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SELFCONSISTENT EQUILIBRIA AND CURRENT LIMITATION IN RELATIVISTIC ELECTRON BEAMS

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SELFCONSISTENT EQUILIBRIA AND CURRENT LIMITATION IN RELATIVISTIC ELECTRON BEAMS

P. L. Auer †

ABSTRACT

Several classes of selfconsistent equilibria are examined with a view towards establishing their general character. It is found that within the context of selfconsistent equilibria current limitation in the sense of Alfven and Lawson is not present *per se*. Practical considerations, however, would argue that this limit has rather broad validity in the strictly relativistic regime providing externally generated magnetic fields are absent. An example of potential practical interest in which field reversal occurs inside the beam while the bulk of it remains nearly force-free is discussed in some detail.

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I. Introduction

Intense relativistic electron beams have created considerable interest in certain quarters as a new energy source with potentially bright prospects for useful applications. In particular, a number of individuals engaged in controlled thermonuclear research are examining the possibility of utilizing the large energy densities and sizeable magnetic fields produced by such beams in plasma heating and confinement experiments. One of the early questions faced by workers in the field was whether any limitation existed to the current which could be carried by a beam. The experimental evidence to date suggests that under a variety of conditions there does not exist a limit to the primary beam current. Both the reasons for the possible existence of a limit and the absence of one are understood reasonably well on the basis of various theoretical arguments, and a considerable body of literature on the topic exists by now.

It seems, nevertheless, of some possible interest to reexamine the question of current limitation in the light of certain general features exhibited by selfconsistent equilibria. While there is little guarantee that a given physical experiment will correspond to conditions described by selfconsistent calculations, the existence and character of such equilibria may serve as a useful guide to the experimenter.

The intent here is to examine certain features along with some specific examples of stationary equilibria involving current carrying electrons and fixed ions which serve to cancel exactly any space charge. Electric fields and collisions are assumed to be absent. Inclusion of ion motion and space charge fields would affect the bulk of our results only in fine detail. We begin with a brief review of arguments leading to the concept of current limitation. Conventional z-pinch equilibria are discussed next, followed by equilibria in which externally generated magnetic fields are included. The entire discussion is restricted to cylindrical geometry.

II. Current Limitation

Alfven[1] and subsequently Lawson[2] have shown that the self-magnetic forces produced by a beam of electrons places an upper bound on the amount of current which may propagate. The argument is briefly as follows. Consider a uniform beam of monoenergetic electrons with radial edge at $r = a$, N is the number of electrons per unit length and the axial drift speed is $\beta_{\parallel} c$, where c is the velocity of light.

The current may be expressed as

$$-I = \left(\frac{ec}{r_o}\right) \nu \beta_{\parallel} \simeq 17{,}000\, \nu \beta_{\parallel} \text{ amps.}$$

$$\nu \equiv N r_o$$

where $r_o = e^2/mc^2$ is the classical radius of an electron and e, m are the charge and mass, respectively and we adopt the convention $\beta_{\parallel} > 0$.

The energy of a given electron is γmc^2 where $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$ and is a constant. An electron with radial turning point at the beam edge must satisfy energy conservation according to the relation

$$\begin{aligned} \beta^2 &= \beta_r^{\,2}(r) + (a/r)^2 \beta_{\perp}^{\,2} + \left[1 - \frac{\nu}{\gamma}(1 - r^2/a^2)\right]^2 \beta_{\parallel}^{\,2} \\ &= \beta_{\perp}^{\,2} + \beta_{\parallel}^{\,2} = \text{const.} \end{aligned} \tag{1}$$

In the above $a\beta_{\perp}$ and $\beta_{\parallel}$ represent the angular momentum and axial velocity at the beam edge and are constants of the electron's motion. The orbital equation follows readily

$$\frac{dz}{dr} = \frac{\beta_z(r)}{\beta_r(r)}$$

$$\beta_z(r) = \left[1 - \frac{\nu}{\gamma}(1 - r^2/a^2)\right]\beta_{\parallel} \tag{2}$$

$$\beta_r^2(r) = (1 - r^2/a^2)\left\{\left[2\frac{\nu}{\gamma} - \left(\frac{\nu}{\gamma}\right)^2 (1 - r^2/a^2)\right]\beta_{\parallel}^{\,2} - (a^2/r^2)\beta_{\perp}^{\,2}\right\}.$$

The orbits may be expressed in terms of elliptic integrals and we wish to make the following observations. Consider first the case where the electrons have zero angular momentum. As long as $\nu < 2\gamma$ both turning points occur at the beam edge; however, two categories of trajectories are to be distinguished. The first contains all trajectories in which electrons have a net drift in the direction of beam propagation, while the second contains those trajectories in which electrons become reflected at some z-plane and have net drifts opposite to the direction of beam propagation. The sepratrix between these two classes is a trajectory describing a figure eight with vertex centered on the origin; it represents a trapped particle with zero net drift and the corresponding $\nu \simeq 1.5\,\gamma$. Trajectories for $\nu > 2\gamma$ behave as the second category above but their second turning point occurs inside the beam at some $r > 0$. From the above one may conclude that beam propagation is limited to $\nu < 1.5\,\gamma$, with a corresponding limit to the current, so long as electrons with significant angular momenta are excluded. This argument implies, of course, that a properly defined beam may contain only electrons with net drift in the same direction as beam propagation, a qualification which need not apply to current carrying plasmas by contrast.

