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THE COUPLED MODE APPROACH TO NON-LINEAR WAVE INTERACTIONS AND PARAMETRIC INSTABILITIES

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ABSTRACT

The non-linear interaction of a few coherent, weakly damped longitudinal and transverse waves in an unmagnetized plasma is considered by means of the coupled mode theory. Using the equations describing the non-linear wave interactions, the threshold electric fields and initial growth rates of a number of parametric instabilities are calculated. The following four instabilities, all of which are relevant to laser fusion are considered: (i) the parametric ion-acoustic Langmuir wave interaction, (ii) the two plasmon decay, (iii) stimulated Raman scattering and (iv) stimulated Brillouin scattering. All these instabilities are driven by a large amplitude electro-magnetic wave (the pump). The first example includes the oscillating two stream instability which is shown to give rise to purely growing ion waves only in the special case of a standing wave pump.

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1. INTRODUCTION

When an intense electromagnetic field is excited in a plasma it can cause the growth of other plasma waves out of the background thermal noise. These effects, which only occur when the exciting field exceeds some threshold, are called parametric instabilities. The plasma waves grow at the expense of the driving electromagnetic field which is usually referred to as the "pump".

Parametric instabilities can result in a greatly enhanced absorption rate of the electromagnetic energy giving rise to plasma heating. They are therefore relevant both to laser fusion and to fusion by magnetic confinement.

There has already been a great deal of work on these effects (Silin, 1965; Dubois and Goldman 1965, 1967; Jackson 1967; Nishikawa 1968 a.b; Kaw and Dawson 1969; Sanmartin 1970). Most of these treatments work to all orders in the pump amplitude but neglect any spatial variation of the pump. In this paper we shall give a much simpler treatment, which although restricted to smaller pump amplitudes, gives the threshold fields and instability growth rates of the more complicated theories. In addition, this simpler approach makes the physical mechanisms clearer and may help in future work on the non-linear saturation of these instabilities.

The treatment we give uses coupled mode theory (see e.g. W.H. Louisell 1960) which treats the pump and excited waves on the same footing. Effects due to frequency mis-match, amplitude (pump) dependent frequency shift and spatial variation of the pump are naturally included.

In this paper we shall treat only an unmagnetized plasma. The effects we shall describe are (i) the parametric Langmuir-ion acoustic and oscillating two stream instabilities, (ii) the interaction of a transverse electromagnetic wave with two Langmuir waves, (iii) stimulated Raman scattering (the decay of a transverse wave into another transverse wave and a Langmuir wave), and (iv) stimulated Brillouin scattering (the decay of a transverse wave into another transverse wave and an ion-acoustic wave). All these are examples of parametric interactions and can be described very simply by the coupled mode equations which we shall derive below.

2. THE NORMAL MODES

We shall consider an unmagnetized uniform plasma of infinite extent. Before we can describe the various parametric interactions already mentioned we must obtain the normal modes which for the present model are the transverse electromagnetic waves, the Langmuir waves (electron plasma waves) and the ion acoustic waves.

We describe the plasma by means of the two fluid model. This suffices to describe many of the basic effects although it does leave some important ones out (we shall return to this point later).

In order to obtain the normal modes we shall need the following equations:

$$\frac{d\underset{\sim}{v}_e}{dt} + \frac{\gamma_e \kappa T_e}{n_e m_e} \underset{\sim}{\nabla} n_e + \nu_e \underset{\sim}{v}_e = -\frac{e}{m_e}(\underset{\sim}{E} + \underset{\sim}{v}_e \times \underset{\sim}{B}) \tag{1}$$

$$\frac{\partial n_e}{\partial t} + \underset{\sim}{\nabla}\cdot(n_e \underset{\sim}{v}_e) = 0 \tag{2}$$

$$\frac{d\underset{\sim}{v}_i}{dt} + \frac{\gamma_i \kappa T_i}{n_i m_i} \underset{\sim}{\nabla} n_i + \nu_i \underset{\sim}{v}_i = \frac{e}{m_i}(\underset{\sim}{E} + \underset{\sim}{v}_i \times \underset{\sim}{B}) \tag{3}$$

$$\frac{\partial n_i}{\partial t} + \underset{\sim}{\nabla}\cdot(n_i \underset{\sim}{v}_i) = 0 \tag{4}$$

$$\underset{\sim}{\nabla} \times \frac{\underset{\sim}{B}}{\mu_o} = \underset{\sim}{J} + \epsilon_o \frac{\partial \underset{\sim}{E}}{\partial t} \tag{5}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t} \tag{6}$$

where

$$\underset{\sim}{J} = e(n_i \underset{\sim}{v}_i - n_e \underset{\sim}{v}_e) .$$

Equations (1) and (3) are the momentum equations for the electron and ion fluids and (2) and (4) the corresponding continuity equations. These equations could represent a partially ionized plasma in which case ν_e and ν_i would be the electron-neutral and ion-neutral collision rates for loss of momentum. They could also represent a fully ionized plasma in which case ν_e would be the electron-ion collision rate. For this case the ion wave damping is not well described since the main damping mechanism for the ion waves would be ion viscosity and ion Landau damping. Nevertheless, the ν_i term will simulate the qualitative effects of damping for this case.

In the remainder of this section we will obtain the equations for the normal modes. The reasons for choosing this method are twofold. Firstly, we shall obtain <u>first order</u> differential equations instead of second order equations. This helps to make clear which modes are responsible for the various interactions to be considered. Secondly, when the modes are normalized so that they represent the total energy in a mode, the equations describing the non-linear interaction between the modes will satisfy conservation relations for energy, momentum and action. This will ensure that these non-linear equations take their simplest form.

2 (i) The Transverse Normal Modes

We take the linearized forms of equations (1) - (6) to obtain the various normal modes. We choose a plane polarized transverse wave with the following polarization

$$\underset{\sim}{E}^T = (0,\ E_y^T,\ 0)$$

$$\underset{\sim}{B}^T = (0,\ 0,\ B_z^T)$$

$$\underset{\sim}{k} = (k_T,\ 0,\ 0)$$

and the transverse frequency is denoted by ω_T. We now take a linear combination of the y-component of equation (1), the y-component of equation (5) and the z-component of equation (6) to obtain

$$\frac{\partial}{\partial t}\left\{ v_y^T + \alpha_T B_z^T + \beta_T \epsilon_o E_y^T \right\} + \nu_e v_y^T + \frac{e}{m_e} E_y^T + i\, k_T \alpha_T E_y^T + i\, k_T \beta_T \frac{B_z^T}{\mu_o} - n_o e \beta_T v_y^T = 0 . \qquad (7)$$

The superscripts T on the fields are written to denote their transverse nature. This will be helpful when we consider the coupling of transverse and longitudinal fields. The normal mode equations are obtained by choosing α_T and β_T such that equation (7) has the required form

$$\frac{\partial a_T^{\pm}}{\partial t} \pm i\omega_T a_T^{\pm} + \frac{\nu_e}{2} \frac{\omega_{pe}^2}{\omega_T^2} a_T^{\pm} = 0 \qquad (8)$$

where we have assumed weak collisional damping ($\nu_e \ll \omega_T$) . The normal mode amplitudes are now given by

$$a_T^{\pm} \equiv v_y^T + \alpha_T^{\pm} B_z^T \mp \beta_T^{\pm} \epsilon_o E_y^T \tag{9}$$

where
$$\alpha_T^{\pm} = -\frac{ik_T}{n_o e \mu_o}\left(1 \mp \frac{i\nu_e}{\omega_T}\right) \tag{10}$$

$$\beta_T^{\pm} = \frac{i\omega_T}{n_o e}\left\{1 \mp \frac{i\nu_e}{\omega_T}\left(1 - \frac{1}{2}\frac{\omega_{pe}^2}{\omega_T^2}\right)\right\} \tag{11}$$

and
$$\omega_T = (\omega_{pe}^2 + c^2 k_T^2)^{\frac{1}{2}} . \tag{12}$$

Note that a_T^+ has the following dependence

$$a_T^+ \sim e^{i(k_T x - \omega_T t)} e^{-\gamma_T t}$$

and
$$a_T^- \sim e^{i(k_T x + \omega_T t)} e^{-\gamma_T t}$$

where
$$\gamma_T = \frac{\nu_e}{2}\frac{\omega_{pe}^2}{\omega_T^2} .$$

We shall always take $\omega_T > 0$ (and also the frequencies of the other normal modes). We allow the wave numbers to have either sign so that a_T^+ and a_T^- are forward and backward propagating transverse waves with respect to the transverse wave vector $\underset{\sim}{k}_T$.

