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J W CONNOR
R J HASTIE

CULHAM LABORATORY
Abingdon Berkshire

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J.W. CONNOR

and

R.J. HASTIE

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ABSTRACT

Neo classical diffusion in a stellarator is investigated in detail by introducing a consistent expansion in $\lambda^{1/3}$ (where λ is the inverse aspect ratio of the torus) for the magnetic fields and particle distribution function. As a result of the expansion, the various characteristic bounce and transit frequencies of the particle orbits appear in different orders, and various regimes of collision frequency (analogous to the 'banana', and 'plateau' regimes of the axisymmetric analysis) are considered separately. It is found that over most of the range of collision frequencies the diffusion is the same as that of an equivalent axisymmetric configuration with the sole role of the helical fields being that of supplying the rotational transform. At sufficiently low collision frequencies particle trapping in the helical field modulations becomes important and localised particle diffusion dominates. Expressions for the diffusion flux and ion heat flux are derived in this limit.

UKAEA Research Group,
Culham Laboratory,
Abingdon,
Berks.

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I. INTRODUCTION

While much attention has been devoted to the neo-classical diffusion of plasma in axisymmetric tori[1-5], the analogous problem in the non-symmetric torus has not been investigated to the same extent. At low collision frequencies, comparable to the drift frequency of particles around the minor azimuth, superbanana effects dominate and the resulting diffusion has been calculated[6]. For collision frequencies comparable with the bounce frequencies of particles moving along the field lines, a rigorous treatment has been given by Frieman[7]. He considers a general non-axisymmetric magnetic field but does not discuss the usual assumption that a large aspect ratio stellarator behaves as an equivalent axisymmetric torus with additional effects superimposed on the diffusion from the helical field modulation. Stringer[8] has observed that the helical modulations in a stellarator can produce a diffusion analogous to that produced by the toroidal curvature, and regarding the diffusion as the sum of a toroidal part from an equivalent axisymmetric torus and a part from the helical modulations has estimated the total diffusion in a stellarator. In addition Kovrizhnykh[5,9] has given a comprehensive discussion of stellarator diffusion in a weakly ionised plasma. His averaging technique for dealing with the complicated magnetic field structure in a stellarator is similar to the one we shall discuss but cannot be as systematically applied to higher order in the averaging parameter and can not be applied in all the collision regimes we shall discuss later.

Our purpose in this paper is to give a rigorous treatment of the diffusion in a large aspect ratio $\ell = 3$ stellarator for collision frequencies comparable with the various bounce and transit frequencies. We adopt the approach of Frieman[7] but introduce from the beginning an explicit form for the magnetic field structure. This appears as an expansion in the strengths of the toroidal and helical modulations of the field, using the ordering procedure of Frieman and Dobrott[10] which we discuss in Section II. The advantage of introducing an expansion of this type is that it separates out the characteristic bounce and transit frequencies connected with the different scale lengths along the field lines of the helical and toroidal modulations. This enables us to discuss the detailed variation of diffusion with respect to collision frequency. In addition the ordering introduced by Frieman and Dobrott[10] assumes that helical and toroidal modulations are comparable (which is the usual experimental situation) and enables us to compare the contributions to diffusion from particles trapped in either type of modulation.

Section III follows Frieman[7] in expanding the Fokker-Planck equation in m/e (i.e. a small larmor radius expansion) with the collision frequency comparable to a typical transit or bounce frequency. This results in a guiding-centre like equation for the relevant part of the distribution function. An expression for the diffusion flux in terms of the solution of this equation is given.

As mentioned above, introducing the small aspect ratio expansion, with the Dobrott-Frieman ordering, enables us to distinguish between the various characteristic frequencies of the particle orbits and different solutions of the guiding-centre equation now result from different orderings of the collision frequency ν with respect to these frequencies within this subsidiary aspect ratio expansion. The characteristic frequencies are: Ω_t, the transit frequency through a helical field period of a particle with $q \sim v_{th}$ (q is the velocity of the particle along the field line, and v_{th} the thermal velocity); ω_t the transit frequency of the same particle through a connection length; Ω_b the bounce frequency of a particle trapped in the helical modulations (or the transit frequency of a 'slow', i.e. just passing, particle through a helical period); and ω_b the bounce frequency of a 'blocked' particle trapped by the toroidal modulation of the field (or the transit frequency of a slow particle through a connection length). In Section IV we consider all the possible orderings of ν or ν_{eff} (ν_{eff} is the effective collision frequency of a slow particle caused by the diffusive nature of the Fokker-Planck term) with respect to these frequencies - that is we order ν (or ν_{eff} as appropriate) in the aspect ratio expansion. We then systematically expand the guiding-centre equation and distribution function in the inverse aspect ratio deriving an equation for the significant order of the distribution function (i.e. the order giving the dominant contribution to the diffusion flux). The solution of this equation is then used to calculate the diffusion flux so that its variation with collision frequency may be examined in detail. Finally our results are discussed in Section V.

II. THE MAGNETIC FIELD GEOMETRY

The coordinates r, θ, ζ are measured with respect to the minor axis of a torus of major radius R as shown in Fig.1. With these coordinates the metric coefficients are

$$h_1 = 1, \quad h_2 = r, \quad h_3 = 1 - \frac{r}{R} \cos \theta .$$

The $\ell = 3$ helical winding has a period $2\pi R/P$ where P is the total number of periods around the machine.

The superposition of solenoidal and helical windings produces a magnetic field $\underline{B}$ whose lines twist about a magnetic axis. As one follows a field line around the torus the strength of the magnetic field B is modulated as in Fig.2. The rapid modulation arises from the helical component while the long scale modulation arises from the variation in the toroidal component as the field line passes from the inside to the outside of the torus. These variations result in the existence of three classes of particles: the localised particles which are trapped within a helical period, the blocked particles which pass over a number of helical periods before being reflected by the toroidal modulation, and passing particles which traverse the entire torus. Some particles may make transitions between the localised and blocked classes since a particle trapped in a helical well may drift into regions where the magnetic field is lower due to the toroidal variation and may then escape, or vice versa[10,11]. This effect occurs on the drift timescale and is unimportant for the collision frequencies considered here.

In the discussion of the magnetic field below, we closely follow Dobrott and Frieman[10] and reproduce a number of their results. A natural expansion parameter is the inverse aspect ratio $\lambda \ll 1$. A dimensionless minor radius ρ, of order one, may then be defined

$$r = R\lambda\rho \cdot \qquad (1)$$

The vacuum magnetic field consists of a lowest order constant field B_0 in the ζ-direction together with toroidal and helical perturbations. The toroidal perturbation is $0(\lambda)$, of course, and the 'optimal' ordering is to choose the $\ell = 3$ helical modulation of $0\ (\lambda)$. A finite total rotational transform then requires that $P \sim 0(\lambda^{-2/3})$ and so we define p :

$$P = p\,\lambda^{-2/3} \tag{2}$$

and a new length in the toroidal direction, s, by

$$\zeta = \frac{R\,s\,\lambda^{2/3}}{p}\,. \tag{3}$$

Thus two natural length scales, that of a single helical period s and of the total distance around the torus $s\lambda^{-2/3}$, arise, roughly corresponding to the trapping distance for localised and blocked particles respectively. It should be noted that the present ordering results in a typical radius ρ being small compared with the length of a helical period, unlike the conventional stellarator ordering[12].