Electrons with finite angular momenta always have a second turning point inside the beam. Evaluating the axial velocity at this point one finds

$$2\beta_z = \{\frac{\nu^2}{\gamma^2}\,\beta_{\parallel}^{\,2} + 4(\beta^2 - \frac{\nu}{\gamma}\,\beta_{\parallel}^{\,2})\}^{\frac{1}{2}} - \frac{\nu}{\gamma}\,\beta_{\parallel} \,.$$

From this we conclude it is sufficient, though not necessary, that

$$\nu < (1 + \beta_{\perp}^{\,2}/\beta_{\parallel}^{\,2})\,\gamma$$

in order to assure propagation.

One may easily deduce the existence of a critical ν/γ on intuitive grounds. Returning to the earlier discussion dealing with electrons without angular momentum, their orbit's local radius of curvature at the beam edge is

$$R = \gamma mc^2\beta_{\parallel}/eB(a) = \gamma mc^3\beta_{\parallel}a/2eI$$

which may be rearranged to yield

$$\frac{\nu}{\gamma} = a/2R \,.$$

Assuming the electron orbits in the self magnetic field B are similar to Larmor orbits (they are not actually) one may conclude that R must be of the order of a . Most authors choose unity for the critical value of ν/γ , leading to the now familiar expression

$$-I_A = 17{,}000 \ \beta_{\parallel}\gamma \text{ amps} \tag{3}$$

for the Alfven-Lawson limiting current.

We close this section by noting that the arguments leading to the concept of current limitation, while physically sound, are not based on self-consistent field calculations. Consequently, the model leading to a critical ν/γ cannot be expected to yield a precise value but only an order of magnitude at best. Finally, there is a hint from the above elementary considerations that beams containing electrons with finite angular momenta could lead to results appreciably different from eq. (3).

III. Conventional Pinch Equilibria

A. General Comments

Bennett[3] first introduced the concept of selfconstricted electron streams; his results lead to the well known pinch relation

$$I_B^2 = 2c^2 NkT \tag{4}$$

where T is the transverse beam temperature, to be made more precise subsequently, and k is Boltzmann's constant. In order to compare eqs. (3) and (4) we write

$$-I_B = eN\bar{\beta}_{\parallel}c$$

where $\bar{\beta}_{\parallel}$ represents a suitably averaged drift speed. It then follows that a current satisfying the Bennett pinch relation may be expressed in terms of the Alfven-Lawson limiting current as

$$I_B = \frac{2kT}{\gamma mc^2\bar{\beta}_{\parallel}^2} I_A \ . \tag{5}$$

Lawson has already shown[4] that any equilibrium beam $I < I_A$ with finite temperature must satisfy eq. (4), implying it would seem that in beams of practical interest $2kT < \gamma mc^2 \bar{\beta}_\parallel^2$. In fact, were the latter inequality to be reversed, a conflict would arise between eqs. (3) and (4).

It is well known that the pinch relation given by eq. (4) follows from rather general stress balance considerations with results equally valid for beams as for plasma. For the simple axisymmetric situation under discussion the axial current density and azimuthal magnetic field produce a radial $\underline{J} \times \underline{B}$ force balanced by the pressure gradient which takes the form

$$\frac{B}{4\pi r} \frac{d}{dr} (rB) = - \frac{d}{dr} P_\perp \quad . \tag{6}$$

Define

$$I(r) = \int_0^r J(r_o) 2\pi r_o \, d r_o$$

as the current flowing in a cross-section of radius r and note $B(r) = 2I(r)/cr$. Inserting the latter expression into eq. (6) and integrating leads to

$$I^2(r) = 2c^2 \int_0^r [P_\perp(r_o) - P_\perp(r)] 2\pi r_o \, d r_o \, . \tag{7}$$

The above is a somewhat more general statement of stress balance than the one given by eq. (4). The latter follows in the event

$$P_\perp = n(r)kT \ ,$$

with a constant temperature, and the integration of eq. (7) is carried to values of r large enough to assure the pressure (number density) at that point is vanishingly small.

A rather self-evident corollary to the above statement is that any self-consistent equilibrium - in the absence of trapped particles - must satisfy eq. (7) identically. Distributions corresponding to selfconsistent equilibria are constructed from the appropriate constants of motion; and in this instance we have

$$f(r, p_\parallel, p_\perp) = F[\gamma mc^2; p_\parallel - \frac{e}{c} A(r)] \tag{8}$$

$$n(r) = \int_{-\infty}^{\infty}\int_{0}^{\infty} f\ 2\pi\ p_\perp\ dp_\perp\ dp_\parallel$$

where $\gamma mc^2 = c[m^2c^2 + p_\perp^{\ 2} + p_\parallel^{\ 2}]^{\frac{1}{2}}$ is the energy and p is a mechanical momentum of a given electron; F is any arbitrary function of its arguments obeying the usual rules for constructing probability distribution functions. We have also introduced the axial vector potential A where $B = - dA/dr$ and n is the beam number density.

The moments of interest are

$$n < \beta_\parallel > = \int_{\infty}^{\infty}\int_{0}^{\infty} \frac{p_\parallel}{\gamma mc}\ f\ 2\pi\ p_\perp\ dp_\perp\ dp_\parallel$$

$$P_\perp = \tfrac{1}{2}nc < p_\perp\ \beta_\perp > \qquad (9)$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{p_\perp^2}{2\gamma m}\ f\ 2\pi p_\perp\ dp_\perp\ dp_\parallel\ ;$$

and we note in passing

$$kT = \tfrac{1}{2} < p_\perp \beta_\perp c > . \qquad (9')$$

The above definition of transverse temperature corresponds to the conventional one defined for a laboratory frame at rest.