2(ii) The Langmuir Normal Mode

In order to obtain this normal mode we denote its wave vector by $\underset{\sim}{k}_\ell$. We now take the scalar product of equation (1) (linearized) and equation (5) with $\underset{\sim}{k}_\ell$ and combine these equations with equation (2) to obtain the desired normal mode equation:

$$\frac{\partial a_\ell^{\pm}}{\partial t} \pm i\omega_\ell a_\ell^{\pm} + \frac{\nu_e}{2} a_\ell^{\pm} = 0 \tag{13}$$

where
$$a_\ell^{\pm} \equiv n_{e1}^\ell \pm \alpha_\ell^{\pm} \underset{\sim}{k}_\ell \cdot \underset{\sim}{v}_e^\ell + \beta_\ell \epsilon_o \underset{\sim}{k}_\ell \cdot \underset{\sim}{E}^\ell \tag{14}$$

$$\beta_\ell = -\frac{i n_o e}{\epsilon_o k_\ell^2 \gamma_e \kappa T_e} \tag{15}$$

$$\alpha_\ell^\pm = \frac{n_o \, \omega_\ell}{\gamma_e k_\ell^2 v_{Te}^2} \left(1 \mp \frac{i}{2} \frac{\nu_e}{\omega_\ell} \right) \tag{16}$$

and $$\omega_\ell = (\omega_{pe}^2 + \gamma_e k_\ell^2 \, v_{Te}^2)^{\frac{1}{2}} \tag{17}$$

$a_\ell^\pm$ have the following space and time variation

$$a_\ell^+ \sim e^{i(\underset{\sim}{k}_\ell \cdot \underset{\sim}{x} \, - \, \omega_\ell t)} \, e^{-\gamma_\ell t}$$

$$a_\ell^- \sim e^{i(\underset{\sim}{k}_\ell \cdot \underset{\sim}{x} \, + \, \omega_\ell t)} \, e^{-\gamma_\ell t}$$

where $$\gamma_\ell = \frac{\nu_e}{2} \, .$$

For the Langmuir mode we are free to choose the direction of $\underset{\sim}{k}_\ell$ and its sense. Finally we must obtain the ion acoustic normal mode.

2(iii) The Ion Acoustic Normal Mode

We denote the wave vector for this mode by $\underset{\sim}{k}_s$. We then take a suitable linear combination of the two linearized continuity equations (2) and (4) and equations (1), (3) and (5) scalar multiplied by $\underset{\sim}{k}_s$. Choosing the coefficients in the linear combination such that the equation takes the normal mode form we obtain

$$\frac{\partial a_s^\pm}{\partial t} \pm i \, \omega_s \, a_s^\pm + \frac{1}{2} \left(\nu_i + \nu_e \frac{m_e}{m_i} \right) a_s^\pm = 0 \tag{18}$$

where

$$a_s^\pm \equiv n_{e1}^s \pm \alpha_s^\pm \, \underset{\sim}{k}_s \cdot \underset{\sim}{v}_e^s + \beta_s^\pm \, n_{i1}^s \pm \delta_s^\pm \, \underset{\sim}{k}_s \cdot \underset{\sim}{v}_i^s + \xi_s^\pm \, \epsilon_o \, \underset{\sim}{k}_s \cdot \underset{\sim}{E}^s \tag{19}$$

where

$$\alpha_s^\pm = \frac{n_o}{\gamma_e k_s^2 v_{Te}^2} \left\{ \omega_s \mp \frac{i}{2} \left(\nu_i + \nu_e \frac{m_e}{m_i} \right) \right\} \tag{20}$$

$$\beta_s^\pm = \frac{\gamma_i T_i}{\gamma_e T_e} \left\{ 1 \pm \frac{i \, \omega_s}{\omega_{pe}^2} \, (\nu_i - \nu_e) \right\} \tag{21}$$

$$\delta_s^{\pm} = \frac{n_o}{k_s^2 c_s^2} \left\{ \omega_s \mp \frac{i}{2} \left(\nu_i + \nu_e \frac{m_e}{m_i} \right) \right\} \tag{22}$$

$$\xi_s^{\pm} = \frac{i}{e} \left\{ 1 \pm \frac{i \omega_s}{\gamma_e k_s^2 v_{Te}^2} (\nu_i - \nu_e) \right\} \tag{23}$$

and

$$\omega_s = |k_s| c_s \; (1 + \gamma_i T_i / \gamma_e T_e)^{\frac{1}{2}} \tag{24}$$

where $c_s \equiv (\gamma_e \kappa T_e / m_i)^{\frac{1}{2}}$ and we have assumed $k_s^2 v_{Te}^2 / \omega_{pe}^2 \ll 1$, $k_s^2 v_{Ti}^2/\omega_{pi}^2 \ll 1$. We also assume the ion acoustic waves to be lightly damped ($T_e \gg T_i$). We are again free to choose the sense and direction of $\underset{\sim}{k}_s$. The $a_s^{\pm}$ modes have the space time dependence

$$a_s^+ \sim e^{i(\underset{\sim}{k}_s . \underset{\sim}{x} - \omega_s t)} \; e^{-\gamma_s t}$$

$$a_s^- \sim e^{i(\underset{\sim}{k}_s . \underset{\sim}{x} + \omega_s t)} \; e^{-\gamma_s t}$$

where we take $\gamma_s = \frac{\nu_i}{2}$.

3. THE COUPLED MODE EQUATIONS

In obtaining the equations for the normal modes we neglected quadratic terms in the wave variables. To obtain the non-linear interactions we now retain these terms. We shall be concerned with the interaction between a few coherent normal modes and shall look for solutions of the coupled modes of the form

$$a_n^{\pm} (\underset{\sim}{x}, t) = A_n^{\pm}(t) \; e^{i(\underset{\sim}{k}_n . \underset{\sim}{x} \mp \omega_n t)} \tag{25}$$

where $A_n^{\pm}(t)$ is a time dependent amplitude due to the effect of the linear damping and the non-linear mode interactions. The mode amplitudes will be assumed to change slowly on the time scale of the high frequency modes but not necessarily slowly on the time scale of the ion acoustic mode which is a very low frequency mode compared with the Langmuir and transverse modes.

3(i) The Parametric Langmuir-Ion Acoustic Interaction

This effect is due to the decay of a transverse electromagnetic wave into a Langmuir wave and an ion acoustic wave

$$T \to L + S.$$

This interaction was first described by Dubois and Goldman, 1965 and later by various others (e.g. Jackson, 1967; Nishikawa, 1968; Lee and Su, 1966; Sanmartin, 1970).

We now return to equations (1) to (6) and assume that the waves to be described have small but finite amplitude. For the longitudinal modes this means

$$\epsilon_o E^2 \ll n_o \kappa T_e \tag{26}$$

and for the transverse modes

$$\frac{e^2 E^2}{m_e^2 \omega_T^2} \ll v_{Te}^2 . \tag{27}$$

Taking the same linear combinations of equations (1) to (6) that we took to obtain equations (8), (13) and (18) but this time keeping the quadratic terms we obtain

$$\frac{\partial a_T^{\pm}}{\partial t} \pm i\omega_T a_T^{\pm} + \frac{\nu_e}{2}\frac{\omega_{pe}^2}{\omega_T^2} a_T^{\pm} = -\frac{e}{m_e} v_{ez} B_x - (\underset{\sim}{v}_e \cdot \underset{\sim}{\nabla}) v_{ey} + \frac{\gamma_e \kappa T_e}{n_o^2 m_e} n_{el} \frac{\partial n_{el}}{\partial y} + e\,\beta_T^{\pm} n_{el} v_{ey} \tag{28}$$

$$\frac{\partial a_\ell^{\pm}}{\partial t} \pm i\omega_\ell a_\ell^{\pm} + \frac{\nu_e}{2} a_\ell^{\pm} = -\underset{\sim}{\nabla} \cdot (n_{el} \underset{\sim}{v}_e) - (\underset{\sim}{v}_e \cdot \underset{\sim}{\nabla}) \alpha_\ell^{\pm} \underset{\sim}{k}_\ell \cdot \underset{\sim}{v}_e + \frac{\gamma_e \kappa T_e}{n_o^2 m_e} n_{el} (\underset{\sim}{k}_\ell \cdot \underset{\sim}{\nabla}) \alpha_\ell^{\pm} n_{el} - \alpha_\ell^{\pm} \frac{e}{m_e} k_\ell \cdot (\underset{\sim}{v}_e \times \underset{\sim}{B}) + \beta_\ell e\, n_{el} \underset{\sim}{k}_\ell \cdot \underset{\sim}{v}_e \tag{29}$$