With this ordering Dobrott and Frieman calculate the magnetic fields and flux surfaces as functions of ρ, θ and s. Defining $\underline{b} = \underline{B}/B_0$ they obtain

$$b_\rho = \frac{3\,\alpha\,\lambda^{2/3}}{p}\,\rho^2\cos(3\theta+s) + \frac{5}{16}\,\alpha\,\lambda^{4/3}\,p\rho^4\cos(3\theta+s)$$
$$+ \frac{\alpha\lambda^{5/3}}{p}\,\rho^3\cos(2\theta+s) + \frac{7}{640}\,\alpha\lambda^2 p^3\rho^6\cos(3\theta+s) + \,..$$

$$b_\theta = -\frac{3\,\alpha\,\lambda^{2/3}}{p}\,\rho^2\sin(3\theta+s) - \frac{3}{16}\,\alpha\lambda^{4/3}p\,\rho^4\sin(3\theta+s)$$
$$- \frac{\alpha\lambda^{5/3}}{2p}\,\rho^3\sin(2\theta+s) - \frac{3}{640}\,\alpha\lambda^2 p^3\rho^6\sin(3\theta+s) + \,..$$

$$b_s = 1 + \lambda[\rho\cos\theta - \alpha\rho^3\sin(3\theta+s)] - \frac{\alpha\lambda^{5/3}p^2}{16}\,\rho^5\sin(3\theta+s)$$
$$- \frac{\alpha\lambda^2}{4}\,\rho^4\,[3\sin(2\theta+s) + 2\sin(4\theta+s)] + \lambda^2\rho^2\cos^2\theta + 0(\lambda^{7/3}) + \,... \tag{4}$$

The parameter α characterises the relative strengths of helical and toroidal modulations. The flux surfaces ψ = constant are given by

$$\psi = \rho^2 - \frac{6\,\alpha\,\lambda^{1/3}}{p^2}\,\rho^3\sin(3\theta+s) - \frac{5}{8}\,\lambda\,[\rho^5\sin(3\theta+s) + 2\rho^3\cos\theta]$$
$$+ \frac{\alpha\lambda^{4/3}}{4p^2}\,\rho^4[46\sin(2\theta+s) + 39\sin(4\theta+s)] + \,... \tag{5}$$

which may be inverted to yield the radii of the flux surfaces:

$$\rho^2 = \psi \left[1 + \frac{6\,\alpha\,\lambda^{1/3}}{p^2} \psi^{1/2} \sin(3\theta + s) + \frac{54\alpha^2\lambda^{2/3}}{p^4} \psi \sin^2(3\theta + s) \right.$$

$$\left. + \lambda \left(\frac{567\alpha^3}{p^6} \psi^{3/2} \sin^3(3\theta + s) + \frac{5}{8}\alpha\psi^{3/2} \sin(3\theta + s) + \frac{5}{4}\psi^{1/2} \cos\theta \right) + \ldots \right] \tag{6}$$

These fields produce a total rotational transform

$$\iota\!\!\!- \equiv \frac{\iota}{2\pi} = -\frac{18\alpha^2\psi}{p^3} + 0(\lambda^{1/3}) \,. \tag{7}$$

III. THE GUIDING-CENTRE EQUATIONS AND DIFFUSION FLUX

The distribution function f satisfies the kinetic equation

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f + \frac{e}{m} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f}{\partial \underline{v}} = C\,(f,f) \tag{8}$$

with the electric field $\underline{E}$ derivable from a potential Φ, $\underline{E} = -\nabla\Phi$ and where $C(f,f)$ is the collision term. It is more convenient to use the $\varepsilon,\mu,\varphi,\sigma$ velocity variables. Writing

$$\underline{v} = \sigma\, q\, \underline{n} + v_\perp\, (\underline{m} \cos\varphi + \underline{p} \sin\varphi)$$

where $\underline{n}$ is a unit vector along the magnetic field and $\underline{m}$ and $\underline{p}$ are unit vectors orthogonal to $\underline{n}$ and each other, then

$$\varepsilon = \frac{1}{2} v^2 + \frac{e}{m}\Phi \quad ; \quad \mu = \frac{1}{2}\frac{v_\perp^{\,2}}{B}$$

$$q = |\underline{v}\cdot\underline{n}| = [2(\varepsilon - \mu B - \frac{e\Phi}{m})]^{\frac{1}{2}} \;;\; \sigma = \frac{\underline{v}.\underline{n}}{q} = \pm 1 \tag{9}$$

and $\int d^3\underline{v}$ is replaced by $\sum_\sigma \int \frac{B}{q}\, d\mu d\varepsilon d\varphi$. In these variables the kinetic equation (8) takes the form

$$\omega_c \frac{\partial f}{\partial \varphi} = [Df - C(f,f)] \tag{10}$$

where $\omega_c = eB/mc$ is the inverse expansion parameter and details of the D operator can be found in references 7, 13.

We expand f as a series of powers in m/e and following Frieman[7], obtain equations for f_o, f_1, f_2 ... Using standard notation in which we write a periodic function $X(\varphi) = \tilde{X} + \bar{X}$ where $\partial\bar{X}/\partial\varphi = 0$ and $\oint d\varphi\tilde{X} = 0$, the equations for f_o become

$$f_o = \bar{f}_o$$

$$\sigma\, q\, \underline{n}\cdot\nabla f_o = \bar{C}(f_o, f_o) \qquad (11)$$

which have the Maxwellian solution chosen by Frieman

$$f_o = \left(\frac{m}{2\pi T}\right)^{3/2} N(\psi)\ \exp\left(-\frac{m\varepsilon}{T}\right) \equiv \left(\frac{m}{2\pi T}\right)^{3/2} n(\psi)\ \exp\left(-\frac{mv^2}{2T}\right) \qquad (12)$$

where

$$n = N\ \exp\left(-\frac{e\Phi}{T}\right)$$

constant over each magnetic surface $\psi(\rho,\theta,s) = $ const. In first order f_1 is given by

$$f_1 = \frac{1}{\omega_c}\int^{\varphi} d\varphi D f_o + \bar{f}_1 \qquad (13)$$

with

$$\sigma q\, \underline{n}\cdot\nabla\, \bar{f}_1 + \underline{v}_d\cdot\nabla\, f_o = \bar{C}\,(f_o,\bar{f}_1) + \bar{C}\,(\bar{f}_1,\, f_o) \qquad (14)$$

where we have introduced the drift velocity $\underline{v}_d$:

$$\underline{v}_d = \frac{1}{\omega_c}\,\underline{n} \times (\mu\nabla B + q^2\underline{n}\cdot\nabla\underline{n}) - \frac{\nabla\Phi \times \underline{B}}{B^2}\ . \qquad (15)$$

Finally from second order we require the result

$$f_2 = \frac{1}{\omega_c}\int^{\varphi} d\varphi\ [D\ f_1 - C(f_o,f_1) - C(f_1,f_o)] + \bar{f}_2\ . \qquad (16)$$

The flux of particles out of the volume V interior to a magnetic surface S (i.e. ψ = constant) is obtained by integrating the kinetic equation (8) over velocity space and physical space inside S. This gives

$$\frac{\partial}{\partial t}\int_V n(\psi)d^3x + \int\!\!\int_S J\,|\nabla\psi|\,d\theta ds \sum_{\sigma} \iiint \frac{B}{q}\, d\mu d\varepsilon d\varphi\, \underline{p}\cdot\underline{v}_{\perp}\tilde{f}_2 = 0 \qquad (17)$$

where

$$J = \rho(1-\lambda\rho\ \cos\ \theta)/\frac{\partial\psi}{\partial\rho}$$

is the jacobian of the transformation from a cartesian system to the ψ, θ, s coordinates, and the unit vector $\underline{p}$ has been chosen to lie along the normal to the magnetic surface. After some manipulation and several partial integrations the surface term may be written, using equation (16) for f_2 in the form

$$\sum_{\sigma} \iint J d\theta ds \iint \frac{B}{q} d\mu d\varepsilon d\varphi \left\{ \bar{f}_1 \underline{v}_d \cdot \nabla\psi + \mu \frac{B\sigma q}{\omega_c^2} \left(\underline{mm} : \nabla \underline{n} - \underline{pp} : \nabla \underline{n} \right) |\nabla\psi| \frac{\partial f_o}{\partial \psi} \right\}$$

$$+ \sum_{\sigma} \iint J d\theta ds \iiint \frac{B}{q} d\mu d\varepsilon d\varphi \frac{\underline{v} \times \underline{n} \cdot \nabla\psi}{\omega_c} [\tilde{C}(f_o, \tilde{f}_1) + \tilde{C}(\tilde{f}_1, f_o)] . \quad (18)$$

Of these terms, the last represents the classical diffusion flux, the second vanishes if f_o is an even function of σ but yields an additional classical contribution if $f_o^+ \neq f_o^-$ (longitudinal flow and current), while the first term represents the anomalous part of the diffusion flux, Γ_a which we wish to investigate here. Thus Γ_a satisfies the equation

$$\frac{\partial n}{\partial t} + (\oint J \, d\theta d\zeta)^{-1} \frac{\partial \Gamma_a}{\partial \psi} = 0 \quad (19)$$

where

$$\Gamma_a = \iint J d\theta ds \sum_{\sigma} \int \frac{B}{q} d\mu d\varepsilon d\varphi \; \bar{f}_1 \; \underline{v}_d \cdot \nabla\psi . \quad (20)$$