One may transform the variable of integration from $p_\perp$ to γ and after an integration by parts find

$$\frac{dP_\perp}{dA} = - e\ n < \beta_\parallel > . \qquad (10)$$

When the above is combined with Ampere's law it leads directly to eq. (6) and completes the proof. Consequently, we are led to the following conclusion.

<u>Theorem</u>: All suitably defined selfconsistent equilibria satisfy stress balance in the form given by eq. (7) and in a form equal to or resembling eqs. (4) and (5). As such, selfconsistent equilibria do not predict current limitation in the sense of Alfven and Lawson as given by eq. (3). Within the context of self consistent equilibria current limitation can only arise if one arbitrarily limits the ratio of transverse energy per particle (kT) to drift kinetic

energy $(\sim \gamma\, mc^2\, \overline{\beta}_{\parallel}^{\,2})$.

As mentioned earlier the concept of current limitation may still be of physical validity in that the act of preparation may produce beams more representative of the non-selfconsistent models used by Alfven and Lawson than any selfconsistent equilibria. Furthermore, there may be good reasons why in physically realizable systems kT must be limited. The foregoing discussion serves largely as a warning to the experimenter that the concept of current limitation does not follow from universal considerations. Some illustrative examples of equilibria may be of interest in that they reveal certain features which may be a useful guide to experimenters.

B. The Bennett Pinch

The familiar distributions resulting from Bennett's original analysis may be put in the form

$$-I_B(r) = \left(\frac{ec}{r_o}\right) \gamma_o \overline{\beta}_{\parallel} \left[\frac{(r/2\lambda)^2}{1 + (r/\lambda_B)^2}\right]$$

$$\lambda^2 = \gamma_o mc^2 / 4\pi n_o e^2 \qquad (11)$$

$$\lambda_B^{\,2} = (8kT/\gamma_o mc^2\, \overline{\beta}_{\parallel}^{\,2})\, \lambda^2 .$$

In the above γ_o represents an average energy, one may take $\gamma_o = (1 - \overline{\beta}_{\parallel}^{\,2})^{-\frac{1}{2}}$, while $\overline{\beta}_{\parallel}$ is the constant drift speed, in units of c. It will be seen that λ is the characteristic scale length, the collision-less magnetic skin depth; but the Bennett distributions decay on the scale λ_B. In the limit $r \gg \lambda_B$ one recovers eqs. (4) and (5) with $N = \pi\lambda_B^2 n_o$ and n_o is the beam density on axis.

The radial profiles of macroscopic quantitites as given by eq. (11) follow from a microscopic selfconsistent distribution which is a suitably modified drifted Maxwellian[5]. Two objections may be raised in so far as the standard Bennet pinch profiles are applied to beams. In the first place integration of the distribution leading to eq. (11) is unrestricted; consequently the distribution contains counterstreaming electrons. It may be argued that this class of electrons must be excluded from any beam-like distribution. In the second place the constant temperature appearing in the Maxwellian represents not only the transverse temperature but also the parallel temperature. Again, it may be argued proper beams are relatively cold in the direction of flow and these considerations may serve to distinguish beams from current carrying plasmas.

The significant feature of the Bennett current profile given by eq. (11) which we wish to stress is that ultimately the current I_B will be greater than or less than I_A depending on whether λ_B is greater than or less than λ. In either case the profile describes what we term a 'solid' distribution in that the entire current channel supports current in proportion to its cross-section.

C. The Longmire Sheath

Longmire[6] has given a tutorial example of a selfconsistent equilibrium based on a monoenergetic (delta function) distribution. The relativistic version of the results is contained in Hammer and Rostoker[7] and takes the form

$$-I_L(r) = \left(\frac{ec}{r_o}\right) \gamma_o \beta_{\parallel} (r/2\lambda)\, I_1(r/\lambda) \tag{12}$$

where now γ_o and $\beta_{\parallel}$ represent the electron energy and drift speed on axis, while I_1 is a Bessel function of imaginary argument.

In contrast with the previous example, Longmire's model leads to a radial profile in which the current is concentrated within a sheath of dimensions λ at the beam edge; we term this a 'hollow' beam. Whereas distributions with velocity dispersion usually lead to beam models in which pressure and density decay to infinity on some scale length, monoenergetic distributions lead to beams with a well defined edge, at say $r = a$. The location of the beam edge is usually given by the requirement that the pressure vanish there. Since the density resulting from these models is often nearly constant, monoenergetic distributions have the somewhat unphysical consequence of a radially varying effective temperature.

Consequently, it is not surprising that instead of eq. (4) one obtains for the Longmire sheath

$$I_L{}^2 = c^2 \gamma_o mc^2 \beta^2 N[I_1(a/\lambda)/I_o(a/\lambda)]^2$$
$$\gamma_o = (1 - \beta^2)^{-\frac{1}{2}} \qquad (13)$$
$$N = \pi a^2 n_o$$

The corresponding current I_L may be arbitrarily greater than I_A, as previously observed[7], providing a is appropriately greater than λ. (In this instance the ratio of Bessel functions appearing in eq. (13) is of order unity). This, in turn, implies $\beta_\perp^2 = \beta^2 - \beta_\parallel{}^2$ is large compared to $\beta_\parallel{}^2$ since the two are related via the condition $\beta = \beta_\parallel I_o(a/\lambda)$.

The criterion that the selfconsistent current given by the Longmire sheath model exceed the Alfven-Lawson limit is fundamentally the same as in Bennett's model. In both cases one requires the transverse energy per particle to exceed the drift energy on axis.