$$\begin{aligned}\frac{\partial a_s^{\pm}}{\partial t} \pm i\omega_s a_s^{\pm} + \frac{\nu_i}{2} a_s^{\pm} = &-\underset{\sim}{\nabla} \cdot (n_{el} \underset{\sim}{v}_e) - \frac{e}{m_e} \alpha_s^{\pm} \underset{\sim}{k}_s \cdot (\underset{\sim}{v}_e \times \underset{\sim}{B}) - \alpha_s^{\pm} (\underset{\sim}{v}_e \cdot \underset{\sim}{\nabla}) \underset{\sim}{k}_s \cdot \underset{\sim}{v}_e \\ &+ \frac{\gamma_e \kappa T_e}{n_o^2 m_e} \alpha_s^{\pm} n_{el} (\underset{\sim}{k}_s \cdot \underset{\sim}{\nabla} n_{el}) - \beta_s^{\pm} \underset{\sim}{\nabla} \cdot (n_{il} \underset{\sim}{v}_i) + \frac{e}{m_i} \delta_s^{\pm} \underset{\sim}{k}_s \cdot (\underset{\sim}{v}_i \times \underset{\sim}{B}) \\ &- \delta_s^{\pm} (v_i \cdot \underset{\sim}{\nabla}) \underset{\sim}{k}_s \cdot \underset{\sim}{v}_i + \frac{\gamma_i \kappa T_i}{n_o^2 m_i} \delta_s^{\pm} n_{il} (\underset{\sim}{k}_s \cdot \underset{\sim}{\nabla} n_{il}) + \xi_s^{\pm} e\, n_{el} \underset{\sim}{k}_s \cdot \underset{\sim}{v}_e - \xi_s^{\pm} e\, n_{il} \underset{\sim}{k}_s \cdot \underset{\sim}{v}_i .\end{aligned} \tag{30}$$

For this interaction we have chosen the following polarization for the transverse modes

$$\underset{\sim}{E} = (0,\ E_y^T,\ 0)$$

$$\underset{\sim}{B} = (B_x,\ 0,\ 0)$$

$$\underset{\sim}{k}_T = (0,\ 0,\ k_T)\ .$$

We must now pick out those terms on the right hand sides of equations (28) - (30) which have the same space and time dependence as the mode on the left hand side of the equation. For example, to obtain the coupled mode equation for a_ℓ^+ we must find which non-linear terms in equation (20) vary approximately as

$$\exp\ i(\underset{\sim}{k}_\ell \cdot \underset{\sim}{x} - \omega_\ell\ t)$$

where we take the following wave vector and frequency selection rules

$$\underset{\sim}{k}_T = \underset{\sim}{k}_\ell + \underset{\sim}{k}_s \tag{31}$$

$$\omega_T \approx \omega_\ell + \omega_s\ . \tag{32}$$

Since we are assuming spatial uniformity we take equation (31) to be satisfied exactly. However, since we are allowing for the mode amplitudes to be time dependent and since we also wish to consider the effects of frequency mis-matching, we will take equation (32) to be satisfied only to within $\Delta\omega \sim \omega_s$ $(\omega_s \ll \omega_T,\ \ \omega_s \ll \omega_\ell)$.

Comparing the magnitudes of the terms on the right hand side of equation (29) we find that the first and last terms are the dominant ones. We then use the fact that the electron fluid velocity and density perturbation result from the influence of each of the normal modes

$$\begin{aligned}\underset{\sim}{v}_e = \mathrm{Re}\Big\{ &\hat{i}_y\, v_{ey+}^T\, e^{i(\underset{\sim}{k}_T \cdot \underset{\sim}{x} - \omega_T t)} + \hat{i}_y\, v_{ey-}^T\, e^{i(\underset{\sim}{k}_T \cdot \underset{\sim}{x} + \omega_T t)} \\ &+ \underset{\sim}{v}_{e+}^\ell\, e^{i(\underset{\sim}{k}_\ell \cdot \underset{\sim}{x} - \omega_\ell t)} + \underset{\sim}{v}_{e-}^\ell\, e^{i(\underset{\sim}{k}_\ell \cdot \underset{\sim}{x} + \omega_\ell t)} + \underset{\sim}{v}_{e+}^s\, e^{i(\underset{\sim}{k}_s \cdot \underset{\sim}{x} - \omega_s t)} \\ &+ \underset{\sim}{v}_{e-}^s\, e^{i(\underset{\sim}{k}_s \cdot \underset{\sim}{x} + \omega_s t)} \Big\}\end{aligned} \tag{33}$$

$$n_{el} = \mathrm{Re}\left\{ n^{\ell}_{el+}\, e^{i(\underset{\sim}{k}_\ell\cdot\underset{\sim}{x} - \omega_\ell t)} + n^{\ell}_{el-}\, e^{i(\underset{\sim}{k}_\ell\cdot\underset{\sim}{x} + \omega_\ell t)} + n^{s}_{el+}\, e^{i(\underset{\sim}{k}_s\cdot\underset{\sim}{x} - \omega_s t)} + n^{s}_{el-}\, e^{i(\underset{\sim}{k}_s\cdot\underset{\sim}{x} + \omega_s t)} \right\} \tag{34}$$

where, of course, there is no density perturbation due to a transverse wave in a uniform plasma. Having found which terms give a response with the required space and time dependence we must finally express all the variables on the right hand side of equation (29) in terms of a_T and a_s. Since we assume that the mode amplitudes are small we can obtain the dependence of the original variables (v_{ey}, n_{el}, n_{il} etc.) on the mode amplitudes with the aid of the linearized forms of equations* (1) - (6) and the definitions of the normal modes (equations (9), (14) and (19)). We must follow a similar procedure for each mode. We note that the first term on the right hand side of equation (28) does not contribute a response to $a_T^{\pm}$ and that the dominant non-linear terms in equation (30) are the first, the third and the ninth. (NB. We neglect the ion response to both the Langmuir wave and the transverse wave). Finally, we note that since $\omega_T \approx \omega_\ell$, $|\underset{\sim}{k}_T| \ll |\underset{\sim}{k}_\ell|$ and therefore $\underset{\sim}{k}_s \approx - \underset{\sim}{k}_\ell$. Taking $\underset{\sim}{k}_s$ and $\underset{\sim}{k}_\ell$ to be predominantly in the y-direction we finally obtain for the coupled mode equations

$$\frac{\partial a_\ell^{\pm}}{\partial t} \pm i\omega_\ell a_\ell^{\pm} + \frac{\nu_e}{2} a_\ell^{\pm} = - \frac{i\eta_\ell}{16\eta_s\eta_T} k_\ell \frac{\omega_{pe}^2\omega_{pi}^2}{\omega_s^2\,\omega_T^2} \left\{ (a_s^{+})^* a_T^{\pm} + (a_s^{-})^* a_T^{\pm} \right\} \tag{35a,b}$$

$$\frac{\partial a_T^{\pm}}{\partial t} \pm i\omega_T a_T^{\pm} + \frac{\nu_e}{2}\frac{\omega_{pe}^2}{\omega_T^2} a_T^{\pm} = - \frac{i\eta_T}{16\eta_s\eta_\ell} \omega_T \frac{\gamma_e k_s^2 c^2 k_\ell}{n_o^2\omega_\ell\omega_s^2} v_{Te}^2 \left\{ a_s^{+} a_\ell^{\pm} + a_s^{-} a_\ell^{\pm} \right\} \tag{35c,d}$$

$$\frac{\partial a_s^{\pm}}{\partial t} \pm i\omega_s a_s^{\pm} + \frac{\nu_i}{2} a_s^{\pm} = \frac{\pm\, i\eta_s}{16\eta_T\eta_\ell} k_s \frac{\omega_{pe}^2}{\omega_T^2}\frac{\omega_s}{\omega_\ell} \left\{ (a_\ell^{+})^* a_T^{+} - (a_\ell)^* a_T^{-} \right\} \tag{35e,f}$$

where we have normalized the mode amplitudes by multiplying a_T by η_T, a_ℓ by η_ℓ and a_s by η_s such that $|a_T^{+}|^2$ represents the energy in the mode a_T^{+} and so on. The quantities η are given by

$$\eta_T = \frac{1}{2\sqrt{2}} \frac{n_o e}{\omega_T\, \epsilon_o^{\frac{1}{2}}} \tag{36}$$

$$\eta_\ell = \frac{1}{2\sqrt{2}} \frac{\gamma_e k_\ell v_{Te}^2}{\omega_\ell\,\omega_{pe}} \frac{e}{\epsilon_o^{\frac{1}{2}}} \tag{37}$$