At this point we depart from the general geometry approach of Frieman by introducing the small aspect ratio expansion to facilitate the solution of equation (14) for $\bar{f}_1$, and to separate out the different collisional regimes by analogy with the axisymmetric situation. Thus the operators $\underline{n} \cdot \nabla$ and $\underline{v}_d \cdot \nabla$ appearing in equation (14) become

$$\underline{n} . \nabla = \frac{\lambda^{1/3}}{R\lambda} \left\{ p \frac{\partial}{\partial s} + \sum_{m=1}^{\infty} \lambda^{m/3} L_m \right\} \quad (21)$$

$$\underline{v}_d . \nabla = \frac{2\kappa - \mu B_o}{\omega_c R^2 \lambda} \sum_{m=0}^{\infty} \lambda^{m/3} M_m \quad (22)$$

where $\kappa = \varepsilon - e\Phi/m$, $\omega_c = eB_o/m$ and we shall explicitly make use of

$$\left. \begin{aligned} L_1 &= - \frac{3\alpha\psi^{\frac{1}{2}}}{p} \sin(3\theta + s) \frac{\partial}{\partial \theta} \\ L_2 &= - \frac{9\alpha^2 \psi}{p^3} \sin^2(3\theta + s) \frac{\partial}{\partial \theta} \end{aligned} \right\} \quad (23)$$

$$M_o = 2[\psi^{\frac{1}{2}} \sin\theta + 3\alpha\psi^{3/2} \cos(3\theta + s)] \frac{\partial}{\partial\psi}$$
$$+ [\cos\theta - 3\alpha\psi \sin(3\theta + s)] \frac{1}{\psi^{\frac{1}{2}}} \frac{\partial}{\partial\theta} \quad (24)$$

It will therefore be necessary to expand $\bar{f}_1$ in powers of $\lambda^{1/3}$.

To calculate the diffusion flux from equation (20) we require the λ expansion of $\iint J d\theta ds \; B/q \; \underline{v}_d \cdot \nabla\psi$:

$$\iint J d\theta \, ds \frac{B}{q} \underline{v}_d \cdot \underline{\nabla\psi} = \frac{2\kappa - \mu B_o}{\omega_c R} \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{ds}{2\pi} \frac{B}{q} \sum_{m=0}^{\infty} \lambda^{m/3} H_m(\psi,\theta,s) \quad (25)$$

with

$$\left.\begin{aligned} H_o &= \sin\theta + 3\alpha\,\psi \cos(3\theta + s) \\ H_1 &= \frac{3\,\alpha\psi^{1/2}}{p^2} [9\alpha\psi \sin(6\theta + 2s) - \cos(2\theta + s) - 2\cos(4\theta + s)] \end{aligned}\right\} \quad (26)$$

At this point it is convenient to discuss the role played by the radial electric field. From the expression (20) for the flux Γ_α it is clear that the ions will, in general diffuse out more rapidly than the electrons unless an appropriate radial electric field exists. This field is such that it balances the ion-pressure gradient[7] in order that the term proportional to ∇f_{oi} in equation (14) for the ions is small, thus reducing the magnitude of f_{1_i}. The ambipolar flux can then be calculated from the electron distribution function in the presence of this radial electric field.

IV. SOLUTION OF THE KINETIC EQUATION AND EVALUATION OF THE DIFFUSION FLUX

The frequencies $\Omega_t, \Omega_b, \omega_t$ and ω_b are ordered in the aspect ratio λ from their definitions as follows:

$$\Omega_t \sim \lambda^{-2/3} \frac{v_{th}}{R}, \quad \Omega_b \sim \lambda^{-1/6} \frac{v_{th}}{R}, \quad \omega_t \sim \frac{v_{th}}{R}, \quad \omega_b \sim \lambda^{1/2} \frac{v_{th}}{R} .$$

The collision frequency can then be ordered with respect to these frequencies, noting that for fast passing particles (those with $q \sim v_{th}$) it is the inverse 90^0 collision time ν, while for slow particles ($q \ll v_{th}$) one uses an effective collision frequency $\nu_{eff} \sim \nu v_{th}^2/q^2$, as a result of the diffusive character of the Fokker-Planck collision operator. Thus we find four critical values of the collision frequency ($\nu \sim \Omega_t, \omega_t$ for fast particles with $q \sim v_{th}$ and $\nu_{eff} \sim \Omega_b, \omega_b$ for slow particles with $q \sim \lambda^{1/2} v_{th}$, occuring at values of ν given by $\nu \sim \lambda^{-2/3}\omega_t, \omega_t, \lambda^{5/6}\omega_t, \lambda^{3/2}\omega_t$) separating five collisional regimes in which different diffusion phenomena might be expected.

To investigate each of these regimes we shall order ν in λ so that it coincides with these four critical frequencies and then perform subsidiary expansions with ν lying on either side of each one.

(a) $\nu > \Omega_t$

We examine this case by ordering $\nu \sim \Omega_t$ in the λ expansion, and then by making a subsidiary expansion in Ω_t/ν, which we label by η. Only fast particles $q \sim v_{th}$ are important, since slow particles receive many scatters before a transit of a helical period. The discussion will be presented in some detail as it exhibits similar features to the other cases to be considered.

We write the kinetic equation (14) for the electron distribution function f as

$$\frac{\sigma q}{R}\left(p\frac{\partial}{\partial s} + \sum_{m=1}^{\infty}\lambda^{m/3} L_m\right) f + \frac{(2\kappa-\mu B_0)}{\omega_c R^2}\sum_{m=0}^{\infty}\lambda^{\frac{m-1}{3}} M_m F = \eta^{-1}\lambda^{2/3} C(f,F) \quad (27)$$

where we have used the notation $f_0 \equiv F$, $\bar{f}_1 \equiv f$, and have dropped the subscript on the distribution functions, gyrofrequency etc., since these now refer to electrons only. Since $\nu \sim \Omega_t \sim \lambda^{-2/3}\omega_t$ the R.H.S. of equation (27) is $O(\lambda^0)$, in the λ expansion and $O(\eta^{-1})$ in the η expansion.

To obtain a solution for f, we expand in powers of $\lambda^{1/3}$ and η in the form

$$f = \lambda^{\alpha}\eta^{\beta}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda^{m/3}\eta^{n} f_m^{(n)} . \quad (28)$$

The ordering of f against F in λ and η (i.e. the choice of the indices α and β in equation (28)) is as yet to be determined. This freedom corresponds to the choice of the order in which the inhomogeneous $\underline{v}_d.\nabla F$ term is introduced into the hierarchy of equations which results from substitution of (28) into (27). The earliest order at which the inhomogeneous term can appear is governed by certain constraints on f and $C(f)$, viz that f be a periodic function of θ and s, and that the collision operator conserve the number of particles in the system.

Substituting (28) into (27) the general equation is

$$\frac{\sigma q}{R}\left(p\,\frac{\partial f_m^{(n)}}{\partial s} + \sum_{\ell=1}^{m} L_\ell\, f_{m-\ell}^{(n)}\right) + \delta_{n,-\beta}\left(\frac{2\kappa-\mu B_o}{\omega_c\, R^2}\right) M_{m+1+3\alpha}\, F = \lambda^{2/3}\, C(f_m^{(n+1)},F) \tag{29}$$

where δ is the Kronecker delta, and $M_n = 0$ for $n < 0$ is understood. The lowest order (in η) equation is

$$\lambda^{2/3}\ C(f_m^{(0)},F) = \delta_{\beta,1}\ \left(\frac{2\kappa-\mu B_o}{\omega_c\ R^2}\right) M_{m+1+3\alpha}\ F\ . \tag{30}$$

However, integrating equation (30) over velocity space, particle conservation requires that

$$\delta_{\beta,1}\ \int d^3v \left(\frac{2\kappa-\mu B_o}{\omega_c\ R^2}\right) M_{m+1+3\alpha}\ F = 0 \tag{31}$$

and consequently that $\beta < 1$. Equation (30) then has the Maxwellian solution

$$f_m^{(0)} = \left(\frac{m}{2\pi T_e}\right)^{3/2} N_m^{(0)}\ (\psi,\theta,s)\ \exp\ (-m\varepsilon/T_e)\ . \tag{32}$$

In the next order of the Ω_t/ν expansion we have

$$\lambda^{2/3}\ C(f_m^{(1)},F) = \frac{\sigma q}{R}\left(p\,\frac{\partial f_m^{(0)}}{\partial s} + \sum_{n=1}^{m} L_n\ f_{m-n}^{(0)}\right) + \delta_{\beta,0}\,\frac{(2\kappa-\mu B_o)}{\omega_c\ R^2}\ M_{m+1+3\alpha}\ F\ . \tag{33}$$

Once again, integrating this equation over velocity space, and noting from (32) that the $f_m^{(0)}$ are even in σ, particle conservation requires $\beta < 0$.