D. A Finite Temperature Equilibrium

The question of whether the standard Bennett model is appropriate to beams has already been raised. Some object to the 'hollow' beam results on physical grounds. It is argued the 'hollow' beam may be viewed as a collection of beams arranged in a ring where each individual 'member' obeys the Alfven-Lawson limit. The larger the radius of the ring the greater the number of such 'members' which can be accommodated, and I_L as given by eq. (13) is simply the sum of the individual contributions. The implication seems to be one would actually have to superpose beams with $I < I_A$ in this fashion to produce a hollow distribution.

While the subject seems hardly worth debating, certain apparently artificial aspects of monoenergetic distributions can be removed if we add velocity dispersion. Rather than modify the drifted Maxwellian so as to remove the objections raised in sec. III B, we offer the relativistic version of an equilibrium which has appeared earlier[8] as a further example.

We choose

$$f = \frac{n_o}{2\pi\gamma_o mkT}\left(\frac{\gamma_o mc^2}{kT + \gamma_o mc^2}\right)\delta(p_\parallel - \gamma_o mc\beta_o - \frac{e}{c}A)\, e^{-\frac{mc^2}{kT}(\gamma - \gamma_o)}$$

$$\gamma_o = (1 - \beta_o^2)^{-\frac{1}{2}} \tag{14}$$

$$\gamma mc^2 = c[m^2c^2 + p_\perp^2 + p_\parallel^2]^{\frac{1}{2}} \quad ;$$

and we find

$$-I(r) = \left(\frac{ec}{2r_o}\right)\gamma_o\beta_o\,[\rho\eta'(\rho)]$$

$$r = \rho\lambda(1 + kT/\gamma_o mc^2)^{\frac{1}{2}} \tag{15}$$

$$\lambda^2 = \gamma_o mc^2/4\pi n_o e^2 \, ,$$

where prime denotes differentiation with respect to argument; and $\beta_o\eta = eA/\gamma_o mc^2$ satisfies the selfconsistent field equation

$$\frac{1}{\rho}[\rho\eta']' = (1 + \eta)\exp\{-\frac{\gamma_o mc^2}{kT}[(1 + 2\beta_o^2\eta + \beta_o^2\eta^2)^{\frac{1}{2}} - 1]\} \tag{16}$$

$$\eta(0) = \eta'(0) = 0 \, .$$

The above set satisfies eq. (7) and eq. (4) according to

$$I^2 = 2c^2(n_o\pi\lambda^2)kT \int_0^\infty \left[\frac{kT}{\gamma_o mc^2} + (1 + 2\beta_o^2\eta + \beta_o^2\eta^2)^{\frac{1}{2}}\right] \exp\{-\frac{\gamma_o mc^2}{kT}[(1 + 2\beta_o^2\eta + \beta_o^2\eta^2)^{\frac{1}{2}} - 1]\}\, d\rho^2 \, . \tag{17}$$

The integral may be approximated by letting $\eta \sim (\rho/2)^2$, whereupon we find

$$\frac{I}{I_A} = \begin{cases} \dfrac{\sqrt{2}kT}{\gamma_o mc^2} \; ; & \text{limit } \beta_o \to 1 \\[2ex] (2\pi)^{\frac{1}{4}}\left(\dfrac{kT}{\gamma_o mc^2\beta_o^2}\right)^{\frac{3}{4}} \; ; & \text{limit } \beta_o \to 0 \end{cases} \tag{18}$$

and we have assumed $\gamma_o mc^2\beta_o^2/kT$ finite in the second of the above limits. The first of these corresponds to the fully relativistic regime while the second describes the non-relativistic limit.

Examples of $I(r)$ obtained by numerical integration of eq. (16) for several values of the parameters are given in figure 1. We also compare the radial profile as given by this model with the Bennett model and Longmire sheath in figure 2 for a case where $I > I_A$. The transition from 'hollow' to 'solid' form may be observed in this illustration.

The division between $I \gtrless I_A$ occurs approximately at $\gamma_o mc^2 \beta_o^2 \sim kT$ as in previous models. We should note, however, that $\gamma_o mc^2$ is required to be large compared to kT on physical grounds. It would seem difficult to satisfy conditions appropriate to $I > I_A$ within the bounds of this model in the fully relativistic regime. By contrast, no such difficulty appears in the non-relativistic regime obtained by letting $\gamma_o = 1$, $kT/\gamma_o mc^2 = 0$, while $2kT/\gamma_o mc^2 \beta_o^2$ remains finite and represents the ratio of transverse thermal energy to drift kinetic energy on axis. We conclude that on the basis of physical considerations current limitation in the sense of Alfven and Lawson should be of fairly general validity for relativistic beams resembling simple z-pinch equilibria.

The lack of direct experimental observation of current limitation does not necessarily imply that our concern with the subject is purely academic. Most experiments to date have been dominated by partial magnetic neutralization due to transient return currents induced by the primary beam. In many potential applications this phenomenon will have disappeared on the time scales of interest. Finally, certain theories[9] proposed for plasma heating by means of large ν/γ beams implicitly rely on the existence of current limitation.

IV. Helical Field Configurations

A. General Remarks

A considerably enlarged class of equilibria exist if, in addition to the azimuthal self magnetic field, an axial component to the magnetic field is allowed. In fact, for toroidal equilibria, the latter component is often essential. Generally speaking, part of the axial magnetic field may be of external origin and need not be produced by currents flowing in the beam or plasma.

