this P-T curve with the saturation P-T curve the pressure, temperature, thickness and

time of formation of the initial vapour layer may be predicted. In our analysis we assume that physical properties such as thermal conductivity, specific heat, etc. are constant. Since the values of these constants are markedly different in the vapour our analysis is valid only up to the time when vapour is first formed. To proceed further the temperature dependence of physical properties must be taken into account.

In section 2 we derive a set of nonlinear PDEs describing the system. The linearized version of these equations has been considered by Epstein [5]. To deal with the nonlinear equations themselves we reduce the PDEs to ODEs; the variety of ways of doing this as well as the motivation is the subject of section 3. In section 4 we discuss the results and in section 5 we consider a situation where the slug of cold liquid is completely trapped.

2. THE PDE METHOD

We derive the nonlinear PDEs describing the system illustrated in Fig.1. On the assumption that conduction is the only heat-transfer mechanism present, the net heat gained by a slab of liquid in unit time is $k \partial^2 T / \partial x^2$ per unit area. Hence from the first law of thermodynamics we have

$$\rho_0 c \frac{\partial T}{\partial t} + \frac{\gamma}{\kappa} T \frac{\partial u}{\partial x} = k \frac{\partial^2 T}{\partial x^2} \quad (2.1)$$

where ρ_0 , c , γ , κ , and k are respectively the liquid density, specific heat, coefficient of expansion, coefficient of compressibility and conductivity and all are assumed to be constant. u is the liquid velocity. The equation of state provides a second equation

$$\frac{\partial P}{\partial t} = \frac{\gamma}{\kappa} \frac{\partial T}{\partial t} - \frac{1}{\kappa} \frac{\partial u}{\partial x} \quad (2.2)$$

and finally we have the equation of motion

$$\frac{\partial u}{\partial t} = - \frac{1}{\rho_0} \frac{\partial P}{\partial x} \quad (2.3)$$

It is not difficult to eliminate u from these equations, the result being an inhomogeneous wave equation for the pressure

$$\kappa \rho_o \frac{\partial^2 P}{\partial t^2} - \frac{\partial^2 P}{\partial x^2} = \gamma \rho_o \frac{\partial^2 T}{\partial t^2} \quad (2.4)$$

and a third order equation for the temperature.

$$\rho_o c \frac{1}{T} \frac{\partial^2 T}{\partial t^2} - \frac{\rho_o c}{T^2} \left(\frac{\partial T}{\partial t} \right)^2 - \frac{\gamma}{\rho_o \kappa} \frac{\partial^2 P}{\partial x^2} = \frac{k}{T} \frac{\partial^3 T}{\partial t \partial x^2} - \frac{k}{T^2} \frac{\partial T}{\partial t} \frac{\partial^2 T}{\partial x^2} \quad (2.5)$$

By using Eq.(2.4) this third order equation may be reduced to

$$\left(\rho_o c + \frac{\gamma^2}{\kappa} T \right) \frac{\partial T}{\partial t} - \gamma T \frac{\partial P}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \quad (2.6)$$

Due to the nonlinear nature of these equations further analytical work in this direction is difficult. Two alternative courses of action are :

1. Linearize the equations.
2. Retain the nonlinearity but reduce the equations to ODEs by means of some approximation.

The second of these alternatives is the main subject matter of this paper and will be considered in section 3.

The nonlinearity of Eq.(2.6) arises from the terms $T \partial T / \partial t$ and $T \partial P / \partial t$ and so the equations may be linearized by neglecting these terms. The source of the nonlinear terms is $T \partial u / \partial x$ in Eq.(2.1), thus a linear theory is produced if the energy equation is assumed to that of simple heat conduction. Since the physical basis of the non-linear term is the fact that the internal energy is a function of volume as well as temperature, we see that a linear theory will be accurate in the case of substances for which this volume dependence is slight. The linearized equations, which may be solved using Laplace transform techniques*, are the subject of a paper by Epstein [5]. However, there are two main disadvantages in dealing with the linearized equations:

* Epstein gives no details of the method. For convenience we give details of the solution in Appendix 1.

1. The Laplace transform method can only be used if the length of the system is infinite.
2. The only heat-transfer process that is considered is simple heat conduction. In fact, it is unlikely that this is the only significant process; turbulence is observed to be present in thermal interactions and this will certainly complicate any heat-transfer mechanism.

For these reasons we consider an alternative approach.

3. THE ODE METHOD

In this section we consider how a solution to the general problem posed in the introduction may be obtained by reducing the PDEs (2.1 - 2.3) to nonlinear ODEs. This may be done by assuming that within a given slab of liquid the temperature and pressure are uniform, i.e. space independent. Apart from mathematical expediency there are two reasons for taking this step:

1. A numerical solution to the problem involves dividing the liquid into cells within which the temperature and pressure are assumed to be uniform. Thus an analytic solution should assist the interpretation of numerical results where, of course, the nonlinearity is even more pronounced since the properties of the materials are assumed temperature dependent.
2. The results will be used in a future paper in the development of an analytic model of thermal interactions.

As a justification of the ODE method we note that

1. In a paper by Cho, Ivins and Wright [6] uniform temperature and pressure is also assumed within a certain zone. The results they obtain do not rule out this approximation.

2. The results obtained by Epstein [5] indicate that uniformity of temperature and pressure is not a gross assumption behind the pressure peak except for very early times.

The equations describing the system depend precisely on what mechanism is used to constrain the slab and also on whether constant or variable mass of slab is assumed. We consider first the acoustic approximation. We expect this approximation to be valid up until the time taken for a sound wave to travel the length of the system and back again. Thereafter, the system will be inertially contained.

A. Acoustic Approximation.

Consider a slab of liquid next to a hot wall (Fig.2). The liquid/hot-wall interface is stationary at all times but the opposite end of the slab is free to move via the excess pressure and also expansion due to temperature rise. The velocity of this end is u . Beyond the slab lies the constraining liquid whose temperature, pressure and density remain constant in time. The cross-sectional area is A . From the first law of thermodynamics we have

$$Mc \frac{dT}{dt} + T \frac{\gamma}{\kappa} \frac{dV}{dt} = RA \quad (3.1)$$

where $M = \rho \times A$ is the mass of the slab and R is the rate of heating per unit time and area. V is the volume of the slab. The pressure is given by

$$\frac{dP}{dt} = \frac{\gamma}{\kappa} \frac{dT}{dt} - \frac{1}{\kappa V} \frac{dV}{dt} \quad (3.2)$$

For the acoustic approximation of infinite extent we have the equation of motion

$$\frac{dV}{dt} = \frac{A(P-P_0)}{\rho_0 s} \quad (3.3)$$

where s is the sonic speed in the unheated liquid. Thus Eqs.(3.1) and (3.2) become

$$\dot{T} + \frac{\gamma}{\kappa m \rho_0 s c} T (P-P_0) = \frac{R}{mc} \quad (3.4)$$

$$\dot{P} = \frac{\gamma}{\kappa} \dot{T} - \frac{(P-P_0)}{\kappa \rho_0 s x} \quad (3.5)$$

where $m = \rho x$ is the mass per unit area of the slab. The initial conditions are

$$P(0) = P_0 \quad T(0) = T_0 .$$

To proceed further we must specify whether the mass, m , being heated is constant or not. The well-known solution of the Fourier heat conduction law in one dimension is

$$T = T_W \operatorname{erfc}\left(\frac{x}{2\sigma\sqrt{t}}\right) \quad (3.6)$$

$$\sigma^2 = k/\rho_0 c \quad \text{and} \quad T(0,t) = T_W .$$

Although the solution (3.6) implies heating throughout all space, the effective sphere of influence is usually taken to be $x = 2\sigma\sqrt{t}$. Thus in Eqs.(3.4) and (3.5) we define the variable mass approximation to be such that

$$m = \rho_0 x_0 \sqrt{t}$$

where $x_0 = 2\sigma^*$. Eqs.(3.4) and (3.5) become

$$\dot{T} + \frac{\gamma}{\kappa \rho_0^2 s c x_0} t^{-\frac{1}{2}} T Q = \frac{R}{\rho_0 c x_0} t^{-\frac{1}{2}} \quad (3.7)$$

$$\dot{Q} = \frac{\gamma}{\kappa} \dot{T} - \frac{t^{-\frac{1}{2}} Q}{\kappa \rho_0 s x_0} . \quad (3.8)$$

(Henceforth we shall use $Q = P-P_0$ as the pressure variable. The initial condition is $Q(0) = 0$).