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* Since we only consider weakly damped waves we shall always neglect the effect of collisions on the non-linear terms.

$$\eta_s = \frac{1}{2\sqrt{2}}\left(\frac{m_i}{n_o}\right)^{\frac{1}{2}} \frac{|k_s| c_s^2}{\omega_s} . \tag{38}$$

The coupled mode equations contain the phase variation of the normal modes. We wish to obtain the coupled mode equations for the slowly varying amplitudes (slow on the time scale of ω_ℓ and ω_T) . We therefore write

$$a_\ell^+(\underset{\sim}{x},t) = A_\ell^+(t) e^{i(\underset{\sim}{k}_\ell \cdot \underset{\sim}{x} - \omega_\ell t)} \tag{39}$$

and similar expressions for the other modes. Substituting equation (39) and equivalent expressions for the other modes, into equations (35) and dividing throughout each equation by the phase factor for that mode we obtain the final form for the coupled mode equations

$$\frac{\partial A_\ell^+}{\partial t} + \frac{\nu_e}{2} A_\ell^+ = -i\,c_{sT}\left[(A_s^+)^* A_T^+ e^{-i(\delta-\omega_s)t} + (A_s^-)^* A_T^+ e^{-i(\delta+\omega_s)t}\right] \tag{40a}$$

$$\frac{\partial A_\ell^-}{\partial t} + \frac{\nu_e}{2} A_\ell^- = -i\,c_{sT}\left[(A_s^+)^* A_T^- e^{i(\delta+\omega_s)t} + (A_s^-)^* A_T^- e^{i(\delta-\omega_s)t}\right] \tag{40b}$$

$$\frac{\partial A_T^+}{\partial t} + \frac{\nu_e}{2}\frac{\omega_{pe}^2}{\omega_T^2} A_T^+ = -i\,c_{\ell s}\left[A_s^+ A_\ell^+ e^{i(\delta-\omega_s)t} + A_s^- A_\ell^+ e^{i(\delta+\omega_s)t}\right] \tag{40c}$$

$$\frac{\partial A_T^-}{\partial t} + \frac{\nu_e}{2}\frac{\omega_{pe}^2}{\omega_T^2} A_T^- = -i\,c_{\ell s}\left[A_s^+ A_\ell^- e^{-i(\delta+\omega_s)t} + A_s^- A_\ell^- e^{-i(\delta-\omega_s)t}\right] \tag{40d}$$

$$\frac{\partial A_s^+}{\partial t} + \frac{\nu_i}{2} A_s^+ = -i\,c_{\ell T}\left[(A_\ell^+)^* A_T^+ e^{-i(\delta-\omega_s)t} - (A_\ell^-)^* A_T^- e^{i(\delta+\omega_s)t}\right] \tag{40e}$$

$$\frac{\partial A_s^-}{\partial t} + \frac{\nu_i}{2} A_s^- = +i\,c_{\ell T}\left[(A_\ell^+)^* A_T^+ e^{-i(\delta+\omega_s)t} - (A_\ell^-)^* A_T^- e^{i(\delta-\omega_s)t}\right] \tag{40f}$$

where $\delta = \omega_T - \omega_\ell$ and we have taken $k_\ell > 0$ and put $k_s = -k_\ell$. We have also neglected contributions to the coupling term $\sim k_\ell^2 \lambda_d^2$. The coupling coefficients are as follows

$$c_{sT} = \frac{\eta_\ell}{16\eta_T\,\eta_s} k_\ell \frac{\omega_{pe}^2\,\omega_{pi}^2}{\omega_s^2\,\omega_T^2}$$

$$c_{\ell s} = \frac{\eta_T}{16\eta_s\,\eta_\ell}\,\omega_T\,\frac{k_s^2 c_s^2\,k_\ell\,\gamma_e\,v_{Te}^2}{n_o^2\,\omega_\ell\,\omega_s^2}$$

$$c_{\ell T} = \frac{\eta_s}{16\eta_T\eta_\ell}\,k_\ell\,\frac{\omega_{pe}^2\,\omega_s}{\omega_T^2\,\omega_\ell}.$$

As they stand these equations describe the non-linear resonant interaction of coherent transverse and longitudinal waves. However we can obtain the parametric effects produced in a plasma driven by an external electromagnetic wave by assuming that

$$|A_T^{\pm}| \gg |A_\ell^{\pm}| \;,\quad |A_T^{\pm}| \gg A_s^{\pm}.$$

We then assume that $A_T^{\pm}$ is constant and need only solve for $A_\ell^{\pm}$ and $A_s^{\pm}$. Furthermore, equations 40 (a, b) and 40(e, f) now form a set of four coupled <u>linear</u> equations with time dependent coefficients. These equations can be solved very simply by making the following transformation

$$\alpha_\ell^{+} = A_\ell^{+} \;;\qquad \alpha_\ell^{-} = e^{-\,i2\delta t}\,A_\ell^{-}$$

$$\alpha_s^{+} = e^{-i(\delta-\omega_s)t}\,(A_s^{+})^{*} \;;\quad \alpha_s^{-} = e^{-i(\delta+\omega_s)t}\,(A_s^{-})^{*}.$$

The four coupled equations describing the parametric interaction then become

$$\frac{\partial\alpha_\ell^{+}}{\partial t} + \frac{\nu_e}{2}\,\alpha_\ell^{+} = -i\,c_{sT}\,A_T^{+}\,(\alpha_s^{+} + \alpha_s^{-}) \tag{41a}$$

$$\frac{\partial\alpha_\ell^{-}}{\partial t} + i(2\delta)\alpha_\ell^{-} + \frac{\nu_e}{2}\,\alpha_\ell^{-} = -i\,c_{sT}A_T^{-}\,(\alpha_s^{+} + \alpha_s^{-}) \tag{41b}$$

$$\frac{\partial\alpha_s^{+}}{\partial t} + i(\delta-\omega_s)\,\alpha_s^{+} + \frac{\nu_i}{2}\,\alpha_s^{+} = i\,c_{\ell T}\left\{(A_T^{+})^{*}\,\alpha_\ell^{+} - (A_T^{-})^{*}\,\alpha_\ell^{-}\right\} \tag{41c}$$

$$\frac{\partial\alpha_s^{-}}{\partial t} + i(\delta+\omega_s)\,\alpha_s^{-} + \frac{\nu_i}{2}\,\alpha_s^{-} = -i\,c_{\ell T}\left\{(A_T^{+})^{*}\,\alpha_\ell^{+} - (A_T^{-})^{*}\,\alpha_\ell^{-}\right\}. \tag{41d}$$

These equations may now be solved in the usual way assuming an exp $(-i\omega t)$ dependence when the following dispersion relation results

$$(\omega + \frac{i\nu_e}{2}) \; (\omega - 2\delta + \frac{i\nu_e}{2}) \; (\omega - \delta + \omega_s + \frac{i\nu_i}{2}) \; (\omega - \delta - \omega_s + \frac{i\nu_i}{2})$$

$$+ 2 \, c_{sT} \, c_{\ell T} \, \omega_s \left\{ (\omega + \frac{i\nu_e}{2}) \; |A_T^-|^2 - (\omega - 2\delta + \frac{i\nu_e}{2}) \; |A_T^+|^2 \right\} = 0 \; . \quad (42)$$

$|A_T^+|$ and $|A_T^-|$ give a measure of the amplitude of the pump. If we take

$$|A_T^+| = |A_T^-| = |A_T|$$

then we have a standing wave as the pump and if we take

$$|A_T^-| = 0$$

we have a travelling pump wave. First consider the standing wave case and put

$$\Omega = \omega - \delta \; .$$

The dispersion equation (42) then becomes

$$(\Omega + \delta + \frac{i\nu_e}{2}) \; (\Omega - \delta + \frac{i\nu_e}{2}) \; (\Omega^2 - \omega_s^2 + i\nu_i\Omega) + 4c_{sT} \, c_{\ell T} |A_T|^2 \, \omega_s \delta = 0 \quad (43)$$

where we have used the fact that we have previously taken $\nu_i \ll \omega_s$. Equation (43) is exactly the form of dispersion relation Nishikawa (1968b) has obtained for this process. Nishikawa (1968a) has pointed out the existence of two exponentially growing solutions to equation (43). The first is the parametric instability ($\delta > 0$) where the frequency of the pump is higher than the frequency of the Langmuir wave and the second is the so-called oscillating two stream instability ($\delta < 0$) where the frequency of the pump is less then the unperturbed frequency of the Langmuir wave. In both cases there exist two growing Langmuir waves (one propagating in the positive y-direction and one in the negative y-direction) and two growing ion acoustic waves. However, in the oscillating two stream case, the ion waves have both undergone a frequency shift so that they are purely growing (non-propagating) waves whereas in the parametric case the ion waves propagate at or near the ion acoustic frequencies. Evidently the oscillating two stream instability requires the presence of <u>four</u> waves, two Langmuir and two ion acoustic whereas the parametric instability requires only one Langmuir wave to be coupled to one ion acoustic wave.