We continue to treat the simple axisymmetric cylinder with quantities varying only in the radial direction. Instead of eq. (6) stress balance follows from

$$\frac{1}{4\pi r} B_\theta \frac{d}{dr}(rB_\theta) + \frac{1}{4\pi} B_z \frac{d}{dr} B_z = -\frac{d}{dr} P_{rr} - \frac{1}{r}(P_{rr} - P_{\theta\theta}) + \frac{1}{r^2} n \langle\beta_\theta\rangle \langle p_\theta\rangle c \tag{19}$$

where in general the last two terms are peculiar to beams and usually absent in the treatment of plasma equilibria. As will be shown subsequently, even in the axisymmetric situation one may find $P_{rr} \neq P_{\theta\theta}$. The centrifugal term which is also included is an electron inertia term that is not negligible in beam equilibria and may be of the same order as the pressure terms.

Replacing $B_\theta(r)$ with $I(r)$ as earlier we find instead of eq. (7)

$$I^2(r) = 2c^2 \int_o^r \{[\tfrac{1}{2} P_{rr}(r_o) + \tfrac{1}{2} P_{\theta\theta}(r_o) + \frac{1}{2r_o} n(r_o) \langle\beta_\theta\rangle\langle p_\theta\rangle c - P_{rr}(r)] + \frac{1}{8\pi}[B_z^2(r_o) - B_z^2(r)]\} 2\pi r_o \, d r_o . \tag{20}$$

We now observe by comparison of eq. (20) with eq. (7) that in general we may expect the above to be less than or greater than the conventional pinch formula of eq. (4) depending on whether the beam or plasma is grossly diamagnetic or paramagnetic, since the terms in B_z^2 can either add or subtract from I^2.

In either case the presence of B_z will affect the arguments of sec. II on current limitation. Hammer and Rostoker[7] have considered a non-selfconsistent example in which B_z is taken uniform and shown that the resulting I may exceed I_A whenever B_z is suitably large compared with the maximum self magnetic field B_θ. An example of selfconsistent equilibria has been given by

Hieronymus and Rostoker[10] in which they use a solid rotor model. These tend to give diamagnetic behavior and cannot be generalized to the fully relativistic regime. Nevertheless, it is a useful model for beams of experimental interest. In the plasma literature there are several examples of equilibria which are admixtures of conventional z-pinch and θ-pinch models in connection with screw pinch and related experimental configurations. Needless to say, there is a rich variety of equilibria corresponding to the situation described by eqs. (19) and (20). Our principal concern, however, will be with equilibria which tend to describe paramagnetic behavior in B_z and are closely related to force-free configurations. For a variety of plasma confinement applications this class seems to be of particular interest.

B. Force-Free Configurations

Strictly speaking force-free magnetic field configurations require the right hand side of eq. (19) to vanish. This, in turn, implies force-free configurations cannot support beam or plasma pressure (finite energy density). Yoshikawa[11] has shown one may achieve equilibria which are arbitrarily close to force-free configurations and the resulting currents may be arbitrarily greater than I_A . A specific example will be discussed in some detail in the following section. For the present we wish to collect some features of force-free configurations.

As a subclass of possible configurations we choose a model in which the drift velocity lies on a helix of constant pitch

$$J_\theta = \kappa r J_z \tag{21a}$$

where $2\pi\kappa^{-1}$ is the pitch length. It follows that

$$B_z = B_{zo} - \kappa r B_\theta \tag{21b}$$

where B_{zo} is the value on axis. We also observe that $2\pi\kappa^{-1}$ is the ratio of I to the azimuthal current per unit length. Substituting into eq. (19) we obtain a standard force-free result in the absence of beam terms

$$B_z = \frac{B_{zo}}{1 + \kappa^2 r^2} = \frac{B_\theta}{\kappa r} \tag{22a}$$

and it will be recognized the above is the same as given by Longmire[12] for a helical field of constant pitch. The paramagnetic behavior in B_z is apparent. Corresponding to the above results one obtains

$$I(r) = \left(\frac{ec}{2r_o}\right) \frac{eB_{zo}}{mc^2\kappa} \left(\frac{\kappa^2 r^2}{1 + \kappa^2 r^2}\right) \tag{22b}$$

which may become arbitrarily large compared with I_A .

In order to obtain a simple estimate of finite beam energy density effects we follow Yoshikawa's example and include the centrifugal term with the additional assumptions that the beam density is uniform and we may approximate $\langle p_\theta \rangle = \bar{\gamma}\, mr \langle \beta_\theta \rangle c$ with $\bar{\gamma}$ a constant. Replacing $\langle \beta_\theta \rangle$ by $-J_\theta / enc$ we obtain from eqs. (19)

$$\frac{B_\theta}{r} \frac{d}{dr}(rB_\theta) + B_z \frac{dB_z}{dr} = \frac{\lambda^2}{r}\left(\frac{dB_z}{dr}\right)^2$$

$$\lambda^2 = \bar{\gamma} mc^2 / 4\pi n_o e^2 . \tag{23}$$

There are an arbitrary number of solutions to the above which will satisfy realistic boundary conditions; but, we are interested in pursuing the consequences of a finite λ introduced by electron inertia on the helical drift model given by eqs. (21). We note in passing beam pressure effects as such are not represented by the approximate manner in which we included the centrifugal force. We shall pursue pressure effects in the following subsection.

Eliminating B_θ via eq. (21b) we re-write eq. (23) in the form

$$k \frac{df}{dx} = \left(1 + \frac{1}{x}\right) f + 1 \tag{24}$$

where $B_z = B_{zo}(1 + f)$, $x = \kappa^2 r^2$ and $k = 2\kappa^2\lambda^2$. We look for solutions which are well behaved at the origin and f vanishes as some positive power of x for small argument. It would appear that eq. (24) represents an example of classical boundary layer phenomenon, but note in eq. (23) λ does not accompany a higher derivative, only a higher power of the first derivative.