As an alternative we may use the constant mass approximation $m = \text{constant}$. Numerical solutions which use a Lagrangian mesh have constant mass cells, thus this method provides a more direct comparison with computer codes. In this method, however, we still have an unknown, x , in Eq.(3.5) to account for. (x may be replaced by m/ρ but ρ is also unknown). To circumvent this problem we replace

*The value of x_0 is a matter of definition. Some may prefer to use $x_0 = 4\sigma$ thus extending the region effectively heated. The effect on the numerical results is slight and certainly the overall conclusions are unaltered.

x by m/ρ_0 . This approximation is equivalent to replacing $-\frac{1}{\kappa V} \delta V$ in the equation of state by $-\frac{1}{\kappa V_0} \delta V$ where V_0 is the initial volume. Assuming that V is a monotonic increasing function of time the effect of this approximation is to decrease the pressure from its correct value. With the constant mass approximation Eqs.(3.4) and (3.5) become

$$\dot{T} + \frac{\gamma}{\kappa m \rho_0 s c} T Q = \frac{R}{m c} \quad (3.9)$$

$$\dot{Q} = \frac{\gamma}{\kappa} \dot{T} - \frac{Q}{\kappa s m} \quad (3.10)$$

B. Inertial Constraint.

In this case the interacting slab of Fig.2 is constrained by a mass M_i of unheated liquid. Let the length of the unheated slab be D . ($D \gg x$, hence D is constant). The acceleration of the constraining slab is $AQ(t)/M_i$ and so

$$\frac{dV}{dt} = \frac{A^2}{M_i} \int_0^t Q(t') dt'$$

thus Eqs.(3.1) and (3.2) become

$$\dot{T} + \frac{\gamma}{\kappa m c \rho_0 D} T \int_0^t Q(t') dt' = \frac{R}{m c} \quad (3.11)$$

$$\dot{Q} = \frac{\gamma}{\kappa} \dot{T} - \frac{1}{\kappa \rho_0 D x} \int_0^t Q(t') dt' \quad (3.12)$$

For the constant mass approximation we have to solve the equations

$$\dot{T} + \frac{\gamma}{\kappa m c \rho_0 D} T \Omega = \frac{R}{m c} \quad (3.13)$$

$$\ddot{\Omega} = \frac{\gamma}{\kappa} \dot{T} - \frac{1}{\kappa D m} \Omega \quad (3.14)$$

where $\Omega = \int_0^t Q(t') dt'$ and $\Omega(0) = \dot{\Omega}(0) = 0$.

With the variable mass approximation the relevant equations are

$$\dot{T} + \frac{\gamma}{\kappa \rho_0^2 x_0 D c} t^{-\frac{1}{2}} T \Omega = \frac{R}{\rho_0 c x_0} t^{-\frac{1}{2}} \quad (3.15)$$

$$\ddot{\Omega} = \frac{\gamma}{\kappa} \dot{T} - \frac{1}{\kappa \rho_0 D x_0} t^{-\frac{1}{2}} \Omega \quad (3.16)$$

Before any of these sets of equations can be solved the rate of heating, R , must be specified. We use two approximations:

1. Constant heating $R = r$ (3.17)

2. As the temperature of the slab increases we expect the rate of heating to decrease, thus a second approximation is (variable heating)

$$R = r_1 - r_2 T \quad (3.18)$$

r_1 and r_2 are constants. It should be noted that the heat-transfer mechanisms are not specified. A detailed estimation of r_1 and r_2 might take into account conduction, convection and radiation as well as the modifications of these processes due to turbulence, but in our calculation r_1 and r_2 are simply input parameters. Average heating rates of 10^7 j/s/m² have been measured but heat transfer rates much larger than this may occur during the initial stages.

We now discuss how the various sets of equations may be solved. The simplest case is the acoustic constraint, constant mass and constant heating approximation (Eqs.(3.9), (3.10), and (3.17)).

Elimination of T and \dot{T} gives

$$\ddot{\Omega} + \frac{\gamma}{\kappa m \rho_o s c} \dot{\Omega}^2 + \left[\left(\frac{1}{\kappa s m} + \frac{\gamma^2 T_o}{\kappa^2 m \rho_o s c} \right) + \frac{\gamma}{\kappa^2 s^2 m^2 \rho_o c} \Omega \right] \dot{\Omega} - \frac{r_1 \gamma}{m c \kappa} = 0 \quad (3.19)$$

where $\Omega = \int_0^t Q(t') dt'$. Equations similar to (3.19) are discussed in most books on nonlinear differential equations and it is known that solutions with fixed critical points exist only for certain values of the coefficient of $\dot{\Omega}^2$ (see, for example, Murphy [7] or Davis [8]). Unfortunately, in this case the coefficient of $\dot{\Omega}^2$ does not fall within this class. Little work has been done on equations whose critical points are not necessarily fixed. To obtain an approximate solution we solve the temperature equation (3.9) with the nonlinear term set equal to zero and then use this solution in the full set of equations. This method is not equivalent to linearizing

the equations. Comparison with numerical solutions shows that this method of obtaining the temperature solution is reasonably accurate. Hence we find a first order equation for Q and this is easily solved to give

$$Q = C_1 \sqrt{\frac{2}{B_1}} \left\{ D_a \left(\sqrt{\frac{B_1}{2}} t + \frac{A_1}{2} \sqrt{\frac{2}{B_1}} \right) - \exp \left(-A_1 t - \frac{1}{2} B_1 t^2 \right) D_a \left(\frac{A_1}{2} \sqrt{\frac{2}{B_1}} \right) \right\} \quad (3.20)$$

$$T = T_0 + \frac{r_1}{mc} t \quad (3.21)$$

where

$$A_1 = \frac{1}{\kappa s m} + \frac{\gamma^2 T_0}{\kappa^2 m \rho_0 s c}$$

$$B_1 = \frac{\gamma^2 r_1}{\kappa^2 m^2 c^2 \rho_0 s}$$

$$C_1 = \frac{r_1 \gamma}{m c \kappa}$$

$D_a(x)$ is Dawson's integral [9]

$$D_a(x) = e^{-x^2} \int_0^x e^{z^2} dz$$

The other sets of equations may be dealt with in a similar manner. In all cases we can obtain a single differential equation for Q (or some integral of Q times a weighting factor*) but in no case have we solved the resulting equation. For the acoustic constraint we always find a second order differential equation whose critical points are not necessarily fixed whilst in the case of the inertial constraint the differential equation turns out to be third order nonlinear. Approximate solutions may be obtained as before. We find the following† :

* See Appendix 2.

† See Appendix 2 for more details.

A. Acoustic Constraint.

1. Constant mass

(a) Constant heating

Eqs. (3.20) and (3.21)

(b) Variable heating

$$T = \frac{r_1}{r_2} - \left(\frac{r_1}{r_2} - T_0 \right) \exp(-D_2 t) \quad (3.22)$$

$$Q = \frac{C_2}{B_2} \left\{ \exp\left(\frac{B_2}{D_2} - A_2 t - \frac{B_2}{D_2} e^{-D_2 t} \right) - 1 + A_2 D_{ag}(t; A_2, B_2/D_2, D_2) \right\} \quad (3.23)$$

$$A_2 = \frac{1}{\kappa s m} + \frac{\gamma^2 r_1}{\kappa^2 m \rho_0 s c r_2}$$

$$B_2 = \frac{\gamma^2}{\kappa^2 m \rho_0 s c} \left(\frac{r_1}{r_2} - T_0 \right)$$

$$C_2 = \frac{r_2 \gamma}{m c \kappa} \left(\frac{r_1}{r_2} - T_0 \right)$$

$$D_2 = \frac{r_2}{m c}$$

$$D_{ag}(x; a, b, c) = \exp(-ax - be^{-cx}) \int_0^x \exp(az + be^{-cz}) dz$$

2. Variable Mass

(a) Constant heating

$$T = T_0 + \frac{2r_1}{\rho_0 c x_0} t^{\frac{1}{2}} \quad (3.24)$$

$$Q = \frac{2C_3}{\sqrt{A_3}} \left\{ D_a \left(\sqrt{A_3 t} + \frac{B_3}{\sqrt{A_3}} \right) - \exp(-A_3 t - 2B_3 t^{\frac{1}{2}}) D_a(B_3/\sqrt{A_3}) \right\} \quad (3.25)$$

$$A_3 = \frac{2 r_1 \gamma^2}{\kappa^2 \rho_o^3 \text{sc}^2 x_o^2}$$

$$B_3 = \frac{1}{\kappa \rho_o \text{sc} x_o} + \frac{\gamma^2 T_o}{\kappa^2 \rho_o^2 \text{sc} x_o}$$

$$C_3 = \frac{r_1 \gamma}{\rho_o \text{c} x_o \kappa}$$

(b) Variable heating

$$T = \frac{r_1}{r_2} - \left(\frac{r_1}{r_2} - T_o \right) \exp\left(-D_4 t^{\frac{1}{2}}\right) \quad (3.26)$$

$$Q = \frac{E_4}{B_4} \left\{ \exp\left(\frac{2B_4}{D_4} - 2A_4 t^{\frac{1}{2}} - \frac{2B_4}{D_4} e^{-D_4 t^{\frac{1}{2}}}\right) - 1 + 2A_4 D_{ag} \left(t^{\frac{1}{2}}; 2A_4, 2B_4/D_4, D_4 \right) \right\} \quad (3.27)$$

$$A_4 = \frac{1}{\kappa \rho_o \text{sc} x_o} + \frac{\gamma^2 r_1}{\kappa^2 \rho_o^2 \text{sc} x_o r_2}$$

$$B_4 = \frac{\gamma^2}{\kappa^2 \rho_o^2 \text{sc} x_o} \left(\frac{r_1}{r_2} - T_o \right)$$

$$D_4 = \frac{2r_2}{\rho_o \text{c} x_o}$$

$$E_4 = \frac{r_2 \gamma}{\kappa \rho_o \text{c} x_o} \left(\frac{r_1}{r_2} - T_o \right)$$