Finally, equation (43) gives the following threshold fields for these two instabilities (see Nishikawa (1968a))

$$|E_y^T|^2 = 16 \frac{m_e}{e^2} \gamma_e \kappa T_e \omega_\ell \nu_e \qquad (44)$$

which is the minimum threshold for the oscillating two stream instability (i.e. when $\omega_T - \omega_\ell = -\frac{\nu_e}{2}$) and

$$K = \frac{\nu_i \nu_e}{4\delta\omega_s} \left\{ 4\delta^2 + \frac{\left(\frac{\nu_e^2}{4} + \nu_e\nu_i/2 + \omega_s^2 - \delta^2\right)^2}{\left(\frac{\nu_i}{2} + \frac{\nu_e}{2}\right)^2} \right\} \qquad (45)$$

for the parametric instability where $K \equiv 4c_{sT}c_{\ell T}|A_T|^2$. For the case where $\nu_e \gg \omega_s$ this can be written in terms of the threshold electric field as

$$|E_y^T|^2 = 16 \frac{m_e m_i}{e^2} \frac{\nu_i}{\nu_e} \frac{\omega_\ell}{k_\ell^2 \delta} \left(\delta^2 + \frac{\nu_e^2}{4}\right)^2 . \qquad (46)$$

We note that our equations for $|E_y^T|^2$ are larger than Nishikawa's (1968b) by a factor 16! The origin of this discrepancy seems to be that Nishikawa neglected to take the real part of terms involving products of the pump amplitude and the ion acoustic or Langmuir wave amplitudes.

We now consider the effect of a travelling pump wave on these two instabilities. In this case the dispersion relation (42) reduces to

$$\left(\Omega + \delta + \frac{i\nu_e}{2}\right)(\Omega^2 - \omega_s^2 + 2i\Omega\gamma_s) - 2c_{sT}\, c_{\ell T}|A_T^+|^2 \omega_s = 0 . \qquad (47)$$

Since we take $A_T^- = 0$ the a_ℓ^- mode is not excited and the dispersion relation reduces to a cubic. (NB. We have written $\gamma_s = \frac{\nu_i}{2}$). In contrast to the previous case we note that equation (47) has no purely growing solutions. Evidently the purely growing solutions depend on the symmetric excitation provided by a standing wave.

Looking for solutions to equation (47)

$$\Omega = x + iy$$

we find that for growing solutions $(y > 0)$ we must have $x < 0$.

For the special case

$$\delta = \omega_s$$

the threshold condition is

$$K = 2\gamma_s \nu_e \qquad (48)$$

(N.B. $K \equiv 4c_{sT}c_{\ell T}|A_T|^2$)
which is the same result as equation (45) for the standing wave pump when $\nu_e \ll \omega_s$. However, if $\nu_e \gg \omega_s$, equation (48) gives a lower threshold than equation (45). In terms of the threshold electric field equation (48) can be written

$$\frac{\epsilon_o |E_y^T|^2}{n_o \gamma_e \kappa T_e} = 32 \frac{\gamma_s}{\omega_s} \frac{\nu_e}{\omega_{pe}} \quad . \qquad (49)$$

This is the parametric instability and its growth rate well above threshold is given approximately by

$$\gamma = \frac{1}{8} (\omega_s \omega_{pe})^{\frac{1}{2}} \left(\frac{\epsilon_o |E_y^T|^2}{n_o \gamma_e \kappa T_e} \right)^{\frac{1}{2}} \qquad (50)$$

where we have neglected the effect of a_s^- and again put $\delta = \omega_s$, which is the condition of perfect matching for the parametric decay process.

Although equation (47) has no purely growing solutions one can still find growing solutions for $\delta < 0$. In other words, for a travelling wave pump, the oscillating two stream instability (since that required $\delta < 0$) gives rise to growing ion waves which are now propagating. There is only one growing Langmuir wave since as we have already mentioned a_ℓ^- is not excited for $A_T^- = 0$.

For $\delta = -\omega_s$ the threshold condition is

$$K = 8\omega_s^2 \gamma_s \frac{(\nu_e/2 + \gamma_s)^2}{(\nu_e/2 + 2\gamma_s)^3} \quad \text{when} \quad \nu_e \ll \omega_s \qquad (51a)$$

and $$K = 2\nu_e \gamma_s \quad \text{when} \quad \nu_e \gg \omega_s \; . \qquad (51b)$$

For $|\delta| \ll \omega_s$, the threshold condition is

$$K = \frac{4\omega_s^2 \gamma_s}{(\nu_e/2 + 2\gamma_s)} \left\{ \frac{|\delta|}{\omega_s} + \frac{1}{(1 + 2\gamma_s/\gamma_e)^{\frac{1}{2}}} \right\} \quad \text{when} \quad \nu_e \ll \omega_s \qquad (52a)$$

and $$K = 2\nu_e \gamma_s \quad \text{when} \quad \nu_e \gg \omega_s \; . \qquad (52b)$$

We note that the threshold for the oscillating two stream instability is now proportional to the damping of the ion acoustic waves whereas for the standing wave pump it was independent of this quantity. This, presumably, is due to the fact that the ion wave solutions are propagating for the travelling wave pump.

The parametric Langmuir-ion acoustic wave and oscillating two stream instabilities require the frequency of the transverse (pump) wave to satisfy $\omega_T \approx \omega_{pe}$. We next consider another parametric process which occurs when $\omega_T \approx 2\omega_{pe}$, namely the parametric decay of a transverse wave into two Langmuir waves.

3(ii) The Decay of a Transverse Wave into Two Langmuir Waves

This process can be written

$$T \rightarrow L + L$$

and has been considered by Jackson (1967) who restricted his analysis to the case where the transverse wavelength was much larger than the longitudinal wavelength. Here we impose no restriction on the transverse wavelength, other than the matching conditions. We shall again specify the polarization of the transverse wave to be

$$\underset{\sim}{E} = (0, E_y, 0)$$

$$\underset{\sim}{B} = (0, 0, B_z)$$

$$\underset{\sim}{k}_T = (k_T, 0, 0) .$$

We choose the two longitudinal wave vectors $\underset{\sim}{k}_{\ell 1}$ and $\underset{\sim}{k}_{\ell 2}$ to be

$$\underset{\sim}{k}_{\ell 1,2} = \left\{ \begin{matrix} (k_{\ell 1x}, k_{\ell 1y}, 0) \\ (k_{\ell 2x}, k_{\ell 2y}, 0) \end{matrix} \right\} .$$

To obtain the coupled mode equations we return to equations (1) to (6) and take the various linear combinations corresponding to the transverse and Langmuir normal modes. We then obtain the following equations

$$\frac{\partial a_T^{\pm}}{\partial t} \pm i\omega_T a_T^{\pm} + \frac{\nu_e}{2} \frac{\omega_{pe}^2}{\omega_T^2} a_T^{\pm} = -(\underset{\sim}{v}.\underset{\sim}{\nabla})v_y + \frac{\gamma_e \kappa T_e}{n_o^2 m_e} n_{1e} \frac{\partial n_{1e}}{\partial y} + n_{1e} e \beta_T^{\pm} v_y \qquad (53)$$

$$\frac{\partial a_\ell^\pm}{\partial t} \pm i\omega_\ell\, a_\ell + \frac{\nu_e}{2} a_\ell^\pm = - \frac{e}{m_e} \alpha_\ell^\pm\, \underset{\sim}{k}_\ell \cdot (\underset{\sim}{v} \times \underset{\sim}{B}) - \alpha_\ell^\pm (\underset{\sim}{v} \cdot \underset{\sim}{\nabla}) \underset{\sim}{k}_\ell \cdot \underset{\sim}{v}$$

$$+ \alpha_\ell^\pm \frac{\gamma_e \kappa T_e}{n_o^2 m_e} n_{1e}\, (\underset{\sim}{k}_\ell \cdot \underset{\sim}{\nabla}) n_{1e} - \underset{\sim}{\nabla} \cdot (n_{1e}\, \underset{\sim}{v}) + \beta_\ell\, n_{1e}\, e\, \underset{\sim}{k}_\ell \cdot \underset{\sim}{v}. \quad (54)$$

First, note that if we consider one-dimensional effects, namely $k_{\ell 1y} = k_{\ell 2y} = 0$, then the a_T mode is decoupled from the two a_ℓ modes ($a_{\ell 1}$ and $a_{\ell 2}$).