For the parameter $k > 1$ we may express the solution as

$$f = F_o x^{1/k} e^{x/k} + \frac{1}{k} \int_o^x \left(\frac{x}{x_o}\right)^{1/k} e^{(x - x_o)/k} \, dx_o$$

$$= F_o \, x^{1/k} \, e^{x/k} + e^{x/k} \sum_o^\infty (-1)^n \frac{(x/k)^{n+1}}{n!(n + 1 - 1/k)} \quad , \tag{25a}$$

where F_o is an arbitrary constant and may be taken as zero. Let us observe for $k > 1$ the functional dependence is on the scale r/λ , is exponentially divergent at large radii and describes diamagnetic behavior. The paramagnetic behavior of the force-free solution is destroyed for $2\kappa^2\lambda^2 > 1$. The power series appearing in eq. (25a) is effectively the incomplete gamma function. When $k = 1$ we obtain a solution in terms of the exponential integral which diverges logarithmically at the origin. When $k = 2$ we obtain the error function, and so on.

Obviously in order to retain the character of the force-free solution we are required to limit $k < 1$, that is $2\kappa^2\lambda^2 < 1$. Discarding the contribution of the homogeneous part proportional to F_o , we write when $k < 1$

$$f = - \int_x^\infty \left(\frac{x}{x_o}\right)^{1/k} e^{(x - x_o)/k} \frac{dx_o}{k} \tag{25b}$$

$$= -x \int_o^\infty \frac{e^{-xu} du}{(1 + ku)^{1/k}} = -x \int_o^\infty e^{-xu - (1/k)\ln(1 + ku)} \, du \quad .$$

Since $(1 + ku)^{-1/k}$ becomes e^{-u} in the limit k vanishes, it is apparent from the above that the force-free solution is recovered in this limit. Furthermore, it can be readily shown that for $k < 1$ there is a convergent expansion in powers of k which yields

$$B_z = \frac{B_{zo}}{1 + \kappa^2 r^2} \left[1 - \frac{(2\kappa^2\lambda^2)\kappa^2 r^2}{(1 + \kappa^2 r^2)^2} + \ldots \right]$$

where the above may be written formally as

$$f = -\sum_{0}^{\infty} k^n \left[z(1-z)^2 \frac{d}{dz}\right]^n z$$
$$z = \frac{x}{1+x} \quad . \tag{25c}$$

We observe the functional behavior is on the scale κr modified by the presence of finite λ .

Since in the helical drift model as given by eqs. (21) the current flowing in some cross-section of radius r is simply proportional to $B_{zo} - B_z$ and therefore to the quantity $-f$, the role of finite λ for $\kappa^2\lambda^2 << \frac{1}{2}$ is to actually increase $I(r)$ from its force-free value[12a]. The influence of the beam, however, vanishes at large enough values of κr . In what follows these effects will be discussed somewhat more precisely. For future purposes it is helpful to note that the results of eq. (25c) could have been obtained directly from eq. (24) by expansion of f in powers of k and iteration.

C. Helical Drift Equilibria

Just as in the discussion of sec. III inclusion of a term $\delta(p_z - \gamma_o mc\beta_o - \frac{e}{c} A_z)$ in the distribution insured axial drift, so the inclusion of a similar term $\delta(p_z + \kappa p_\theta - \gamma_o mc\beta_{\parallel} - \frac{e}{c} A_z - \frac{e}{c} \kappa r A_\theta)$ will guarantee the distribution to provide helical drifts in accordance with eqs. (21). We choose to reexamine the monoenergetic distribution studied by Kan and Lai[13] in a similar context. Whereas the former authors restricted their investigation to configurations in which externally generated fields were absent, we do not do so. The criticism that certain unphysical features may result from delta function distributions should not affect our search for scaling relations between beam and field parameters.

Details of the model under discussion may be found in reference 13. We recall our earlier remark on pressure tensor anisotropy and note in this case

$$P_{rr} = (1 + \kappa^2 r^2) P_{\theta\theta} = P_{\theta\theta} + P_{zz} ;$$

similar anisotropic behavior may be expected in more general equilibria, though we have not attempted to prove it is a necessary consequence of paramagnetic configurations.

Quantities of interest may be obtained in terms of the reduced magnetic field component $b_z = B_z/B_{zo}$ which satisfies the equation

$$\alpha\rho\,[\frac{1}{\alpha\rho}\,(1 + \alpha^2\rho^2)^{3/2}b'_z]' \;=\; (1 + \alpha^2\rho^2)b_z - 1 \;, \tag{26}$$

and the notation is $r = \lambda\rho$, $\lambda^2 = \gamma_o mc^2/4\pi n_o e^2$, $\gamma_o = (1 - \beta^2)^{-\frac{1}{2}}$, $\beta^2 = \beta_\perp{}^2 + \beta_\parallel{}^2$ and $\alpha = \kappa\lambda$. The magnetic skin depth is given in terms of beam energy and density on axis while β represents streaming velocity on axis. As in the case of the Longmire sheath there is a sharp beam edge at $r = a$, outside of which the fields are given by vacuum solutions $B_z = B_z(a)$, $B_\theta = (a/r)B_\theta(a)$.