B. Inertial Constraint

1. Constant mass

(a) Constant heating

T given by Eq.(3.21)

$$Q = \pi F_5 B_5^{-1/3} \left\{ B_i'(-z) \int_{z_0}^z A_i(-y) dy - A_i'(-z) \int_{z_0}^z B_i(-y) dy \right\} \quad (3.28)$$

$$z_0 = -A_5 B_5^{-2/3}$$

$$z = z_0 + B_5^{1/3} t$$

$$A_5 = \frac{1}{\kappa D m} + \frac{\gamma^2 T_0}{\kappa^2 m c^2 \rho_0 D}$$

$$B_5 = \frac{\gamma^2 r_1}{\kappa^2 m^2 c^2 \rho_0 D}$$

$$F_5 = \frac{\gamma r_1}{m c \kappa}$$

A_i and B_i are the Airy functions (ref.[9] p.446) and A_i' and B_i' are their derivatives.

(b) Variable heating

T given by Eq.(3.22)

Q can be found in terms of integrals and derivatives of Bessel functions of imaginary order and argument.

2. Variable mass

(a) Constant heating

T given by Eq.(3.24) .

Q can be expressed in terms of a series solution whose coefficients are determined by a three-term recurrence relation.

(b) Constant heating

T given by Eq.(3.26)

No solution for Q has been obtained.

4. DISCUSSION OF RESULTS

A. Acoustic Constraint.

The solutions (3.20), (3.23), (3.25) and (3.27) can all be written in the form

$$Q = \text{constant } e^{-F(t)} \int_0^t f(t') e^{F(t')} dt' \quad (4.1)$$

where in each case F and f are positive for all $t > 0$. $Q(0) = 0$ and it is not difficult to prove that in each case $\lim_{t \rightarrow \infty} Q = 0$. Since Q is a non-negative function it follows that Q must have at least one maximum in the range $(0, \infty)$. Numerical evaluation indicates that only one maximum exists. If the pressure and temperature equations are used to eliminate t we find that the P-T curve takes the form shown in Fig.3. Also shown is the P-T saturation line for water. The liquid vaporizes when the P-T curve crosses the saturation line. Since the saturation line has only a finite extent it is possible that no intersection occurs. In this case vaporization is suppressed completely until the high density gas is formed at the critical temperature. Tables 1-3 display some results for the case when the liquid is water*. (Appendix 3 gives details concerning the evaluation of the integrals).

The results shown in Tables 1-3 indicate that:

1. Variation of r_2 has little effect provided $r_1/r_2 > 10^3$. Physically this means that if r_1 is large enough the fact that the slab's temperature is rising has little effect on the net rate at which heat is entering the slab. By considering conduction across a slab whose ends are at temperatures T_W and T_O we find

$$\frac{r_1}{r_2} = \frac{T_W + T_O}{2} .$$

* See Appendix 4 for the application to sodium.

In the case of thermal interactions T_W and T_O are typically such that $r_1/r_2 \sim 10^3$. Thus, from the point of view of thermal interactions, it is sufficient to consider only constant heating models. Of course, if the temperature of the liquid, T , becomes a significant proportion of T_W then the variable heat approximation must be used to predict the temperature as a function of time.

2. The temperatures and pressures at which vaporization first occurs differ only slightly from one approximation to another. Thus the constant mass, constant heat approximation (the simplest) may be used to predict the temperatures and pressures at which vaporization occurs in all cases. (Although we have not presented any results for this approximation we have already seen that using variable heating is a refinement which has little effect on the results.)
3. The time at which vaporization occurs is the only result that is significantly model-dependent. Since the temperature and pressure for any of the acoustic approximations may be predicted using the simplest case and since for each approximation we have an explicit expression for the temperature as a function of time, the vaporization time is effectively known by considering only the simplest approximation.
4. As r_1 increases higher temperatures are required before vaporization occurs, i.e. the liquid is heated so rapidly that the pressure rise due to a rapid temperature rise is sufficient to suppress vaporization until the system relaxes by expansion. If r_1 is large enough vaporization is suppressed completely until the high density gas is formed at the critical temperature. It should be observed that quite large values of r_1 are needed before large pressures are generated. In section 5 we will show that if the system is contained, large pressures may be generated for small r_1 .

B. Inertial Constraint

By using asymptotic expansions for $A_i'(-z)$, $B_i'(-z)$, $\int_0^z A_i(-y)dy$ and $\int_0^z B_i(-y)dy$ ([9], p.446), an approximate expression for Eq.(3.28) can be derived

$$Q = F_5 A_5^{-\frac{1}{2}} \left(1 + \frac{B_5 t}{4A_5} \right) \sin A_5^{\frac{1}{2}} t \quad (4.2)$$

(By using Luke's [10] expansions higher terms can be shown to be negligible). Thus the P-T curve has the form shown in Fig.4. Vaporization occurs when the P-T curve first intersects the saturation line. As before we see that vaporization is initially suppressed due to the rapid temperature rise causing a rapid pressure rise. It is found, however, that vaporization can occur at much lower temperatures than in the acoustic case. This is because the constraint due to a column of liquid ($D \sim 1m$) is slight. Thus a rapid expansion of the slab can take place after the initial pressure rise. This results in vaporization at low temperatures.

5. INFINITELY STRONG CONSTRAINT

We have seen how large pressures can occur in a vaporized region by having a large influx of heat. However, it is to be expected that the duration of these pressures is short (see Tables 1-3 and also Epstein [5])^{*}. In this section we show how long-lived pressures can be obtained with quite moderate heat fluxes.

We assume that the system discussed previously is bounded by a rigid wall to the right of the slab of liquid. (As far as thermal interactions are concerned such a situation could occur if jetting takes place and the vaporizable liquid is trapped by the hot fuel). To represent the infinitely strong constraint we assume that the system cannot relax by volume expansion. In this case $\partial u / \partial x = 0$ and Eqs.(2.1) and (2.2) become

$$\rho_0 c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \quad (5.1)$$

$$\frac{\partial P}{\partial t} = \frac{\gamma}{\kappa} \frac{\partial T}{\partial t} \quad (5.2)$$

* See Appendix 1.

Hence

$$P = P_o + \frac{\gamma}{\kappa} (T - T_o) \quad (5.3)$$

For water the P-T curve given by Eq.(5.3) lies wholly above the saturation line. Thus vaporization is suppressed. If we assume that the solution of Eq.(5.1) is simply the solution in the infinite region then (for perfect thermal contact)

$$T = T_o + (T_W - T_o) \operatorname{erfc}\left(\frac{x}{2\sigma\sqrt{t}}\right) \quad (5.4)$$

On substituting Eq.(5.4) into (5.3) we see that the maximum pressure is

$$P_{\max} - P_o = \frac{\gamma}{\kappa} (T_W - T_o) \quad (5.5)$$

which is exactly the same as that calculated by Epstein [5]. However, unlike Epstein's model the pressure in this case does not decay but tends to become equal to P_{\max} throughout the whole region (Fig.5).

For the spatially independent situation we have

$$\rho_o c x \frac{dT}{dt} = r_1 - r_2 T \quad (5.6)$$

$$\frac{dP}{dt} = \frac{\gamma}{\kappa} \frac{dT}{dt} \quad (5.7)$$

these equations being for variable heating. x is $x_o\sqrt{t}$ or a parameter depending on which of the variable or constant mass approximations is used. The solutions to Eqs.(5.6) and (5.7) are given by Eqs.(5.3) and (3.22) or (3.26). For variable mass the maximum pressure is

$$P_{\max} - P_o = \frac{\gamma}{\kappa} \left(\frac{r_1}{r_2} - T_o \right) \quad (5.8)$$

which for $r_1 = 10^6$, $r_2 = 10^3$ and $T_o = 300\text{K}$ gives 4400 bars. Thus very high pressures may be generated by quite moderate heat fluxes. If we use the expression

$$\frac{r_1}{r_2} = \frac{T_W + T_o}{2}$$

with $T_W = 1000\text{K}$ then $P_{\max} - P_o = 2200$ bars.