To obtain the final form of the coupled mode equations we must express the original variables ($n^\ell_{1e\pm}$, $v^\ell_{x\pm}$, $v^\ell_{y\pm}$, $E^T_{y\pm}$ etc.) in terms of the normal mode amplitudes $a_T^\pm$, $a_{\ell 1}^\pm$ and $a_{\ell 2}^\pm$ with the aid of the linearized normal mode equations. For this interaction, the interacting waves all have high frequencies (compared with the frequency shift induced by the pump). We may therefore neglect the coupling between the + and - modes.

We take the following conditions to be satisfied

$$\underset{\sim}{k}_T = \underset{\sim}{k}_{\ell 1} + \underset{\sim}{k}_{\ell 2} \quad (55)$$

$$\omega_T \approx \omega_{\ell 1} + \omega_{\ell 2} \quad (56)$$

but will allow for a small mis-match of frequencies. Proceeding as in the previous case we find the coupled mode equations for a_T^+, $a_{\ell 1}^+$ and $a_{\ell 2}^+$. Writing these modes as

$$a_T^+ = A_T^+(t) e^{i(\underset{\sim}{k}_T \cdot \underset{\sim}{x} - \omega_T t)} \quad (57a)$$

$$a_{\ell 1}^+ = A_{\ell 1}^+(t) e^{i(\underset{\sim}{k}_{\ell 1} \cdot \underset{\sim}{x} - \omega_{\ell 1} t)} \quad (57b)$$

$$a_{\ell 2}^+ = A_{\ell 2}^+(t) e^{i(\underset{\sim}{k}_{\ell 2} \cdot \underset{\sim}{x} - \omega_{\ell 2} t)} \quad (57c)$$

we obtain the required equations

$$\frac{\partial A_T^+}{\partial t} + \frac{\nu_e}{2} \frac{\omega_{pe}^2}{\omega_T^2} A_T^+ = - i\, c_{12}\, A_{\ell 1}^+\, A_{\ell 2}^+\, e^{i \varphi t} \quad (58a)$$

$$\frac{\partial A_{\ell 1}^+}{\partial t} + \frac{\nu_e}{2} A_{\ell 1}^+ = - i\, c_{2T}\, A_T^+\, (A_{\ell 2}^+)^*\, e^{- i \varphi t} \quad (58b)$$

$$\frac{\partial (A_{\ell 2}^{+})^{*}}{\partial t} + \frac{\nu_e}{2} (A_{\ell 2}^{+})^{*} = i\, c_{1T} (A_T^{+})^{*} A_{\ell 1}^{+} e^{i \varphi t} \tag{58c}$$

where

$$c_{12} = \frac{\eta_T}{8\eta_{\ell 1} \eta_{\ell 2}} \frac{\gamma_e^2 v_{Te}^4}{n_o^2} \frac{(k_{\ell 1}^2 k_{\ell 2y} + k_{\ell 2}^2 k_{\ell 1y})}{\omega_{\ell 1} \omega_{\ell 2}}$$

$$c_{1T} = \frac{\eta_{\ell 2}}{16\eta_T \eta_{\ell 1}} k_T \frac{\omega_{pe}^2}{\omega_T^2} \frac{\omega_{\ell 2}}{\omega_{\ell 1}} \frac{k_{\ell 1y}}{k_{\ell 2}^2} (k_{\ell 2x} - k_{\ell 1x})$$

$$c_{2T} = \frac{\eta_{\ell 1}}{16\eta_T \eta_{\ell 2}} k_T \frac{\omega_{pe}^2}{\omega_T^2} \frac{\omega_{\ell 1}}{\omega_{\ell 2}} \frac{k_{\ell 2y}}{k_{\ell 1}^2} (k_{\ell 1x} - k_{\ell 2x})$$

and we have neglected terms of order $k_{\ell 1}^2 \lambda_{de}^2$, $k_{\ell 2}^2 \lambda_{de}^2$ in the coupling terms. The phase φ represents the mis-match

$$\varphi \equiv \omega_T - \omega_{\ell 1} - \omega_{\ell 2} \tag{59}$$

and the quantities η_T , $\eta_{\ell 1}$ and $\eta_{\ell 2}$ which normalize the normal modes to the energy in each mode were defined in equations (36) and (37). Equations (58) represent the non-linear resonant interaction of the two Langmuir waves with a transverse mode. If we neglect the damping and put $\varphi = 0$ these equations conserve energy density, momentum density and action density (Dysthe (1970)).

We can obtain the parametric interaction between a large amplitude tranverse pump wave and two small amplitude Langmuir waves as before by taking

$$A_T^{+} = \text{const.} \qquad \text{and}$$

$$|A_T^{+}| \gg |A_{\ell 1}^{+}| , \quad |A_T^{+}| \gg |A_{\ell 2}^{+}| .$$

Equations (58b) and (58c) are now linear and we can solve them for the growth rate of the two Langmuir waves and the threshold transverse electric field required to excite them.

If we make a similar transformation to the one used in the previous section

$$\alpha_{\ell 1} = A_{\ell 1}^{+}$$

$$\alpha_{\ell 2} = (A_{\ell 2}^{+})^{*} e^{-i \varphi t}$$

equations (58b and c) then have constant coefficients and a solution $\sim \exp(-i\omega t)$ can be found where ω satisfies

$$\left(\omega + \frac{i\nu_e}{2}\right)\left(\omega - \varphi + \frac{i\nu_e}{2}\right) + c_{1T}\, c_{2T}\, |A_T^+|^2 = 0 . \qquad (60)$$

The solution of (60) is

$$\omega = \frac{1}{2}\left\{\varphi - i\nu_e \pm \sqrt{\varphi^2 - 4K}\right\} \qquad (61)$$

where

$$K = c_{1T}\, c_{2T} |A_T^+|^2 .$$

The minimum threshold field is given for perfect matching (i.e. $\varphi = 0$) when

$$K^{\frac{1}{2}} = \frac{\nu_e}{2} \qquad (62)$$

or in terms of the transverse electric field

$$\frac{e\, E_y^T}{m_e\, \omega_{pe}} = 8\, \frac{\nu_e}{k_T}\, \frac{(k_{\ell 1}\, k_{\ell 2})^{\frac{1}{2}}}{|k_{\ell 2x} - k_{\ell 1x}|}\, \left(\cos\theta_1\, \cos\theta_2\right)^{-\frac{1}{2}} . \qquad (63)$$

The growth rate of the Langmuir waves well above threshold is

$$\gamma = \frac{k_T}{16}\, \frac{e}{\omega_{pe}}\, \frac{E_y^T}{m_e}\, |(\cos\theta_1\, \cos\theta_2)|^{\frac{1}{2}}\, \frac{|k_{\ell 2x} - k_{\ell 1x}|}{(k_{\ell 1}\, k_{\ell 2})^{\frac{1}{2}}} \qquad (64)$$

where we have again taken $\varphi = 0$, and

$$\cos\theta_1 = k_{\ell 1y}/k_{\ell 1} \quad ; \quad \cos\theta_2 = k_{\ell 2y}/k_{\ell 2} .$$

Equation (64) gives the growth rate of the Langmuir waves without any restriction on the transverse wave number k_T (other than equation (55)). For the special case where $k_T \ll k_{\ell 1}$, $\underset{\sim}{k}_{\ell 1}$ and $\underset{\sim}{k}_{\ell 2}$ are effectively collinear but in opposite directions. The maximum growth rate in this case occurs when $\underset{\sim}{k}_{\ell 1}$ makes an angle of $\pi/4$ with $\underset{\sim}{k}_T$ (and $\underset{\sim}{k}_{\ell 2}$ an angle of $5\pi/4$) when equation (64) gives

$$\gamma = \frac{k_T}{16}\, \frac{e\, E_y^T}{m\, \omega_{pe}} \qquad (65)$$

which is the result obtained by Jackson (1967).