The beam edge may be found from the pressure relation

$$\langle\beta_z\rangle^2 \;=\; \frac{\beta^2}{1 + \kappa^2 a^2} \quad @ \quad \rho \;=\; \frac{a}{\lambda} \;, \tag{27a}$$

where

$$\langle\beta_z(\rho)\rangle \;=\; b_o\,\frac{(1 + \alpha^2\rho^2)^{\frac{1}{2}}}{\alpha\rho}\;b'_z \tag{27b}$$

$$b_o \;=\; B_{zo}/(4\pi n_o\gamma_o mc^2)^{\frac{1}{2}} \;.$$

It will be recalled that in this model the beam density varies as $(1 + \kappa^2 r^2)^{-\frac{1}{2}}$, while other quantities of interest follow from eqs. (21).

The boundary conditions on b_z may be obtained by allowing $\alpha\rho$ to approach zero and it follows that

$$b_z \;\to\; 1 + (\beta_\parallel/b_o)\alpha\rho I_1(\rho)$$

$$b'_z \;\to\; (\beta_\parallel/b_o)\alpha\rho I_o(\rho)$$

$$\langle\beta_z(0)\rangle \;=\; \beta_\parallel$$

in the vicinity of the axis. One may also infer from the above that in the limit of vanishing α we recover the Longmire sheath results, as is to be expected.

An approximate solution to eq. (26) may be constructed by observing that the transformation

$$b_z = 1 + \frac{\alpha y}{1 + \frac{1}{2}\alpha^2 y^2} \phi(y)$$

$$1 + \alpha^2 \rho^2 = (1 + \tfrac{1}{2}\alpha^2 y^2)^2 \tag{28a}$$

allows the original equation to be written as

$$\frac{1}{y}[y\phi']' - (k^2 + \frac{1}{y^2})\phi = \alpha y$$

$$k^2 = 1 + \frac{\alpha^2 y^2}{4 + \alpha^2 y^2} . \tag{28b}$$

Since k is a slowly varying function of y and is of modulus unity, we treat it as a constant and obtain to this degree of approximation

$$b_z = 1 - \frac{(\alpha y/k)^2}{1 + \frac{1}{2}\alpha^2 y^2} + (\frac{\beta_{\parallel}}{b_o} + \frac{2\alpha}{k^3}) \frac{\alpha y}{1 + \frac{1}{2}\alpha^2 y^2} I_1(ky) . \tag{28c}$$

The first two terms above may be viewed as the modified force-free solution, while the last term is the contribution from a Longmire sheath-like B_θ term. The nature of our approximation inherently assumes that α is small.

From eq. (28c) it is apparent that for $(\beta_{\parallel}/b_o) > 0$, the beam terms are diamagnetic in lowest order of α. Consequently, we must choose the current on axis to flow parallel to the local axial field in order to preserve the paramagnetic near-force-free character. In what follows we shall treat $\beta_{\parallel}$ negative and b_o positive.

To within our order of approximation we can write

$$b'_z(\rho) = - \frac{\alpha\rho}{(1 + \alpha^2\rho^2)^{3/2}} [\frac{2\alpha}{k^2} + (\frac{\beta_{\parallel}}{b_o} + \frac{2\alpha}{k^3})\alpha y I_1(ky)] + k(\frac{\beta_{\parallel}}{b_o} + \frac{2\alpha}{k^3}) \left[\frac{\alpha\rho}{1 + \alpha^2\rho^2} I_o(ky)\right]; \tag{28d}$$

and to lowest order in α we find with the aid of eq. (27b)

$$\langle\beta_z\rangle = \frac{\beta_{\parallel}}{(1 + \alpha^2\rho^2)^{\frac{1}{2}}} [kI_o(ky) - \frac{\alpha y I_1(ky)}{1 + \frac{1}{2}\alpha^2 y^2}] . \tag{28e}$$

We may now combine eqs. (27) with the results of eqs. (28) to solve for the beam edge and related quantities; thus

$$\frac{\beta}{|\beta_{\parallel}|} = \left[kI_o(ky) - \frac{\alpha y I_1(ky)}{1+\frac{1}{2}\alpha^2 y^2}\right]_{y = y_a}$$

$$\alpha^2 y_a^2 = 2[(1+\kappa^2 a^2)^{\frac{1}{2}} - 1] \ . \qquad (29a)$$

Alternatively we may introduce the vacuum axial field $b_{zv} = b_z(a/\lambda)$ via eq. (28c)

$$\frac{\beta}{|\beta_{\parallel}|} = \left\{kI_o(ky) - \frac{b_o}{|\beta_{\parallel}|}\left[1 - \frac{(\alpha y/k)^2}{1+\frac{1}{2}\alpha^2 y^2} - b_{zv}\right]\right\}_{y = y_a} \ . \qquad (29b)$$

The beam current, in turn, is

$$I_z = \left(\frac{ec}{r_o}\right)\frac{\gamma_o b_o}{2\alpha}(1 - b_{zv})$$

$$= \left(\frac{ec}{2r_o}\right)\gamma_o |\beta_{\parallel}| \left[\frac{y}{1+\frac{1}{2}\alpha^2 y^2} I_1(ky) + \frac{\alpha y^2/k^2}{1+\frac{1}{2}\alpha^2 y^2}(b_o/|\beta_{\parallel}|)\right]_{y = y_a} \ . \qquad (29c)$$

In the limit that α vanishes it reduces to the Hammer and Rostoker[7] result quoted in our eq. (12). Kan and Lai[13] have shown for the specific case $b_{zv} = 0$ that the helical drift model leads to larger beam currents than the corresponding simple axial drift model. It may be inferred from eqs. (29) that there is no upper bound in principle on the current except for practical limitations. The contribution of the force-free character of the equilibrium to the current is approximately proportional to the parameter b_o, which in turn is proportional within numerical factors to $\alpha|\beta_{\parallel}|(I_z/I_A)$. This intrinsically paramagnetic contribution reduces the dependence of I_z on $(\beta_{\perp}/\beta_{\parallel})$ which was evident in the Longmire sheath model.