6. CONCLUSIONS

We have seen how high pressures may develop as a result of two different mechanisms. In the first case, if the heat input is high enough large pressures occur initially and vaporization is suppressed. The system then relaxes by expansion and vapour is formed. However, it is expected that any large pressures occurring by this means are short-lived. In the second case, if the system is contained by rigid walls quite moderate heat fluxes can produce extremely large long-lived pressures. There exists the possibility that a large proportion of the shredding that occurs during a thermal interaction is the effect rather than the cause of the interaction. If the cold liquid is trapped by the fuel then the pressures that result from quite moderate heat transfer rates are sufficient to blow the surrounding fuel apart. Thus any model of thermal interactions that claims to predict the area increase must separate out this particular source of shredding and that which is the cause of the coolant being trapped. That is, the area increase that occurs by, for example, jetting, is the means of the coolant being trapped by the fuel; further area increase can then occur by this surrounding fuel being blown apart.

Acknowledgements

It is a pleasure to acknowledge valuable discussions with
Dr J A Reynolds.

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TABLE 1

r_1 (j/s/m ²)	Time at which vapour layer forms (secs)	Temperature at which vapour layer forms (K)	Thickness (mm)	Pressure of layer at formation (n/m ²)
10^6	2.24×10^{-2}	373	1.46×10^{-1}	1.016×10^5
10^7	2.30×10^{-4}	374	1.48×10^{-2}	1.041×10^5
10^8	2.84×10^{-6}	380	1.64×10^{-3}	1.289×10^5
10^9	5.44×10^{-8}	414.5	2.28×10^{-4}	3.767×10^5
10^{10}	1.74×10^{-9}	504	4.07×10^{-5}	28.49×10^5
10^{11}	Vaporizes above critical temperature			

Table 1 Vapour layer formation using the acoustic, variable mass, constant heating approximation. [Eqs. (3.24) and (3.25)]. ($P_0 = 1$ bar, $T_0 = 300$ K). The liquid is water.

TABLE 2

r_1 (j/s/m ²)	r_2 (j/s/ m ² /deg)	Time at which vapour layer forms (secs)	Temperature at which vapour layer forms (K)	Thickness (mm)	Pressure of layer at formation (n/m ²)
10 ⁶	10 ³	5.00 × 10 ⁻²	373	2.18 × 10 ⁻¹	1.013 × 10 ⁵
10 ⁷	10 ³	2.40 × 10 ⁻⁴	373.5	1.51 × 10 ⁻¹	1.04 × 10 ⁵
10 ⁸	10 ³	2.72 × 10 ⁻⁶	380	1.65 × 10 ⁻³	1.29 × 10 ⁵
10 ⁸	10 ⁴	2.89 × 10 ⁻⁶	380	1.67 × 10 ⁻³	1.28 × 10 ⁵
10 ⁹	10 ³	5.52 × 10 ⁻⁸	414.5	2.29 × 10 ⁻⁴	3.76 × 10 ⁵
10 ⁹	10 ⁴	5.56 × 10 ⁻⁸	414.4	2.30 × 10 ⁻⁴	3.75 × 10 ⁵
10 ⁹	10 ⁵	5.82 × 10 ⁻⁸	413.5	2.35 × 10 ⁻⁴	3.65 × 10 ⁵
10 ¹⁰	10 ³	1.77 × 10 ⁻⁹	504	4.11 × 10 ⁻⁵	28.44 × 10 ⁵
10 ¹⁰	10 ⁴	1.78 × 10 ⁻⁹	504	4.12 × 10 ⁻⁵	28.41 × 10 ⁵
10 ¹⁰	10 ⁵	1.78 × 10 ⁻⁹	503	4.12 × 10 ⁻⁵	28.29 × 10 ⁵
10 ¹⁰	10 ⁶	1.82 × 10 ⁻⁹	501	4.16 × 10 ⁻⁵	27.02 × 10 ⁵
10 ¹¹	Vaporizes above critical temperature				

Table 2 Vapour layer formation using the acoustic, variable mass, variable heating approximation. [Eqs. (3.26) and (3.27)]. ($P_0 = 1$ bar, $T_0 = 300$ K). The liquid is water.

TABLE 3

r_1 (j/s/m ²)	Time at which vapour layer forms (secs)	Temperature at which vapour layer forms (K)	Pressure of layer at formation (n/m ²)
10^6	4.80×10^{-1}	375	1.078×10^5
10^7	3.30×10^{-2}	375.5	1.106×10^5
10^8	3.46×10^{-3}	381.5	1.354×10^5
10^9	4.83×10^{-4}	414	3.833×10^5
10^{10}	8.54×10^{-5}	504	28.5×10^5
10^{11}	Vaporizes above critical temperature		

Table 3 Vapour layer formation using the acoustic, constant mass, variable heating approximation. [Eqs. (3.22) and (3.23)]. ($P_0 = 1$ bar, $T_0 = 300$ K, $m = x\rho_0$, $x = 10^{-3}$ m, $r_2 = 10^3$). The cold liquid is water. For other values of x the same temperatures and pressures occur but the times are altered in proportion. For example, if $x = 10^{-4}$ and $r_1 = 10^6$ then the time to vaporize is 4.80×10^{-2} secs. Variation of r_2 affects these results only slightly. cf. Table 2.

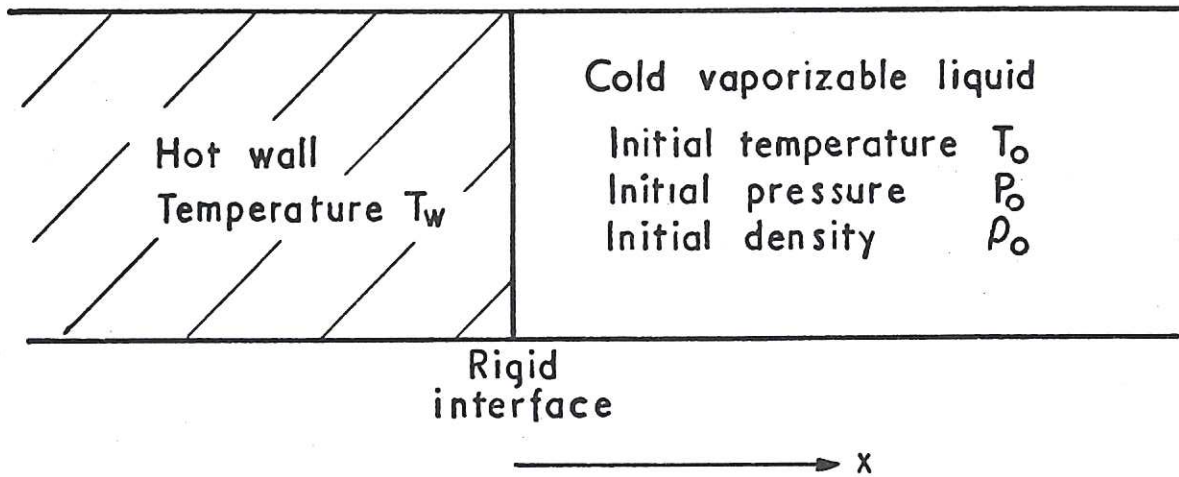


Figure 1. The Initial Conditions.