In the next order however, ($n = 1$ in equation (29)) the presence of the $\underline{v}_d\cdot\nabla F$ term is not inconsistent with the constraint so that $\beta = -1$ and after integrating $\int d^3v$ we obtain

$$p\,\frac{\partial}{\partial s}\left(\frac{U_m^{(1)}}{B}\right) + \sum_{n=1}^{m} L_n\left(\frac{U_{m-n}^{(1)}}{B}\right) + \frac{2(T_i+T_e)}{eB^2R}\ M_{m+1+3\alpha}\ n(\psi) = 0 \tag{34}$$

where we have used the form (12) for F, and the fact that the radial electric field balances the ion pressure gradient. The velocities $U_m^{(1)}$ are defined as

$$U_m^{(1)} = \int d^3v\ \sigma q\ f_m^{(1)}\ . \tag{35}$$

Since we finally require the θ,s dependence of $N_o^{(0)}$ we now construct equations for the $N_m^{(0)}$ in terms of the $U_m^{(1)}$ by integrating equation (33) $\int d^3 v\,\sigma q$. This gives

$$p \frac{\partial}{\partial s} N_m^{(0)} + \sum_{n=1}^{m} L_n N_{m-n}^{(0)} = a\eta_{\parallel} U_m^{(1)} \tag{36}$$

where we have used the equivalence of equation (33) with the Spitzer problem.[14] The resistivity $\eta_{\parallel}$ is defined by

$$\eta_{\parallel} = \lambda^{2/3} \frac{4\sqrt{2\pi}\, m_e^{\frac{1}{2}}\, \ell n\Lambda\, e^2\, ZA(Z)}{3T_e^{3/2}} \tag{37}$$

where A(Z) is a correction factor depending only on $Z = \left|\frac{e_i}{e}\right|$, with

$$0.51 = A(1) \leq A(Z) < A(\infty) = 3\pi/32 .$$

The constant a is given by

$$a = - \frac{Rne^2}{T_e} \exp\,(e\Phi/T_e) .$$

Now returning to equations (36) and (34), if m = r is the first value of m for which $U_m^{(1)}$ is non-zero, the first three equations for the U's are

$$p \frac{\partial}{\partial s} \left(\frac{U_r^{(1)}}{B_o}\right) + \frac{2(T_i + T_e)}{eB_o^2 R} M_{r+1+3\alpha}\, n(\psi) = 0 \tag{38}$$

$$p \frac{\partial}{\partial s} \left(\frac{U_{r+1}^{(1)}}{B_o}\right) + L_1 \left(\frac{U_r^{(1)}}{B_o}\right) + \frac{2(T_i + T_e)}{eB_o^2 R} M_{r+2+3\alpha}\, n(\psi) = 0 \tag{39}$$

$$p \frac{\partial}{\partial s} \left(\frac{U_{r+2}^{(1)}}{B_o}\right) + L_1 \left(\frac{U_{r+1}^{(1)}}{B_o}\right) + L_2 \left(\frac{U_r^{(1)}}{B_o}\right) + \frac{2(T_i + T_e)}{eB_o^2 R} M_{r+3+3\alpha}\, n(\psi) = 0 . \tag{40}$$

In equation (38), $r + 1 + 3\alpha \leq 0$ i.e., $M_{r+1+3\alpha}$ must be either zero or equal to M_o. But since part of $M_o n(\psi)$ is independent of s, $U_r^{(1)}$ can not be a periodic function of s unless $M_{r+1+3\alpha} = 0$. It then follows that

$$U_r^{(1)} = U_r^{(1)}(\theta,\psi) . \tag{41}$$

In equation (39), since $L_1 U_r^{(1)}$ is a periodic function of s the same argument requires that $M_{r+2+3\alpha} = 0$ and therefore

$$U_{r+1}^{(1)} = -\frac{1}{p}\int^{s} L_1 U_r^{(1)}\, ds + \bar{U}_{r+1}^{(1)}(\theta,\psi). \tag{42}$$

Finally in equation (40), $M_{r+3+3\alpha} = M_0$ is not inconsistent with the requirement that $U_{r+2}^{(1)}$ be a periodic function of s, and on integrating this equation over a period in s, and using the expressions (23) for L_1 and L_2 we obtain

$$\iota \frac{\partial U_r^{(1)}}{\partial \theta} + \frac{2(T_i + T_e)}{eB_0} \sin\theta \frac{\partial n}{\partial r} = 0 \tag{43}$$

with ι given by equation (7). We have expressed the derivative with respect to ψ in M_0, equation (24), in terms of r to lowest order in $\wedge$. Now turning to equation (36) we find that r = 0, or r = 1 would violate the periodicity condition for $N_m^{(0)}(\psi,\theta,s)$, so that r = 2 (i.e. the index α of expansion (28) is $\alpha = -5/3$) and we obtain

$$\frac{\partial N_0^{(0)}}{\partial s} = 0 \tag{44}$$

$$p\frac{\partial N_1^{(0)}}{\partial s} + L_1 N_0^{(0)} = 0 \tag{45}$$

$$p\frac{\partial N_2^{(0)}}{\partial s} + L_1 N_1^{(0)} + L_2 N_0^{(0)} = a\, \eta_{\parallel}\, U_2^{(1)} . \tag{46}$$

As for $U_r^{(1)}(\theta,\psi)$, we obtain an equation for $N_0^{(0)}(\theta,\psi)$ by integrating (46) over a period in s, and substituting from (44) and (45) the solutions for $N_1^{(0)}$ and $N_0^{(0)}$. The resulting equation is

$$\iota \frac{\partial N_0^{(0)}}{\partial \theta} = a\, \eta_{\parallel}\, U_2^{(1)} . \tag{47}$$

Combining equations (47) and (43) we obtain

$$\iota^2 \frac{\partial^2 N_0^{(0)}}{\partial \theta^2} = -2\, a\, \eta_{\parallel} \frac{(T_i + T_e)}{eB_0} \frac{\partial n}{\partial r} \sin\theta . \tag{48}$$

This equation completes the determination of $f_0^{(0)}$ the dominant term of the expansion (28) and substituting this solution in equation (20) for Γ_a we have for the flux Γ per unit area (i.e. $\Gamma = \Gamma_a / \oint J d\theta ds$)

$$\Gamma = -2n\, \eta_{\parallel} \frac{(T_i + T_e)}{\iota^2 B_0^2} \frac{\partial n}{\partial r} . \tag{49}$$

This is identical to Pfirsch-Schlüter diffusion as in axisymmetric torii [1] with an effective rotational transform ι arising from the helical windings.

Using similar techniques we can investigate $N_1^{(0)}$ with the result that there is no contribution to the diffusion in next order in $\lambda^{1/3}$. In fact the first 'helical' corrections to the Pfirsch-Schlüter result above occur as $\lambda^{2/3}$ corrections.

(b) $\omega_t < \nu < \Omega_t$

This region can be treated by making the $\nu < \Omega_t$ subsidiary expansion of equation (27). The solution follows similar lines to that in subsection (a) above, except that we must impose the constraint due to periodicity in s before we can deduce that the lowest order significant contribution to f is Maxwellian. The solution is found to be the same as in region (a), thus leading to

the Pfrisch-Schlüter result again, the effect of helical fields being solely to produce the rotational transform. To confirm the validity of these results over the whole region $\omega_t < \nu < \Omega_t$ we have also derived them in subsection (c) by making the subsidiary expansion $\omega_t / \nu \ll 1$ of equations valid for $\nu \sim \omega_t$. In addition the assumption that fast particles $(q \sim v_{th})$ dominate the diffusion in the present range will be substantiated below.