3 (iii) Stimulated Raman Scattering

Raman scattering in a plasma refers to the decay of a tranverse electromagnetic wave into another transverse electromagnetic wave and a Langmuir wave,

$$T \rightarrow T + L .$$

This effect can be described most simply in one dimension. We therefore consider all waves to vary as

$$\exp i(kx - \omega t)$$

and choose the following linear polarization for both transverse waves

$$\underset{\sim}{E}^T = (0, E_y^T, 0)$$

$$\underset{\sim}{B} = (0, 0, B_z) .$$

Forming the usual linear combinations of equations (1) to (6) for the transverse and Langmuir normal modes we obtain the following equations describing the non-linear coupling of the waves

$$\frac{\partial a_T^+}{\partial t} + i\omega_T a_T^+ + \frac{\nu_e}{2} \frac{\omega_{pe}^2}{\omega_T^2} a_T^+ = \frac{e}{m_e} v_x B_z - v_x \frac{\partial v_y}{\partial x} + n_{1e} e \beta_T^+ v_y \quad (66)$$

$$\frac{\partial a_\ell^+}{\partial t} + i\omega_\ell a_\ell^+ + \frac{\nu_e}{2} a_\ell^+ = -\frac{\partial}{\partial x}(n_{1e} v_x) - \alpha_\ell^+ \frac{e}{m_e} v_y B_z - \alpha_\ell^+ v_x \frac{\partial v_x}{\partial x}$$
$$+ \alpha_\ell^+ \frac{\gamma_e \kappa T_e}{n_o^2} n_{1e} \frac{\partial n_{1e}}{\partial x} + \beta_\ell e n_{1e} v_x \quad (67)$$

where a_T^+ and a_ℓ^+ are defined in equations (9) and (14) and the constants β_T^+, β_ℓ and α_ℓ^+ are given by equations (11), (15) and (16).

We only give the equations for a_ℓ^+ and a_T^+ since these modes do not couple to a_ℓ^- and a_T^- in view of the high frequencies of the modes and the weakness of the coupling. The equations for a_ℓ^- and a_T^- would be of exactly the same form as equations (66) and (67) and would not give any additional information.

We note that, of the terms on the right-hand side of equation (67), only one

$$- \alpha_\ell^+ \frac{e}{m_e} v_y \, B_z$$

contributes significantly to a_ℓ^+ i.e. has a variation

$$\sim \exp i(k_\ell x - \omega_\ell t) \; .$$

We assume the frequencies and wave numbers of the two transverse waves and the Langmuir mode satisfy the conditions

$$\omega_{To} \approx \omega_{T1} + \omega_\ell$$

$$k_{To} = k_{T1} + k_\ell \, .$$

Expressing n_{1e} , v_x in terms of a_ℓ^+ , v_{yo} , B_{zo} in terms of a_{To}^+ and v_{y1} , B_{y1} in terms of a_{T1}^+ , from the linearized normal mode equations we obtain the coupled mode equations for a_ℓ^+ , a_{To}^+ and a_{T1}^+ . Writing these amplitudes as a product of a slowly varying amplitude and a rapidly varying phase

$$a_{To}^+ = A_{To}^+(t) \; e^{i(k_{To}x - \omega_{To}t)} \tag{68a}$$

$$a_{T1}^+ = A_{T1}^+(t) \; e^{i(k_{T1}x - \omega_{T1}t)} \tag{68b}$$

$$a_\ell^+ = A_\ell^+(t) \; e^{i(k_\ell x - \omega_\ell t)} \tag{68c}$$

we obtain the final form for the coupled mode equations

$$\frac{\partial A_{To}^+}{\partial t} + \frac{\nu_e}{2} \frac{\omega_{pe}^2}{\omega_{To}^2} A_{To}^+ = - i \, c_{1\ell} \, A_\ell^+ \, A_{T1}^+ \, e^{i \varphi t} \tag{69a}$$

$$\frac{\partial A_{T1}^+}{\partial t} + \frac{\nu_e}{2} \frac{\omega_{pe}^2}{\omega_{T1}^2} A_{T1}^+ = - i \, c_{o\ell} (A_\ell^+)^* \, A_{To}^+ \, e^{- i \varphi t} \tag{69b}$$

$$\frac{\partial}{\partial t} (A_\ell^+)^* + \frac{\nu_e}{2} (A_\ell^+)^* = i \, c_{o1} \, (A_{To}^+)^* \, A_{T1}^+ \, e^{i \varphi t} \tag{69c}$$

where

$$c_{1\ell} = \frac{\eta_{To}}{16 \eta_{T1} \eta_\ell} \; \frac{\gamma_e k_\ell^2 v_{Te}^2}{n_o} \; \frac{\omega_{pe}^2 \omega_{To}}{\omega_\ell^2 \, \omega_{T1}^2}$$

$$c_{o\ell} = \frac{\eta_{T1}}{16 \eta_{To} \eta_\ell} \; \frac{\gamma_e k_\ell^2 v_{Te}^2}{n_o} \; \frac{\omega_{pe}^2 \omega_{T1}}{\omega_\ell^2 \, \omega_{To}^2}$$

$$c_{o1} = \frac{\eta_\ell}{16\eta_{To}\,\eta_{T1}}\,\frac{n_o\,\omega_\ell}{\gamma_e v_{Te}^{\,2}}\,\frac{\omega_{pe}^{\,4}}{\omega_{To}^{\,2}\,\omega_{T1}^{\,2}}$$

and

$$\varphi = \omega_{To} - \omega_{T1} - \omega_\ell \;.$$

η_{To} , η_{T1} and η_ℓ are the normalization constants for the modes defined by equations (36) and (37).

Equations (69a - c) have previously been derived by Sjolund and Stenflo (1967) but without the damping or allowance for any mismatch. These equations are identical in form to those for the interaction between a transverse electromagnetic wave and two Langmuir waves given by equations (58a - c). This is, of course, to be expected since the basic process is that of the resonant interaction of three waves. As before we can describe the parametric decay of a transverse travelling electromagnetic wave A_{To}^{+} by assuming

$$A_{To}^{+} = \text{const.}$$

and

$$|A_{To}^{+}| \gg |A_{T1}^{+}| \;;\quad |A_{To}^{+}| \gg |A_{\ell}^{+}| \;.$$

Equations (69b, c) can then be solved by assuming

$$A_{T1}^{+} \propto e^{-i\omega t}$$

$$(A_{\ell}^{+})^{*} e^{-i\varphi t} \propto e^{-i\omega t}$$

where ω is a solution of

$$(\omega + i\gamma_{T1})\,(\omega - \varphi + \frac{i\nu_e}{2}) + K = 0 \qquad (70)$$

where $\gamma_{T1} = \frac{\nu_e}{2}\,\frac{\omega_{pe}^{\,2}}{\omega_{T1}^{\,2}}$; $K = c_{o1}\,c_{o\ell}\,|A_{To}^{+}|^{2}$.

The threshold condition for growing Langmuir and scattered electromagnetic waves to exist is

$$K = \frac{\nu_e\,\gamma_{T1}}{2} + \frac{\varphi^2\nu_e\,\gamma_{T1}}{2\left(\gamma_{T1} + \frac{\nu_e}{2}\right)^2} \;. \qquad (71)$$

In terms of the threshold transverse electric field E_{yo}^{T} of the initial electromagnetic wave the minimum threshold condition can be written as

$$\frac{e^2 |E_{yo}^T|^2}{m_e^2 \omega_{To}^2 v_{Te}^2} = \frac{16}{k_\ell^2 \lambda_{de}^2} \frac{\nu_e}{\omega_{pe}} \frac{\nu_e}{\omega_{T1}} \qquad (72)$$

where $\varphi = 0$ at minimum threshold. ν_e represents either the electron-ion collision frequency for a fully ionized plasma or the electron-neutral for a partially ionized plasma.

So far we have not mentioned whether the initial transverse wave decays into a forward wave or a backward wave. Equations (69a - c) cover both cases. For the back scattering process the longitudinal wave number k_ℓ is always larger and usually much larger than the corresponding quantity for forward scattering. Since the threshold field is proportional to k_ℓ^{-2} the back scattering process has a much lower threshold than the forward scattering process.

The maximum growth rate of the scattered transverse wave and the Langmuir wave well above threshold is given by

$$\gamma_{max.} = \frac{k_\ell \lambda_{de}}{8} \left(\frac{\omega_{pe}}{\omega_{T1}}\right)^{\frac{1}{2}} \frac{e|E_{yo}^T|}{m_e \omega_{To} v_{Te}} \qquad (73)$$

when $\varphi = 0$.