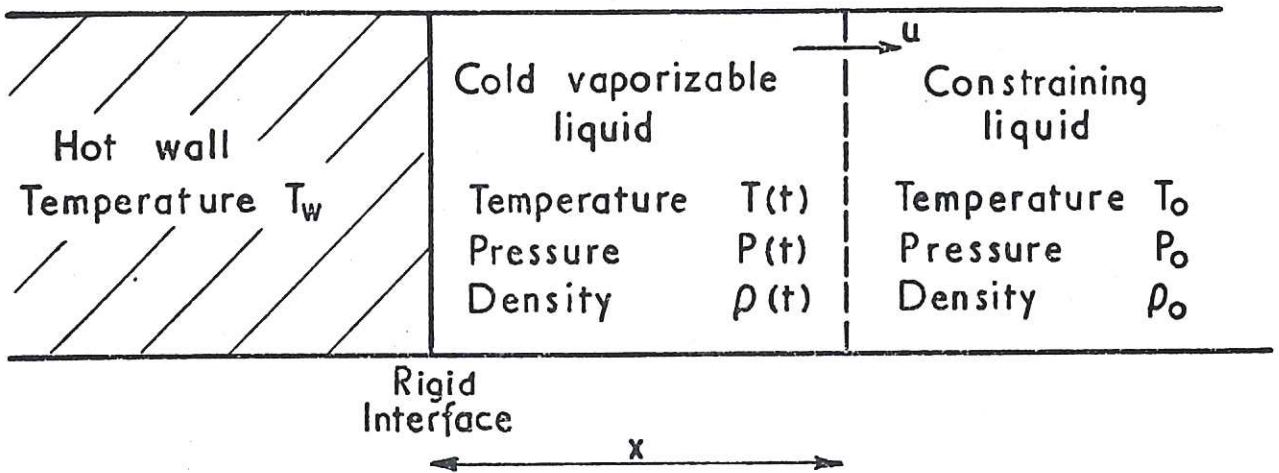


Figure 2. Conditions at time t (less than the vaporization time) for the ODE method.

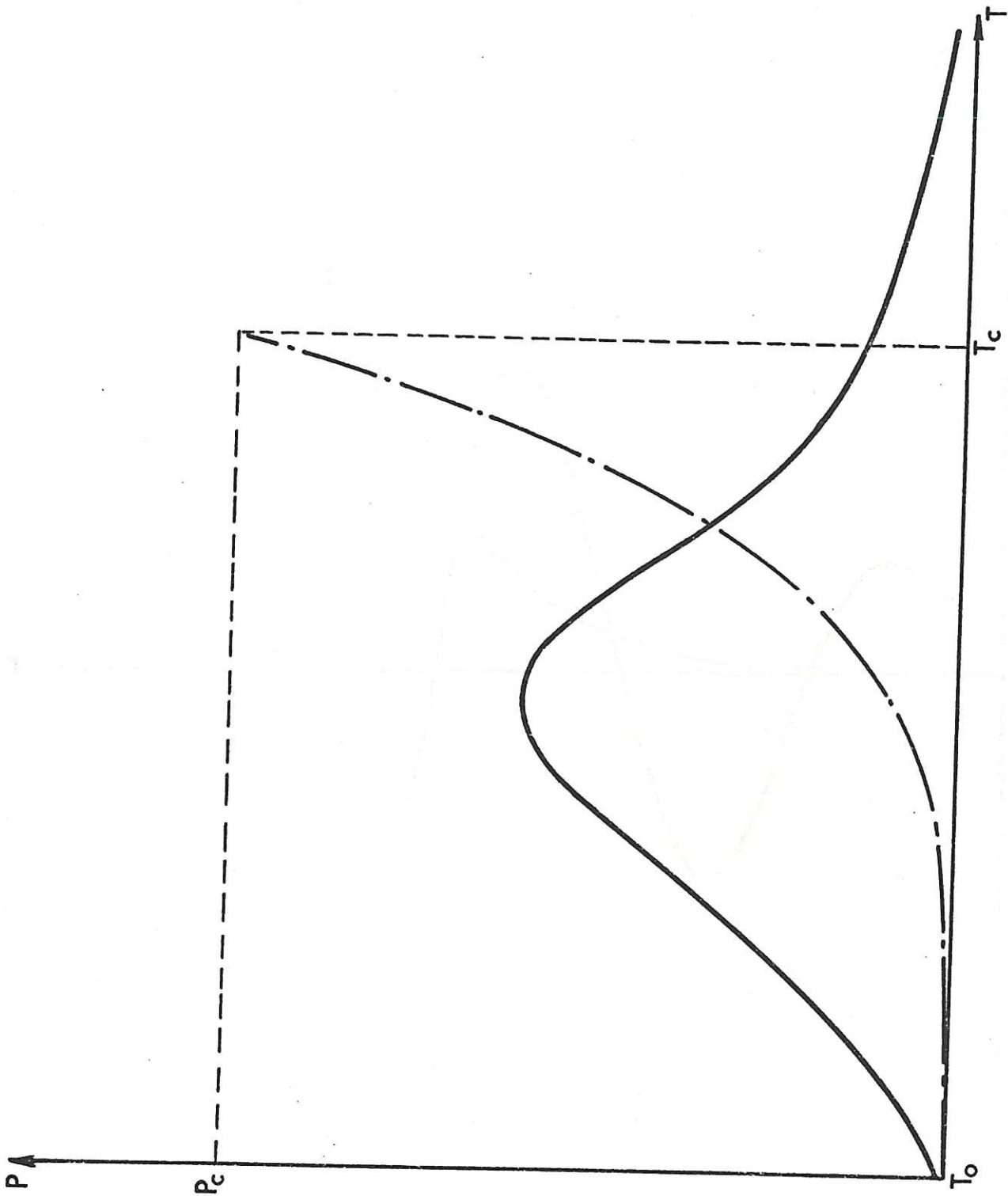


Figure 3.

Typical P-T curve for the acoustic approximation. The dotted curve is the saturation line for water. P_c and T_c are the critical pressure and temperature respectively. The intersection of the two curves gives the pressure and temperature at which vaporization occurs. Notice that $P = P_0 (= 1 \text{ bar})$ when $T = T_0 (= 300\text{K})$. At $T = 300\text{K}$ the saturated vapour pressure of water is 0.0356 bar.

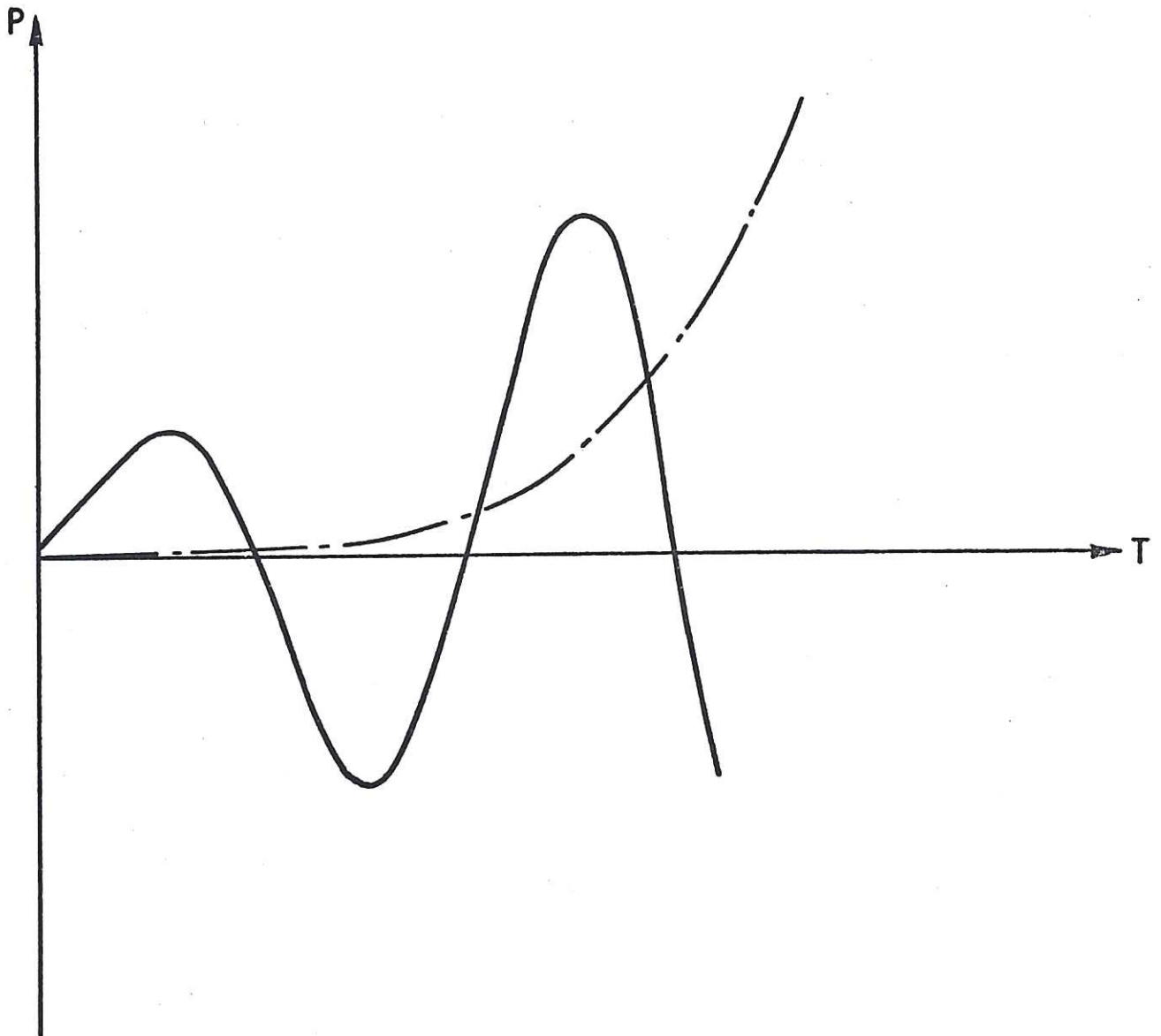


Figure 4. P-T curve for the Airy function solution. The dotted curve is the saturation line for water. The first intersection of the two curves gives the pressure and temperature at which vaporization occurs.