(c) $\nu < \omega_t \; : \; \nu_{eff} > \Omega_b$

In this region, where trapped and slow passing particles suffer significant collisions before executing a helical period, (and a connection length, of course) but fast particles traverse a connection length without scattering, one finds the dominant contribution to the diffusion arises from particles intermediate in longitudinal velocity between the trapped and fast particles, i.e. those particles whose effective collision frequency is comparable with their transit frequency through a connection length. To investigate this regime we consider $\nu \sim \omega_t$. With this scaling of ν the kinetic equation becomes

$$\frac{\sigma q}{R}\left[p\frac{\partial}{\partial s} + \sum_{m=1}^{\infty} \lambda^{m/3} L_m\right] f + \frac{(2\kappa - \mu B_0)}{\omega_c \; R^2} \sum_{m=0}^{\infty} \lambda^{\frac{m-1}{3}} M_m F = \lambda^{2/3} \, C(f,F) \qquad (50)$$

where $C(f,F)$ is now $O(1)$ in the λ expansion. Taking an expansion of the form

$$f = \lambda^{\alpha} \sum_{n=0}^{\infty} \lambda^{n/3} f_n \qquad (51)$$

we find that to satisfy the periodicity conditions in s, α must have the value -1 so that in lowest order in λ

$$\frac{\partial f_0}{\partial s} = 0 \; . \qquad (52)$$

In first order

$$p \frac{\partial f_1}{\partial s} + L_1 f_0 = 0 \qquad (53)$$

with solution

$$f_1 = -\frac{1}{p} \int^{s} L_1 f_0 ds + \bar{f}_1(\psi,\theta) \, . \qquad (54)$$

In second order the $\underline{v}_d . \nabla F$ term appears without violating periodicity in s.

$$\frac{\sigma q}{R} \left(p \frac{\partial f_2}{\partial s} + L_1 f_1 + L_2 f_0 \right) + \frac{2\kappa - \mu B_o}{\omega_c R^2} M_0 F_0 = C(f_0, F_0) \, . \qquad (55)$$

Since we are considering particles which are too 'fast' to be trapped, $q > \lambda^{\frac{1}{2}} v_{th}$ and $1/q \; \partial q/\partial s < \lambda^{\frac{1}{2}}$ so that we may assume q to be independent of s in this equation, and by introducing the solution (54) for f_1 and integrating over a period in s we obtain (cf equation (43)), after expressing ψ in terms of r to lowest order in λ

$$+ \frac{\sigma q}{R} \frac{\partial f_o}{\partial \theta} + \frac{(2\kappa - \mu B_o)}{\omega_c R^2} \sin\theta \frac{\partial F_o}{\partial \rho} = C(f_o, F_o) \, . \qquad (56)$$

This is identical to the equation obtained for the axisymmetric case by Rutherford [4] and hence the diffusion is identical to the equivalent axisymmetric case. However we also note that by employing a subsidiary expansion of f_o when $\nu > \omega_t$

$$f_0 = f_0^{(0)} + \frac{\omega_t}{\nu} f_0^{(1)} + \ldots \qquad (57)$$

we readily show that $f_0^{(0)}$ is Maxwellian, and by taking $\int d^3 v$ and $\int \sigma q \, d^3 v$ moments of equation (56), obtain equations (34) and (36).

Hence the results agree with those obtained by considering $\nu \sim \Omega_t$ in the λ expansion and expanding in the subsidiary small parameter ν/Ω_t, confirming their validity throughout the range (b) $\omega_t < \nu < \Omega_t$.

Returning to the consideration of $\nu < \omega_t$, the solution of equation (56) has been given by Rutherford[4] in the limit $q/v_{th} < 1$ and with only electron-ion collisions taken into account through the Lorentz operator

$$C_{ei}(f) = \nu_{ei}(\kappa)\, q \frac{\partial}{\partial \mu} \frac{q\mu}{B} \frac{\partial}{\partial \mu} f, \quad \nu_{ei} = \frac{\sqrt{2\pi}\; ne^4 \ell n\, \Lambda}{m_e^2\, \kappa^{3/2}} . \qquad (58)$$

The solution for $f = \lambda^{-1} f_0$ is

$$f = 2\, \frac{F_0}{\omega_{c_0}} \frac{1}{n} \frac{\partial n}{\partial r} \frac{\kappa}{t} \int_0^\infty \sin\,(qt - \theta)\, e^{-\frac{u^3 t^3}{3}}\, dt \qquad (59)$$

where $u^3 = \dfrac{R\, \nu_{ei}\, \kappa}{t}$; thus since the reduction of equation (55) to (56) requires $q > \lambda^{\frac{1}{2}} v_{th}$, the solution (58) for f_0 is valid in the region of q space defined by $\lambda^{\frac{1}{2}} v_{th} < q < v_{th}$. However general arguments show that the dominant contribution to the diffusion flux comes from particles with

$$\frac{q}{R} \sim \frac{\nu\, v_{th}^2}{q^2} \qquad \text{or} \qquad q \sim \left(\frac{\nu R}{v_{th}}\right)^{1/3} v_{th}$$

i.e. those whose transit frequency through a connection length is comparable to their effective collision frequency for small angle scatters out of this band. Thus for $\nu_{eff} > \Omega_b$ and $\nu < \omega_t$ this band of particles lies in the range $\lambda^{\frac{1}{2}} < q < v_{th}$ and is correctly described by the distribution function (59). The corresponding diffusion flux per unit area is

$$\Gamma = \frac{\sqrt{(2\pi)}}{2\omega_c^2} \left(\frac{T_e}{m_e}\right)^{3/2} \frac{1}{tR} \frac{\partial n}{\partial r} \left(1 + \frac{T_i}{T_e}\right) \qquad (60)$$

as first given by Galeev and Sagdeev[2]. The result is independent of ν, (the plateau result) implying that the neglect of electron-electron collisions is unimportant.

For collision frequencies in the range $\omega_t < \nu < \Omega_t$ considered in subsection (b) a similar analysis can be applied to those particles suffering a collision during a helical transit. The conclusion is that they contribute only a small correction to the Pfirsch-Schlüter diffusion arising from fast particles.

(d) $\omega_b < \nu_{eff} < \Omega_b$

In this region collisions are so infrequent that localised particles can perform many bounces before being scattered out of their helical well, and thus begin to drift round their superbanana drift surfaces between collisions, yet blocked particles cannot complete a bounce before they are scattered. In this regime the treatment of resonant particles given in subsection (c) is still relevant since the critical frequency $\nu_{eff} = \Omega_t$ has no significance for them but we must also consider localised particles with $q \sim \lambda^{\frac{1}{2}} v_{th}$ and their effect on diffusion; this we do by ordering $\nu_{eff} \sim \Omega_b$ in λ . An expansion in Ω_b/ν_{eff} to investigate the region $\Omega_b < \nu_{eff}$ confirms that these particles produce only a small correction to the plateau result in that regime. The treatment of the region $\Omega_b > \nu_{eff}$ produces a contribution to the diffusion from localised particles scaling like ν^{-1} but which is still less than the plateau result except in the vicinity of $\nu_{eff} \sim \omega_b$. (This point will be discussed later, in subsection (f)). The analysis is equivalent to the treatment of localised particles given in subsection (e) below and thus we postpone the details until then. Thus the 'plateau' of the diffusion curve persists in the present range of ν, although a small contribution to the flux, scaling as ν^{-1} and attributable to trapping in the helical field, is also present.

(e) $\nu_{eff} < \omega_b$

For effective collision frequencies below ω_b blocked and localised particles i.e. those having velocities $q \sim \lambda^{\frac{1}{2}} v_{th}$ perform many of their respective bounces before scattering. Note that the contribution to diffusion from the resonant particles would peak at $q \leqslant \lambda^{\frac{1}{2}} v_{th}$ in the range $\nu_{eff} < \omega_b$ according to expression (59). Thus we expect the dominant contribution to arise from the slow particles with $q \sim \lambda^{\frac{1}{2}} v_{th}$ for which $\nu_{eff} \sim \nu/\lambda$. We treat these particles by ordering $\nu_{eff} \sim \omega_b$ in λ and introducing an expansion of the form (51) in the kinetic equation we obtain the hierarchy

$$p \frac{\partial f_0}{\partial s} = 0 \tag{61}$$

$$p \frac{\partial f_1}{\partial s} + L_1 f_0 = 0 \tag{62}$$

with solution

$$f_1 = - \frac{1}{p} \int^{s} L_1 f_0 ds + \bar{f}_1(\psi,\theta) \qquad (63)$$

$$\frac{\sigma q}{R}\left(p \frac{\partial f_2}{\partial s} + L_1 f_1 + L_2 f_0 \right) + \frac{2\kappa - \mu B_0}{\omega_c R^2} M_0 F_0 = C(f_0) \qquad (64)$$

where the $\underline{v}_d \cdot \nabla F$ term can first be introduced without violating periodicity requirements in θ and s as will be evident later; this implies $\alpha = -1$ in the expansion (51). Let us substitute the solution for f_1 into equation (64) and apply the annihilator $\oint ds/q$ over a period in s - this applies to blocked or passing particles only since localised particles do not sample a whole period.