There is the practical question of how such equilibria may be prepared in the laboratory. Purely selfconstricted helical equilibria in which vacuum fields are absent seem somewhat artificial and may be difficult to achieve.

On the other hand, with an appropriate choice of parameters it should be possible to produce configurations in which the axial field reverses within the beam, that is, equilibria in which $b_{zv} < 0$. This much is evident from eq. (29b). In practical terms, if ξ measures the degree of field reversal so that $B_{zv} = - B_{zo}/\xi$, then

$$
\begin{aligned}
|B_{zv}| &= \frac{2\kappa}{c} \left(\frac{I_z}{1 + \xi}\right) \\
&= \frac{\kappa a}{5a} \left(\frac{I_z}{1 + \xi}\right)
\end{aligned}
$$

where in the second form B is measured in gauss, a in cm. and I_z in amperes. The above suggests that beams with net currents on the order of 10^5 amps should be accompanied by vacuum fields on the order of 10^3 to 10^4 gauss on the premise the beam radius is several cm. and $\kappa a/(1 + \xi)$ is of order unity.

It has been shown[14] in connection with a related plasma problem that equilibria with reversed field configurations have desirable stability properties. This finding may possibly extend to beams. The calculations leading up to eqs. (29) have been crude in parts and a more accurate evaluation of the reversed field case may be in order. By comparison with Christofilos' well-known Astron concept, the specific case discussed above can lead to a high order of field reversal.

In conclusion we find that the presence of externally generated magnetic fields leads to a rich variety of beam equilibria with currents limited only by practical considerations. Relaxing the constraints of a simple z-pinch configuration allows one in principle to exceed the Alfven-Lawson limit to an arbitrary extent. Equilibria which resemble force-free configurations can be constructed with finite beam energy density. The force-free behavior is limited to the region where the beam current is parallel to the axial field. For practical purposes it may be desirable to construct configurations with field reversal occurring inside the beams in the vicinity of its edge and this, in turn, relies on the non force-free character of the beam.

In a number of applications one is interested in toroidal beam configurations rather than the elementary cylindrical cases treated here. The generalization is straightforward but not simple. Continuing to use cylindrical coordinates to describe the torus with θ the minor and z the major axes, toroidal equilibria in general require a vertical field which may be obtained from an $A_\theta = A_\theta\,(r,z)$. Thus, while axisymmetry is retained, z is no longer an ignorable coordinate and specific examples of equilibria constructed here cannot be taken over bodily to the toroidal situation. Nevertheless, we would expect in a large aspect ratio expansion of the torus problem many of our results to retain qualitative validity.

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12a For a *given* B_z distribution eq. (23) predicts increasing I_z with finite λ in general since $I_z^2 = \frac{1}{2} c^2 \int_0^r [\Delta B_z^2 + \lambda^2 (B_z')^2] r_o dr_o$ where $\Delta B_z^2 = B_z^2(r_o) - B_z^2(r)$.

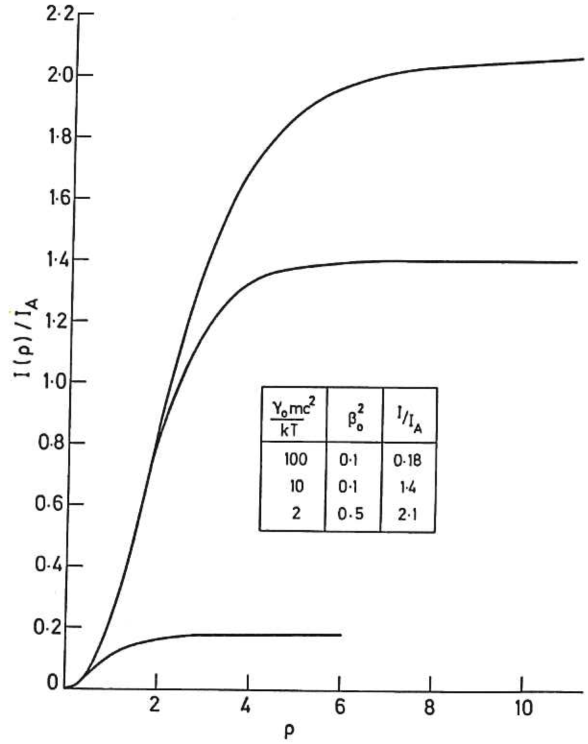

Fig. 1 Radial variation of the current according to eq. (15).

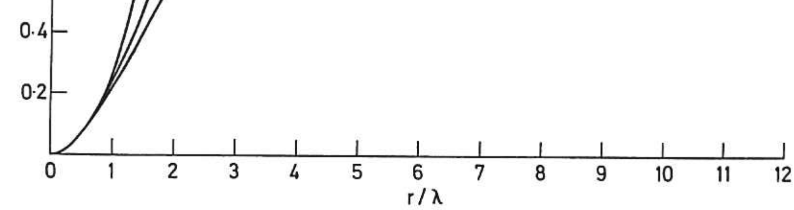

Fig. 2 Comparison of current distribution as given by 1 - eq. (12), 2 - eq. (15), and 3 - eq. (11). Each curve is normalized to the same dimensionless value $I/I_A = 1.4$.