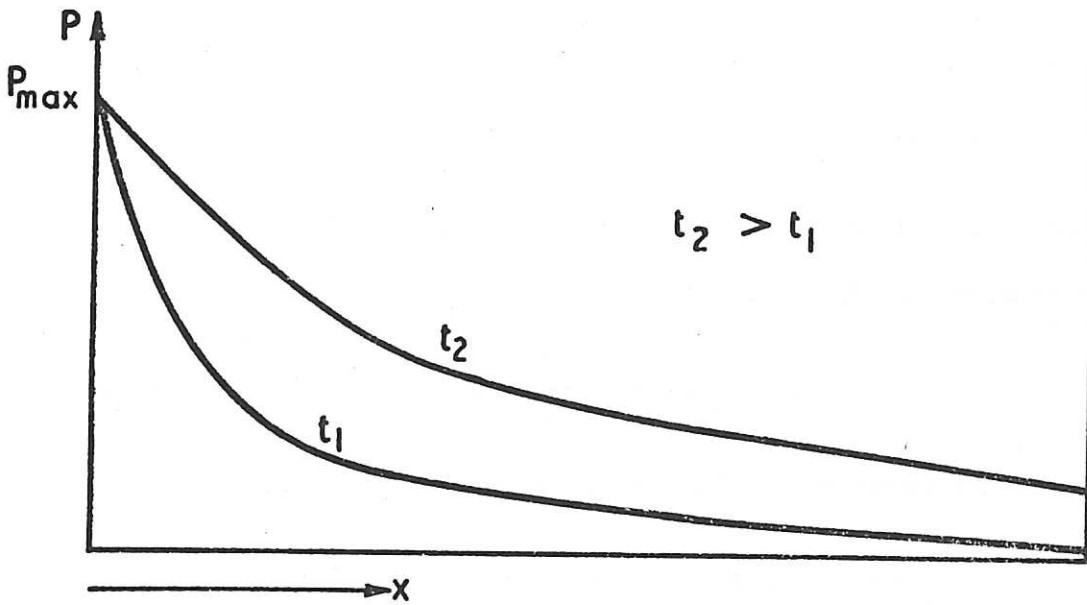


Figure 5. Infinitely strong constraint : pressure as a function of space and time ($t_2 > t_1$). The curves are obtained by eliminating T from Eqs.(5.3) and (5.4).

APPENDIX 1

We indicate how to obtain the solutions of the linearized equations

$$\kappa \rho_o \frac{\partial^2 P}{\partial t^2} - \frac{\partial^2 P}{\partial x^2} = \gamma \rho_o \frac{\partial^2 T}{\partial t^2} \quad (A1.1)$$

$$\frac{\partial T}{\partial t} = \sigma^2 \frac{\partial^2 T}{\partial x^2} \quad (A1.2)$$

$\sigma^2 = k/\rho_o c$ and the boundary and initial conditions are

$$T(o,t) = T_w \quad \left. \frac{\partial P}{\partial x} \right|_{x=0} = 0 \quad P(x,o) = P_o .$$

There is no loss in generality in taking $P_o = 0, T_o = 0$

The solution of Eq. (A1.2) is

$$T = T_w \operatorname{erfc} \left(\frac{x}{2\sigma \sqrt{t}} \right) . \quad (A1.3)$$

Take the Laplace transform of Eq. (A1.1) with respect to t .

$$\kappa \rho_o \xi^2 L - \frac{d^2 L}{dx^2} = \gamma \rho_o T_w \xi \exp(-x\sqrt{\xi}/\sigma) . \quad (A1.4)$$

The solution of Eq. (A1.4) is

$$L = \frac{\gamma \rho_o T_w \exp(-x\sqrt{\xi}/\sigma)}{(\kappa \rho_o \xi - 1/\sigma^2)} - \frac{\gamma \rho_o T_w \exp(-\sqrt{\kappa \rho_o} \xi x)}{\sqrt{\kappa \rho_o} \sigma (\kappa \rho_o \xi - 1/\sigma^2) \xi^{1/2}} . \quad (A1.5)$$

By using tables of inverse Laplace transforms we find that the solution of Eq. (A1.1) is

$$P = \frac{\gamma T_w}{2\kappa} \left\{ \exp\left(\frac{t + \sqrt{\kappa \rho_o} x}{\sigma^2 \kappa \rho_o}\right) \operatorname{erfc}\left(\frac{1}{2} \frac{x}{\sigma} t^{-1/2} + \frac{t^{1/2}}{\sigma \sqrt{\kappa \rho_o}}\right) + \right. \\ \left. + \exp\left(\frac{t - \sqrt{\kappa \rho_o} x}{\sigma^2 \kappa \rho_o}\right) \operatorname{erfc}\left(\frac{1}{2} \frac{x}{\sigma} t^{-1/2} - \frac{t^{1/2}}{\sigma \sqrt{\kappa \rho_o}}\right) \right\} -$$

$$-\frac{\gamma T_w}{\kappa} \exp\left(\frac{t - \sqrt{\kappa\rho_0} x}{\sigma^2 \kappa\rho_0}\right) \operatorname{erfc}\left(\frac{\sqrt{t - \sqrt{\kappa\rho_0} x}}{\sigma \sqrt{\kappa\rho_0}}\right) \Theta(t - \sqrt{\kappa\rho_0} x) \quad (\text{A1.6})$$

where

$$\Theta(x) = \begin{cases} 1 & ; x > 0 \\ 0 & ; x < 0 \end{cases}.$$

If the initial temperature and pressure are T_0 and P_0 (non-zero) then P is given by Eq. (A1.6) plus P_0 and in this equation T_w is replaced by $(T_w - T_0)$. T is given by

$$T = (T_w - T_0) \operatorname{erfc}\left(\frac{x}{2\sigma \sqrt{t}}\right) + T_0.$$

By using the asymptotic expansion for $\operatorname{erfc}(x)$ the time as a function of interface pressure can be shown to be

$$t \approx \frac{\sigma^2 \gamma^2 \rho_0}{\pi \kappa} \left\{ \frac{T_w - T_0}{P(0, t) - P_0} \right\}^2. \quad (\text{A1.7})$$

For water the time to fall from the maximum pressure

$$\frac{\gamma}{\kappa} (T_w - T_0) = 6300 \text{ bars to } 10 \text{ bars is } 1.34 \times 10^{-8} \text{ secs. } (T_w - T_0 = 10^3).$$

Eq. (A1.7) is used by Epstein [5] and Buchanan (see footnote on p.3)

for N_a/UO_2 ; however, the time quoted in both those papers for the interface pressure to fall to 22 bars is incorrect and should read

$$2.77 \times 10^{-7} \text{ secs.}$$

APPENDIX 2

We give a more detailed discussion of the various sets of equations that appear in section 3.

A. Acoustic constraint.

1. Constant mass.

(a) Constant heating.

The equations are

$$\dot{T} + \frac{\gamma}{\kappa m \rho_0} \frac{TQ}{sc} = \frac{r_1}{mc} \quad (A2.1)$$

$$\dot{Q} = \frac{\gamma}{\kappa} \dot{T} - \frac{Q}{\kappa s m} . \quad (A2.2)$$

The formal solution of (A2.2) is

$$\frac{\gamma}{\kappa} T = Q + \frac{1}{\kappa s m} \int Q dt + \frac{\gamma}{\kappa} T_0 .$$

Thus substituting for T and \dot{T} in Eq. (A2.1) we get Eq. (3.19). To obtain an approximate solution we substitute for \dot{T} in Eq. (A2.2) and then use Eq. (3.21) for T . This gives

$$\dot{Q} + (A_1 + B_1 t) Q = C_1 \quad (A2.3)$$

and the solution of this equation is

$$Q = C_1 \exp \left(- A_1 t - \frac{1}{2} B_1 t^2 \right) \int_0^t \exp \left(A_1 x + \frac{1}{2} B_1 x^2 \right) dx \quad (A2.4)$$

(b) Variable heating.