We obtain, expressing ψ in terms of ρ to lowest order in λ,

$$\frac{\pm\sigma}{R} \frac{\partial f_0}{\partial \theta} + \frac{1}{\omega_c R^2 \rho} \frac{\partial J_P}{\partial \theta} \frac{\partial F}{\partial \rho} = \frac{1}{2\pi} \oint \frac{ds}{q} C(f_0) \qquad (65)$$

where $J_p = \frac{1}{2\pi} \oint ds\, q$ and we have utilised $\mu B \sim \kappa$ since $q \sim \lambda^{\frac{1}{2}} v_{th}$. For the localised particles we apply the annihilator $\int_{s_1}^{s_2} ds/q$, where $q(s_1) = q(s_2) = 0$, to obtain

$$\frac{1}{\omega_c R^2 \rho} \frac{\partial J_L}{\partial \theta} \frac{\partial F}{\partial \rho} = \frac{1}{2\pi} \int_{s_1}^{s_2} \frac{ds}{q} C(f_0) \qquad (66)$$

where $J_L = \frac{1}{2\pi} \int_{s_1}^{s_2} ds\, q$.

Now equation (65) describes the passing and blocked particles and can be solved to determine the stellarator analogue of banana diffusion. It is in general dominated by the diffusion from the localised particles satisfying equation (66). We shall however calculate their contribution for the Lorentz model collision operator (58), for comparison purposes. (We note in passing that in the regime $\nu_{eff}/\omega_b > 1$ particles with $1/q\ \partial q/\partial s \ll 1$, $J_p \sim q$ so that we recover the axisymmetric equation (56) and hence the plateau results.) In the region $\nu_{eff} < \omega_b$ we expand in the small parameter ν_{eff}/ω_b obtaining

$$\sigma \pm \frac{\partial f_0^{(0)}}{\partial \theta} + \frac{1}{\omega_c R\rho} \frac{\partial J_P}{\partial \theta} \frac{\partial F}{\partial \rho} \cdot \qquad (67)$$

Thus

$$f_0^{(0)} = - \frac{\sigma}{\omega_c + R\rho} \frac{J_p}{R\rho} \frac{\partial F}{\partial \rho} + g\ (\kappa, \mu, r) \cdot \qquad (68)$$

In next order

$$\sigma + \frac{\partial f_0^{(1)}}{\partial \theta} = R\ \nu_{ei}(\kappa) \frac{\partial}{\partial \mu} \mu \frac{J_p}{B_o} \frac{\partial f_0^{(0)}}{\partial \mu} . \qquad (69)$$

For passing particles we have the solubility condition

$$\oint d\theta \frac{\partial}{\partial \mu} \mu\ J_p \left[\frac{\partial}{\partial \mu} g_p - \frac{\sigma}{\omega_c + R\rho} \frac{\partial F}{\partial \rho} \frac{\partial J_p}{\partial \mu} \right] = 0 \qquad (70)$$

while for blocked particles

$$\int_{\theta_1}^{\theta_2} d\theta \frac{\partial}{\partial \mu} \mu\ J_p \frac{\partial g_b}{\partial \mu} = 0 \qquad (71)$$

where $J_p(\theta_2) = J_p(\theta_1) = 0$.

These equations closely resemble the axisymmetric case with J replacing q. Thus the diffusion may be evaluated in like manner to yield

$$\Gamma_{Block} = \frac{1}{+^2\ \omega_c^2} \frac{1}{n} \frac{\partial n}{\partial r} \left(\frac{R}{r}\right)^2 \oint \frac{d\theta}{2\pi} \sum_{\sigma} \int 2\pi\ Bd\mu\ d\kappa\ F$$

$$\nu_{ei}\ (\kappa)\ J_p \frac{\partial}{\partial \mu} \frac{\mu}{B} J_p \left\{ \frac{H \frac{\partial}{\partial \mu} \left\langle \frac{J_p^2}{2} \right\rangle}{\langle J_p \rangle} - \frac{\partial J_p}{\partial \mu} \right\} \qquad (72)$$

where H = 0 blocked particles

= 1 passing particles

and $\langle A \rangle = \oint d\theta / 2\pi\ A$.

This expression can be simplified in the limit where the helical modulation is much less than the toroidal variation. In this limit we find

$$J \simeq \bar{q}$$

where

$$\frac{\bar{q}^2}{2} = \kappa - \mu B_o \left(1 - \frac{r}{R} \cos\theta\right)$$

that is $\bar{q}$ is the parallel velocity in an axisymmetric term of aspect ratio r/R. This expression can be evaluated to yield the axisymmetric result

$$\Gamma_{\text{Block}} = 0.73 \left(\frac{R}{r}\right)^{3/2} \frac{\nu^*_{ei}}{\iota^2} \rho_e^2 \left(1 + \frac{T_i}{T_e}\right) \frac{dn}{dr} \qquad (73)$$

where

$$\nu^*_{ei} = \frac{4\sqrt{2\pi}\ ne^4 \ell n\ \Lambda}{3\ m_e^{\frac{1}{2}}\ T_e^{3/2}}\ ;\ \rho_e^2 = \frac{2\ m_e\ T_e}{e^2\ B^2}\ . \qquad (74)$$

Note that inclusion of electron-electron collisions[3] modifies the coefficient 0.73 to 1.12.

Now we return to the contribution from localised particles. This term normally dominates the diffusion and we will therefore calculate it with some precision. Since these results no longer resemble the axisymmetric results it is valuable to include temperature gradients and calculate the ion heat loss. We shall need to solve the ion kinetic equation and explicitly calculate the ambipolar radial electric field since in the presence of temperature gradients it does not exactly balance the ion pressure gradient. The solution of the ion equation can then be used to calculate the ion heat flux. In order to obtain a tractable equation we must simplify the full Fokker-Planck operator occurring in equation (66). For electron-ion collisions we use the Lorentz approximation (58) and for like particle collisions we use the form

$$C_{jj}(f_j) = \nu_{jj}(\kappa)\ \frac{q}{B} \frac{\partial}{\partial\mu}\ q\mu\ \frac{\partial}{\partial\mu}\ f_j + \nu_{jj}(\kappa)\ q\ F_j\ \frac{P_j m_j}{T_j} \qquad (75)$$

with

$$P_j = \frac{\int d^3v\ \nu_{jj}(\kappa)\ q\ f_j}{\frac{m_j n_j}{T_j} \int d^3v\ \nu_{jj}(\kappa)\ q^2\ F_j}\ .$$

The term containing P ensures momentum conservation but may be ignored here, for with a localised distribution its contribution is small - in fact it may be included in the analysis, but does not alter the result. (This term is essential in the axisymmetric case where momentum conservation plays an important role. In that case results obtained using this collision operator are identical with those obtained from a more accurate variational treatment[3].) We remark that Reference 6 employs this form of collision operator. Finally we must define the form of the collision frequencies $\nu_{jj}(\kappa)$:

$$\nu_{jj}(\kappa) = \frac{\sqrt{2}\pi \; ne^4 \ell n \; \Lambda}{m_j^{\frac{1}{2}} \; T_j^{3/2}} \; A_{jj}(x_j) \tag{76}$$

where $x_j = m_j\kappa/T_j$ and

$$A_{jj}(x_j) = (\eta_j + \eta_j' - \frac{\eta_j}{2x_j}) \; x_j^{-3/2}$$

with

$$\eta_j(x_j) = \frac{2}{\sqrt{\pi}} \int_0^{x_j} e^{-t} \, t^{\frac{1}{2}} \, dt, \quad \eta_j' = \frac{d\eta}{dx_j} \; .$$