The equations are

$$\dot{T} + \frac{\gamma}{\kappa m \rho_0} \frac{TQ}{sc} = \frac{r_1}{mc} - \frac{r_2}{mc} T \quad (A2.5)$$

$$\dot{Q} = \frac{\gamma}{\kappa} \dot{T} - \frac{Q}{\kappa s m} . \quad (A2.6)$$

T can be eliminated as before giving Eq. (3.19) plus the additional terms

$$\frac{r_2}{mc} \dot{\Omega} + \frac{r_2}{m^2 \kappa s c} \Omega + \frac{r_2 T_0 \gamma}{m \kappa c} .$$

On using Eq. (3.22) we find that Q is given approximately by the solution of

$$\dot{Q} + (A_2 - B_2 e^{-D_2 t}) Q = C_2 e^{-D_2 t} \quad (A2.7)$$

i.e.

$$Q = C_2 \exp \left(-A_2 t - \frac{B_2}{D_2} e^{-D_2 t} \right) \int_0^t \exp \left(-D_2 x + A_2 x + \frac{B_2}{D_2} e^{-D_2 x} \right) dx \quad (A2.8)$$

2. Variable mass.

(a) Constant heating.

We have

$$\dot{T} + \frac{\gamma}{\kappa \rho_0^2 s c x_0} t^{-\frac{1}{2}} T Q = \frac{r_1}{\rho_0 c x_0} t^{-\frac{1}{2}} \quad (A2.9)$$

$$\dot{Q} = \frac{\gamma}{\kappa} \dot{T} - \frac{t^{-\frac{1}{2}} Q}{\kappa \rho_0 s x_0} . \quad (A2.10)$$

On eliminating T and \dot{T} we find

$$\begin{aligned} \frac{d^2 \Omega}{d\tau^2} + \left[2B_3 + \frac{2\gamma}{\kappa^2 \rho_0^3 s^2 x_0^2 c} \Omega \right] \frac{d\Omega}{d\tau} + \\ + \frac{\gamma}{\kappa \rho_0^2 s c x_0} \left(\frac{d\Omega}{d\tau} \right)^2 - 4C_3 = 0 \end{aligned} \quad (A2.11)$$

where $\Omega = \int_0^t t^{-\frac{1}{2}} Q dt$ and $\tau^2 = t$. This equation is of the same type as Eq. (3.19). Using Eq. (3.24) we find that the approximate

solution is

$$Q = C_3 \exp(-A_3 t - 2B_3 t^{\frac{1}{2}}) \int_0^t x^{-\frac{1}{2}} \exp(A_3 x + 2B_3 x^{\frac{1}{2}}) dx \quad (A2.12)$$

(b) Variable heating.

In this case

$$\dot{T} + \frac{\gamma}{\kappa \rho_0^2 s c x_0} t^{-\frac{1}{2}} T Q = \frac{r_1}{\rho_0 c x_0} t^{-\frac{1}{2}} - \frac{r_2}{\rho_0 c x_0} t^{-\frac{1}{2}} T \quad (A2.13)$$

$$\dot{Q} = \frac{\gamma}{\kappa} \dot{T} - \frac{t^{-\frac{1}{2}} Q}{\kappa \rho_0 s x_0} \quad (A2.14)$$

If T and \dot{T} are eliminated we find Eq. (2.11) with the additional terms

$$\frac{2 r_2}{\rho_0 c x_0} \frac{d\Omega}{d\tau} + \frac{4 r_2 \Omega}{\kappa \rho_0^2 c x_0^2 s} + \frac{4 r_2 \gamma T_0}{\kappa \rho_0 c x_0} \quad .$$

Using Eq. (3.26) the approximate solution is found to be

$$Q = E_4 \exp\left(-2A_4 t^{\frac{1}{2}} - \frac{B_4}{D_4} e^{-D_4 t^{\frac{1}{2}}}\right) \int_0^t x^{-\frac{1}{2}} \exp\left(2A_4 x^{\frac{1}{2}} - D_4 x^{\frac{1}{2}} + 2 \frac{B_4}{D_4} e^{-D_4 x^{\frac{1}{2}}}\right) dx \quad (A2.15)$$

B. Inertial Constraint.

1. Constant mass.

(a) Constant heating.

The relevant equations are

$$\dot{T} + \frac{\gamma}{\kappa m c \rho_0 D} T \Omega = \frac{r_1}{m c} \quad (A2.16)$$

$$\ddot{\Omega} = \frac{\gamma}{\kappa} \dot{T} - \frac{1}{\kappa D m} \Omega \quad (\text{A2.17})$$

where

$$\Omega = \int_0^t Q(t') dt' .$$

On eliminating T and \dot{T} we find

$$\ddot{\Theta} + \frac{\gamma}{\kappa m c \rho_0 D} \dot{\Theta} + (A_5 + \frac{\gamma}{\kappa^2 m^2 c \rho_0 D^2} \Theta) \dot{\Theta} - F_5 = 0 \quad (\text{A2.18})$$

$$\Theta = \int_0^t \Omega(t') dt' .$$

There seems little hope of making much progress with this nonlinear third order differential equation.

Using the approximate temperature solution Eq. (3.21) we find that Ω satisfies.

$$\ddot{\Omega} + [A_5 + B_5 t] \Omega = F_5 . \quad (\text{A2.19})$$

The transformation $-B_5^{2/3} z = A_5 + B_5 t$ converts Eq. (A2.19) to

$$\frac{d^2 \Omega}{dz^2} - z \Omega = F B^{-2/3} \quad (\text{A2.20})$$

which is the inhomogeneous version of Airy's equation.

(Ref. [9], p.446). This may be solved by standard methods

(Ref. [7], B1, 12-3) and we find

$$\begin{aligned} \Omega(z) = & \lambda_1 \text{Ai}(z) + \lambda_2 \text{Bi}(z) - \pi F_5 B_5^{-2/3} \text{Ai}(z) \int \text{Bi}(z) dz \\ & + \pi F_5 B_5^{-2/3} \text{Bi}(z) \int \text{Ai}(z) dz \end{aligned} \quad (\text{A2.21})$$

λ_1 and λ_2 are constants to be determined by the initial conditions

$$\Omega(z_0) = \frac{d\Omega}{dz} \Big|_{z=z_0} = 0 .$$

Finally $Q = -B^{1/3} \frac{d\Omega}{dz}$ thus we arrive at Eq. (3.28).

By using Luke's complete asymptotic expansions [10] we can show that a more accurate expression for Eq. (4.2) is

$$\begin{aligned}
 Q = F_5 A_5^{1/2} \left\{ \right. & \left. \sin \omega t - \frac{B_5}{A_5^{3/2}} (1 - \cos \omega t) \right. \\
 + \frac{B_5}{A_5} t \left[\frac{1}{4} \sin \omega t + \frac{B_5}{32 A_5^{3/2}} (71 + \cos \omega t) \right] & \\
 \left. - \frac{3}{32} \frac{B_5^2}{A_5^2} t^2 \sin \omega t \dots \right\} & \quad (A2.22)
 \end{aligned}$$

where
$$\omega = A_5^{1/2} + \frac{1}{4} A_5^{-1/2} B_5 t .$$

(b) Variable heating.

The right hand side of Eq. (A2.15) becomes $\frac{r_1}{mc} - \frac{r_2}{mc} T$ and on eliminating T and \dot{T} we find Eq. (A2.17) plus the additional terms

$$\frac{r_2}{mc} \ddot{\Theta} + \frac{r_2}{\kappa D m^2 c} \dot{\Theta} + \frac{r_2 \gamma T_o}{m \kappa} .$$

Using Eq. (3.22) we find that Ω satisfies

$$\ddot{\Omega} + \left[A_6 - B_6 e^{-D_6 t} \right] \Omega = E_6 e^{-D_6 t} \quad (A2.23)$$

where

$$\begin{aligned}
 A_6 = \frac{1}{KDM} + \frac{\gamma^2 r_1}{\kappa^2 mc \rho_o D r_2} & \quad B_6 = \frac{\gamma^2}{\kappa^2 mc \rho_o D} \left(\frac{r_1}{r_2} - T_o \right) \\
 D_6 = \frac{r_2}{mc} & \quad E_6 = \frac{r_2 \gamma}{mc \kappa} \left(\frac{r_1}{r_2} - T_o \right) .
 \end{aligned}$$

The solution of Eq. (A2.23) can be obtained by standard methods

(Ref. [7] , B1, 12-3) if two linearly independent solutions of the homogeneous equation can be obtained. By making the transformation

$$z = 2 e^{-D_6 t/2} \sqrt{-B_6/D_6}$$

the homogeneous equation may be transformed into Bessel's equation and thus Eq. (A2.23) may be solved.

2. Variable mass.

(a) Constant heating.

In this case

$$\dot{T} + \frac{\gamma}{\kappa \rho_0^2 x_0 Dc} t^{-1/2} T \Omega = \frac{r_1}{\rho_0 c x_0} t^{-1/2} \quad (\text{A2.24})$$

$$\ddot{\Omega} = \frac{\gamma}{\kappa} \dot{T} - \frac{1}{\kappa \rho_0 D x_0} t^{-1/2} \Omega \quad (\text{A2.25})$$

On eliminating T and \dot{T} we find

$$\begin{aligned} t^3 \ddot{\Phi} + \left[1 + \frac{\gamma}{\kappa \rho_0^2 x_0 Dc} t \dot{\Phi} \right] t^2 \ddot{\Phi} + \\ + \left[\left(\frac{1}{\kappa \rho_0 D x_0} + \frac{\gamma^2 T_0}{\kappa^2 \rho_0^2 x_0 Dc} \right) t^{3/2} - \frac{1}{4} \right. \\ \left. + \frac{\gamma}{\kappa^2 \rho_0^3 x_0^2 D^2 c} t^{3/2} \dot{\Phi} \right] t \dot{\Phi} + \\ + \frac{\gamma}{2 \kappa \rho_0^2 x_0 Dc} (t \dot{\Phi})^2 - \frac{\gamma r_1}{\kappa \rho_0 c x_0} t^2 = 0 \end{aligned} \quad (\text{A2.26})$$

$$\Phi = \int_0^t z^{-1/2} \Omega(z) dz \quad .$$

Using equation (3.24) we find

$$\ddot{\Omega} + (A_7 + B_7 t^{-\frac{1}{2}}) \dot{\Omega} = D_7 t^{-\frac{1}{2}} \quad (\text{A2.27})$$

$$A_7 = \frac{2 \gamma^2 r_1}{\kappa^2 \rho_o^2 x_o^2 Dc^2} \quad B_7 = \frac{1}{\kappa \rho_o D x_o} + \frac{\gamma^2 T_o}{\kappa^2 \rho_o^2 x_o Dc}$$

$$D_7 = \frac{\gamma r_1}{\kappa \rho_o c x_o} .$$

Eq. (A2.27) may be solved by means of a series whose coefficients are given by a three term recurrence formula.

(b) Variable heating.

The right hand side of Eq. (A2.24) becomes

$$\frac{r_1}{\rho_o c x_o} t^{-\frac{1}{2}} - \frac{r_2}{\rho_o c x_o} t^{-\frac{1}{2}} T$$

and on eliminating T and \dot{T} we have Eq. (A2.25) plus the additional terms

$$\frac{r_2}{\rho_o c x_o} \left[t^{\frac{1}{2}} t^2 \ddot{\Phi} + \frac{1}{2} t^{\frac{1}{2}} t \dot{\Phi} + \frac{t^2 \Phi}{\kappa \rho_o D x_o} + \frac{\gamma}{\kappa} T_o t^2 \right].$$

Using Eq. (3.26) we find

$$\ddot{\Omega} + \left[A_8 t^{-\frac{1}{2}} - B_8 t^{-\frac{1}{2}} e^{-D_8 t^{\frac{1}{2}}} \right] \dot{\Omega} = E_8 t^{-\frac{1}{2}} e^{-D_8 t^{\frac{1}{2}}} \quad (\text{A2.28})$$

where

$$A_8 = \frac{1}{\kappa \rho_o D x_o} + \frac{\gamma^2 r_1}{\kappa^2 \rho_o^2 x_o Dc r_2}$$

$$B_8 = \frac{\gamma^2}{\kappa^2 \rho_o^2 x_o Dc} \left(\frac{r_1}{r_2} - T_o \right)$$

$$D_8 = \frac{2 r_2}{\rho_o c x_o}$$

$$E_8 = \frac{\gamma r_2}{\rho_o c x_o} \left(\frac{r_1}{r_2} - T_o \right)$$

APPENDIX 3

We describe in more detail how the integrals D_a and D_{ag} are evaluated. Dawson's integral is well-known and may be evaluated using either the ascending series

$$D_a(x) = e^{-x^2} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!}$$

or the asymptotic expansion

$$D_a(x) \sim \frac{1}{2x} \sum_{n=1}^{\infty} \frac{(n-3/2)!}{(-1/2)! x^{2(n-1)}}$$

Dingle's [11,12] theory of converging factors may be used to increase the range of usefulness of the asymptotic expansion.

Evaluation of the integral $D_{ag}(x;a,b,c)$ depends on the sign of c . If c is negative D_{ag} can be expressed simply as the difference of two incomplete gamma functions. In our case, however, c is positive. For positive c , D_{ag} can be expressed as a sum of a finite number of terms plus the difference of two incomplete gamma functions but if a/c is large the number of terms in the series is large and the expression is useless for actual calculation. By expanding $\exp(be^{-cx})$ and integrating term by term the following series is obtained :

$$D_{ag}(x;a,b,c) = \exp(-ax-be^{-cx}) \sum_{n=0}^{\infty} \frac{b^n}{n!} \frac{1}{(a-nc)} \left\{ e^{(a-nc)x} - 1 \right\} .$$

This result assumes that $Nc \neq a$ for any positive integer N . The exceptional case is easily treated by replacing this term by $xb^N/N!$. If b is large the substitution $u = 1-e^{-cx}$ enables us to obtain the series

$$D_{ag}(x;a,b,c) = \frac{\exp(-ax-be^{-cx})}{bc} \sum_{n=0}^{\infty} \frac{(p+n-1)!}{b^n (p-1)! n!} \gamma(n+1, \alpha b)$$

where $\alpha = 1-e^{-cx}$, $p = 1 + a/c$ and $\gamma(n,x)$ is the incomplete gamma function (ref.[9],p.260)

$$\gamma(n,x) = \int_0^x e^{-t} t^{n-1} dt .$$

This series converges if $\alpha < 1$. If cx is sufficiently small D_{ag} can be approximated by a combination of Dawson integrals.

One final point to note is that terms like $D_a\left(\sqrt{\frac{B}{2}}t + \frac{A}{2}\sqrt{\frac{2}{B}}\right)$

are best evaluated by means of a Taylor series about the point

$$\frac{A}{2}\sqrt{\frac{2}{B}}.$$

APPENDIX 4

In this appendix we give some results of applying the foregoing theory to liquid sodium. The following input parameters were chosen (MKS units).

$$\kappa = 1.58 \times 10^{-10}$$

$$S = 2595$$

$$\gamma = 2.8 \times 10^{-5}$$

$$\rho_0 = 830$$

$$c = 1300$$

$$k = 63$$

The vapour pressure curve was taken to be

$$\ln P = 29.2 - \frac{1276.7}{T} - 0.613 \ln T$$

(P in dynes / cm², T in K) (R.B. Duffey, CEGB Report RD/B/N1609) and the critical temperature was assumed to be 2803 K. Tables A4.1 and A4.2 show the results for the constant mass, constant heat and variable mass, constant heat approximations. The system is acoustically constrained.

r_1	Time at which vapour layer forms (secs)	Pressure of layer at formation (n/m^2)	Temperature of layer at formation (K)
10^5	5.67×10^{-1}	1.013×10^5	1153
10^6	5.67×10^{-2}	1.014×10^5	1153
10^7	5.73×10^{-3}	1.019×10^5	1153
10^8	6.81×10^{-4}	1.069×10^5	1163
10^9	1.17×10^{-4}	1.569×10^5	1208
10^{10}	3.37×10^{-5}	6.566×10^5	1412
10^{11}	8.66×10^{-6}	56.43×10^5	1903
10^{12}	1.84×10^{-6}	vaporizes at critical temperature	

Table A4.1 Vapour layer formation of sodium using the acoustic, constant mass, constant heating approximation. $P_o = 1$ bar, $T_o = 1100$ K, $m = x \rho_o$, $x = 10^{-3}$ m.

r_1	time at which layer forms (secs)	thickness of layer (mm)
10^5	1.877×10	6.62×10
10^6	1.877×10^{-1}	6.62
10^7	1.917×10^{-3}	6.69×10^{-1}
10^8	2.708×10^{-5}	7.95×10^{-2}
10^9	7.993×10^{-7}	1.37×10^{-2}
10^{10}	6.631×10^{-8}	3.94×10^{-3}
10^{11}	4.379×10^{-9}	1.01×10^{-3}
10^{12}	1.977×10^{-10}	2.15×10^{-4}

Table A4.2 Vapour layer formation of sodium using the acoustic, variable mass, constant heating approximation. ($\rho_0 = 1$ bar, $T_0 = 1100$ K). The time of formation and thickness of the initial vapour layer are given for various values of r_1 . The pressure and temperature at the time of formation are given in Table A4.1.