Both ions and electrons satisfy an equation of the form (66) and we reintroduce a label $j = i, e$. Since $\kappa \sim \mu B$ for localised particles we may write equation (66) as

$$- \frac{\sin \theta}{R^2 \omega_{cj}} \; \frac{\kappa}{B} \; \frac{\partial J_L}{\partial \mu} \; \frac{\partial F_j}{\partial \rho} = \sum_k \nu_{jk}(\kappa) \, \frac{\partial}{\partial \mu} \frac{\mu}{B} \, J_L \, \frac{\partial}{\partial \mu} \, f_{0j} \; . \tag{77}$$

Thus the solution for $f_j \simeq \lambda^{-1} \, f_{0j}$ is

$$f_j = \frac{\sin \theta}{\omega_{cj} R} \; \frac{\partial F_j}{\partial r} \; \frac{B}{\sum_k \nu_{jk}(\kappa)} \; \left(\mu - \frac{\kappa}{B_{max}} \right) \tag{78}$$

where $B_{max} = B_o \; [1 + \lambda(\rho \cos \theta - \alpha\rho^3)]$, since this solution vanishes at the boundary of the localised region in velocity space and is regular within. Substituting these solutions in the expression (20) for the particle fluxes, performing the μ integration in a straight forward manner and the κ integrations numerically[15] we obtain

$$\Gamma_i = - \frac{64}{9} \; \frac{\varepsilon_h^{3/2}}{(2\pi)^{3/2}} \left(\frac{T_i}{e \, BR} \right)^2 \frac{27.42}{\nu_{ii}} \left[\frac{n'}{n} + \frac{e\Phi'}{T_i} + 3.37 \, \frac{T_i'}{T_i} \right] \tag{79}$$

and

$$\Gamma_e = - \frac{64}{9} \; \frac{\varepsilon_h^{3/2}}{(2\pi)^{3/2}} \left(\frac{T_e}{eBR} \right)^2 \frac{12.78}{\nu_{ei}} \left[\frac{n'}{n} - \frac{e\Phi'}{T_e} + 3.45 \, \frac{T_e'}{T_e} \right] \tag{80}$$

where ν_{ei} is given by equation (58) with $\kappa = T_e/m_e$ and ν_{ii} is a similar expression but with ion parameters replacing the electron ones. The quantity $\varepsilon_h = \lambda\alpha\rho^3$ measuring the helical modulation of the magnetic field is introduced to compare our results with conventional notation[6]. The ion rate is reduced to the electron rate by the ambipolar electric field

$$\Phi' = -\frac{T_i}{e}\left(\frac{n'}{n} + 3.37\,\frac{T_i'}{T_i}\right) \tag{81}$$

leading to an ambipolar diffusion rate per unit area given by

$$\Gamma = -4.34\,\frac{\varepsilon_h^{3/2}}{\nu_{ei}^*}\left(\frac{T_e}{eBR}\right)^2\left[\frac{n'}{n}\left(1+\frac{T_i}{T_e}\right) + 3.37\,\frac{T_e'}{T_e} + 3.45\,\frac{T_i'}{T_e}\right]. \tag{82}$$

A similar calculation for the ion heat flux per unit area, defined by

$$Q_i = \frac{\iint J d\theta ds \sum_\sigma \int \frac{B}{q}\, d\mu d\varepsilon d\varphi m_i \kappa f_i\, \underline{v}_d \cdot \nabla\psi}{\iint J d\theta ds} \tag{83}$$

yields

$$Q_i = -46.5\,\frac{\varepsilon_h^{3/2}}{\nu_{ii}^*}\left(\frac{T_i}{eBR}\right)^2 T_i' . \tag{84}$$

In these expressions

$$\nu_{jk}^* = \frac{4}{3}\,\frac{\sqrt{2\pi}\, ne^4 \ell n\,\Lambda}{m_j^{\frac{1}{2}}\, T_j^{3/2}} . \tag{85}$$

These results extend the work of Reference 6 and are not subject to the restriction there that $\varepsilon_t \ll \varepsilon_h$ where $\varepsilon_t = r/R$ is the toroidal modulation of the magnetic field strength.

(f) The Vicinity of $\nu_{eff} \sim \omega_b$

The contributions to the diffusion from the plateau, expression (60), the blocked particles expression (73) and the localised particles expression (82) are all comparable when $\nu_{eff} \sim \omega_b$ i.e. $\nu \sim v_{th}/\ell R\,(r/R)^{3/2}$. The exact shape of the diffusion curve in the vicinity of $\nu_{eff} \sim \omega_b$ depends on $\varepsilon_h/\varepsilon_t$. The cases $\varepsilon_h < \varepsilon_t$ and $\varepsilon_t < \varepsilon_h$ are displayed on Fig.4. These are obtained by taking the maximum of the contributions to the diffusion from blocked and localised particles when $\nu_{eff} < \omega_b$ and the maximum of the contributions from plateau and localised particles when $\nu_{eff} > \omega_b$. Thus when $\varepsilon_h > \varepsilon_t$ there is a transition from plateau to localised behaviour as ν decreases through a value $\nu \sim (\varepsilon_h/\varepsilon_t)^{3/2} \omega_b$ and when $\varepsilon_h < \varepsilon_t$ the transition from blocked to localised behaviour occurs as ν decreases through a value $\nu \sim (\varepsilon_h/\varepsilon_t)^{3/4} \omega_b$.

V. CONCLUSIONS

In this paper expressions for the diffusion rate in an $\ell = 3$, large aspect ratio, stellarator have been calculated for collision frequencies comparable with the transit and bounce frequencies in the helical and toroidal modulations. Thus we exclude consideration of the 'super-banana' effects occuring for collision frequencies comparable with the drift frequency ω_D, but our ordering does describe the $\nu \gg \omega_D$ limit of this behaviour.

Our results show that in leading order such a stellarator may be regarded as an equivalent axisymmetric torus with effective rotational transform given by equation (7) over much of the collision frequency range considered. This behaviour breaks down when $\nu_{eff} \lesssim \omega_b$ i.e. $\nu_{ei} \lesssim v_{th} \iota\!\!\!- \varepsilon_t^{3/2}/R$ when localised particles begin to dominate the diffusion. (There is some 'fine structure' in the vicinity of $\nu_{eff} \sim \omega_b$ which slightly modifies this statement and we will return to this point later). Reference to figure 3, where the diffusion rate is plotted as a function of collision frequency will aid the qualitative discussion of our results.

At high collision frequencies, where $\nu > v_{th}P/R = \Omega_t$ so that fast particles suffer many collisions before completing a helical period, we find the diffusion rate is given by the Pfirsch-Schlüter expression (49) with a rotational transform $\iota\!\!\!- = - 18\, \varepsilon_h^2/P^3 \varepsilon_t^4$. Corrections to this are at least of order $\lambda^{2/3}$. This behaviour persists into the region $\omega_t = v_{th} \iota\!\!\!-/R < \nu < v_{th}P/R = \Omega_t$ where fast particles suffer a collision within a connection length, but not within a helical period.

Over the whole range defined by $\nu < v_{th} \iota\!\!\!-/R = \omega_t$ and $\nu_{eff} = \nu/\varepsilon_t > \varepsilon_t^{1/2} v_{th} \iota\!\!\!-/R = \omega_b$ we find the dominant contribution to the diffusion arises from the 'resonant' particles, those which suffer one collision per connection length, giving the 'plateau' behaviour of the equivalent axisymmetric torus as calculated by Galeev and Sagdeev [2] (expression (60)). This is despite the fact that for $\nu_{eff} = \nu/\varepsilon_h < v_{th}/R\ \varepsilon_h^{1/2}\ P = \Omega_b$ the slow particles execute a helical bounce without a collision and are thus localised. These particles do indeed contribute to the diffusion but only as a ω_b/ν_{eff} correction to the plateau result. Their contribution scales as ν^{-1} and when $\nu_{eff} = \nu/\varepsilon_t < \varepsilon_t^{1/2}\ v_{th}\ \iota/R = \omega_b$ this dominates the axisymmetric-like contribution from the passing and blocked particles. Thus in this region the stellarator departs from the equivalent axi-symmetric torus and exhibits localised particle effects as superbanana-like behaviour is approached. This ν^{-1} behaviour given by expression (82) matches with the results of previous calculations[6] with $\nu \sim \omega_D$ (where superbananas exist) when one considers the limit $\nu \gg \omega_D$. We have quoted results for the diffusion flux including temperature gradients, and have also evaluated the ion heat flux, using a reliable model collision operator. These results extend previous work in this regime[6]. In the neighbourhood of $\nu_{eff} \sim \omega_b$ the plateau behaviour arising from the resonant particles and the ν^{-1} scaling arising from the localised particle diffusion and blocked particle diffusion, analogous to banana diffusion are competing effects. This is the motivation for our ordering with $\varepsilon_t \sim \varepsilon_h$. The details of the variation of the diffusion with collision frequency in the immediate vicinity of this point depend on the ratio $\varepsilon_t/\varepsilon_h$, the ratio of toroidal and helical modulations; the results are summarized in Fig.4.

The analysis presented here has been specifically for an $\ell = 3$ stellarator but the results are more general so that one need only use the appropriate expression for ι to describe the effect of different ℓ-windings.

In our analysis the helical effects discussed by Stringer[8] can be obtained as higher order corrections (we do not present the analysis for reasons of brevity). These are small compared to the toroidal effects and helical corrections of a geometrical character (i.e. pertaining to the shape of surfaces etc., rather than to the interaction of particle motion with the helical modulations) dominate them anyway.

Finally we comment on the relation of the present work to that of Kovrizhnykh[5,9]. The essential difference is his use of a relaxation time collision operator since he considers the case of a weakly ionised gas, so that the concept of ν_{eff} for small angle collisions does not enter his work. In addition his averaging scheme for dealing with the stellarator fields does not cover all the cases described here. Thus, for example, his final kinetic equation cannot describe the range $\nu > \Omega_b$ since collisions exceed the helical transit frequency.

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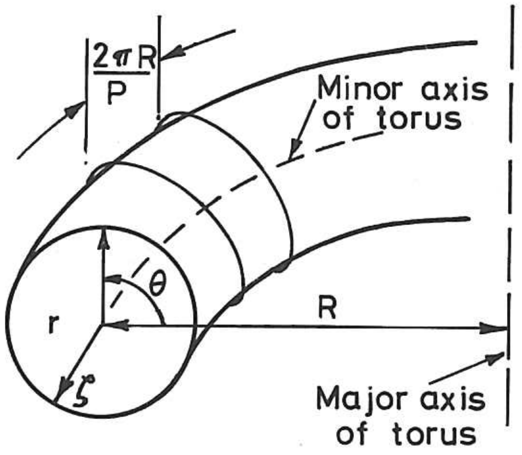

Fig.1 The co-ordinate system.

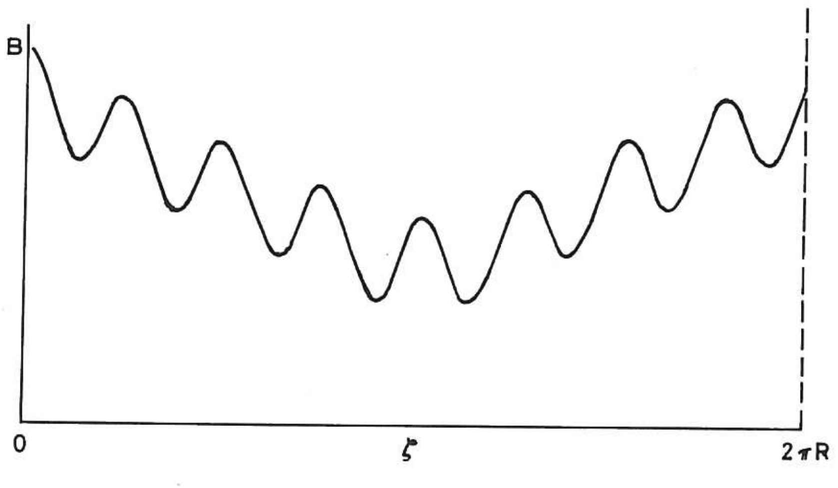

Fig.2 The variation in B, the magnitude of the magnetic field, along a field line. CLM-P 341

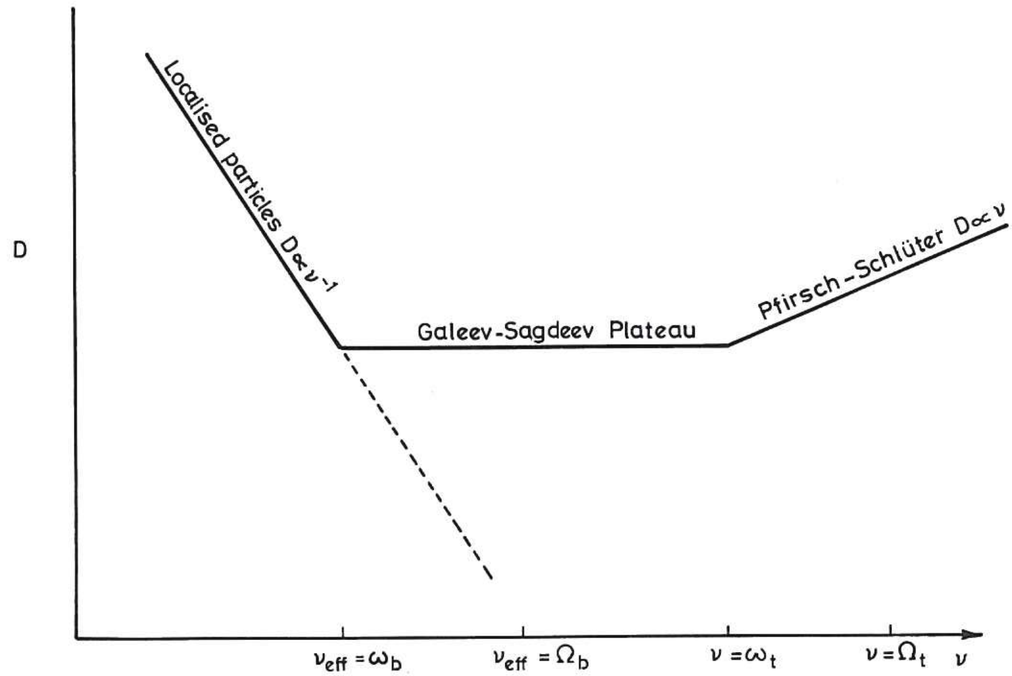

Fig.3 The variation of the diffusion coefficient D with collision frequency ν.

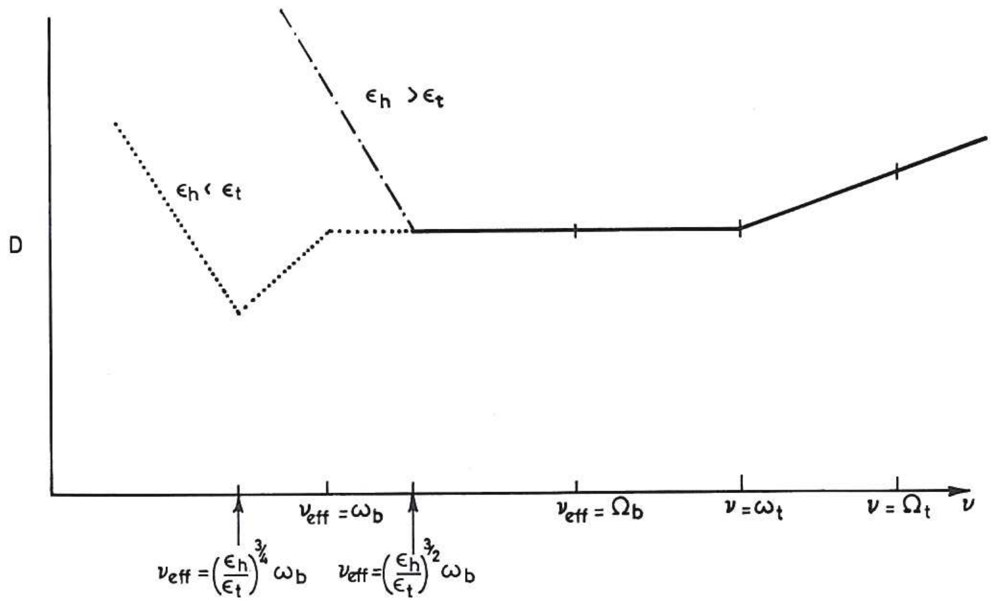

Fig.4 The variation of the diffusion coefficient D with collision frequency ν for the cases $\epsilon_h > \epsilon_t$ and $\epsilon_h < \epsilon_t$.