The Raman scattering conditions can be satisfied for transverse frequencies ω_T varying from $\omega_T \sim \omega_{pe}$ to $\omega_T \gg \omega_{pe}$. However, the higher the value of ω_T the smaller the required threshold since k_ℓ increases as ω_T increases. Thus, the process would occur preferentially at the higher frequencies.

3 (iv) Stimulated Brillouin Scattering

This process differs from the previous one in that the Langmuir wave is replaced by an ion sound wave. In other words it is the decay of a transverse wave into another transverse wave and an ion sound mode

$$T \rightarrow T + S .$$

We can again describe the process in 1-dimension and therefore take all waves to propagate in the x-direction and choose the same linear polarization for both transverse waves as in the previous example.

Taking the various linear combinations of equations (1) - (6) corresponding to the transverse and ion sound normal modes we obtain the following equations for the non-linear interaction of these modes

$$\frac{\partial a_T^{\pm}}{\partial t} \pm i\omega_T a_T^{\pm} + \frac{\nu_e}{2} \frac{\omega_{pe}^{2}}{\omega_T^{2}} a_T^{\pm} = \frac{e}{m_e} v_{ex} B_z - v_{ex} \frac{\partial v_{ey}}{\partial x} \mp \beta_T^{\pm} e n_{el} v_{ey} \pm \beta_T^{\pm} e n_{il} v_{iy} \quad (74)$$

$$\frac{\partial a_s^{\pm}}{\partial t} \pm i\omega_s a_s^{\pm} + \frac{\nu_i}{2} a_s^{\pm} = \mp \frac{e}{m_e} \alpha_s^{\pm} \underset{\sim}{k}_s . (\underset{\sim}{v}_e \times \underset{\sim}{B}) \quad (75)$$

where again the normal modes and the constants have been defined in equations (9), (11), (19) and (20). In equation (74) we neglect the term $\beta_T^{\pm} e n_{il} v_{iy}$ since this involves the ion response to the high frequency transverse wave. In deriving equation (75) we have used the fact that for one-dimension the $(\underset{\sim}{v}_e \times \underset{\sim}{B})$ term is the only non-linear term involving a product of two transverse quantities which is what is required to give a variation in the vicinity of $a_s^{\pm}$.

Since $\omega_s \ll \omega_T$ we cannot neglect the coupling between the + and - modes as was the case for stimulated Raman scattering. Enforcing the usual selection conditions

$$\omega_{To} \approx \omega_{T1} + \omega_s$$

$$k_{To} = k_{T1} + k_s$$

and proceeding as before we obtain the following coupled mode equations for the slowly varying amplitudes

$$\frac{\partial A_{T1}^{+}}{\partial t} + \gamma_{T1} A_{T1}^{+} = - i c_{os} A_{To}^{+} \left\{ (A_s^{+})^{*} e^{-i(\delta - \omega_s)t} + (A_s^{-})^{*} e^{-i(\delta + \omega_s)t} \right\} \quad (75a)$$

$$\frac{\partial A_{T1}^{-}}{\partial t} + \gamma_{T1} A_{T1}^{-} = i c_{os} A_{To}^{-} \left\{ (A_s^{+})^{*} e^{i(\delta + \omega_s)t} + (A_s^{-})^{*} e^{i(\delta - \omega_s)t} \right\} \quad (75b)$$

$$\frac{\partial}{\partial t}(A_s^+)^* + \gamma_s (A_s^+)^* = i\, c_{ol}^s \left\{ (A_{To}^+)^* A_{T1}^+ e^{i(\delta-\omega_s)t} + (A_{To}^-)^* A_{T1}^- e^{-i(\delta+\omega_s)t} \right\} \qquad (75c)$$

$$\frac{\partial}{\partial t}(A_s^-)^* + \gamma_s (A_s^-)^* = -i\, c_{ol}^s \left\{ (A_{To}^+)^* A_{T1}^+ e^{i(\delta+\omega_s)t} + (A_{To}^-)^* A_{T1}^- e^{-i(\delta-\omega_s)t} \right\} \qquad (75d)$$

where

$$c_{os} = \frac{\eta_{T1}}{16\eta_{To}\,\eta_s} \, \frac{\omega_{T1}}{n_o} \, \frac{\omega_{pe}^2}{\omega_{To}^2} \, \frac{k_s^2 c_s^2}{\omega_s^2}$$

$$c_{ol}^s = \frac{\eta_s}{16\eta_{To}\,\eta_{T1}} \, \frac{n_o}{\gamma_e v_{Te}^2} \, \frac{\omega_s \, \omega_{pe}^4}{\omega_{To}^2 \omega_{T1}^2}$$

$$\gamma_{T1} = \frac{\nu_e}{2} \frac{\omega_{pe}^2}{\omega_{T1}^2} \, , \qquad \gamma_s \approx \frac{\nu_i}{2}$$

and

$$\delta = \omega_{To} - \omega_{T1} \, .$$

We have not written down the equations for the amplitudes $A_{To}^{\pm}$ since these equations are not required in order to calculate the threshold condition and growth rates for the parametric decay process. The equations for $A_{To}^{\pm}$ can be obtained from equations (75 a, b) by interchanging the indices 1 and 0.

Equations (75 a - d) are of a similar form to those describing the decay of a transverse wave into a Langmuir wave and an ion acoustic wave in section 3(i). There will therefore be two types of solution corresponding to the parametric decay instability and the oscillating two stream instability. In this section we will assume that the pump wave is a travelling wave and so we take $A_{To}^- = 0$. For a small mismatch ($\delta \approx \omega_s$) the parametric decay process can be adequately described by two coupled equations (75a, c). For this case the solution is obtained as in the previous section and the threshold electric field for the scattered electromagnetic and ion acoustic waves to become unstable is given by

$$\frac{e^2 |E^T_{yo}|^2}{\gamma_e m_e^2 \omega_{To}^2 v_{Te}^2} = 32 \frac{\nu_e}{\omega_{T1}} \frac{\gamma_s}{\omega_s} \tag{76}$$

and the growth rate well above the threshold is

$$\gamma = \frac{1}{8} \frac{\omega_{pe}}{\omega_{T1}} k_s c_s \frac{e|E^T_{yo}|}{m_e \omega_{To} v_{Te}} \left(\frac{\omega_{T1}}{\gamma_e \omega_s}\right)^{\frac{1}{2}} \tag{77}$$

where we have taken $\delta = \omega_s$ in both cases.

It is clear from the dispersion relations for transverse and ion acoustic waves in plasma that the forward scattering process can only occur under conditions of mis-match. The threshold field for forward scattering will therefore be higher than for the corresponding back scattering. Since the condition of perfect matching only allows back scattering, equation (76) is therefore the threshold condition for the back scatter decay. Stimulated Brillouin back-scattering can take place for all transverse frequencies from ω_{pe} upwards.

Equations (75) also give rise to growing wave solutions when the pump frequency is less than the unperturbed scattered wave frequency (the analogue of the oscillating two stream instability). The dispersion relation for this instability is of same form as equation (42). Although the threshold fields for this case are generally higher than for the minimum threshold for the decay instability, these solutions may well be important for fields well above threshold where many modes are excited by the pump. This process will produce a scattered wave at the pump frequency only in the case where the pump field is a standing wave.

4. CONCLUSIONS

We have shown how the theory of parametric instabilities in an unmagnetized plasma can be formulated very simply by means of coupled mode theory (the method can easily be generalized to a magnetized plasma). The coupled mode theory as developed here is valid only for weakly damped waves. In this form, one sees very clearly which modes are responsible for the various parametric interactions i.e. which modes grow and their frequency shifts. In this paper we have only attempted to calculate the threshold fields and initial growth rates of various parametric instabilities. However, a knowledge of which waves are responsible for each instability should be of some help in deciding on possible non-linear saturation mechanisms.

We have considered four basic parametric processes, (i) the Langmuir wave-ion acoustic interaction, (ii) the two Langmuir wave interaction, (iii) stimulated Raman scattering and (iv) stimulated Brillouin scattering. The Langmuir wave ion acoustic interaction has received the most attention where two possibilities have been discovered. The first is the ordinary parametric interaction where the pump frequency is greater than the Langmuir wave frequency and the second is the oscillating two stream instability where the pump frequency is below the Langmuir wave frequency. The characteristic feature of this second case is that the unstable ion modes are non-oscillatory. However, we have shown that this is only true for the special case of a standing wave pump. For a travelling wave pump the ion waves are oscillatory as is the case for the parametric interaction. It appears that the minimum number of modes the pump is required to couple in the two cases is one Langmuir and one ion acoustic for the parametric instability and one Langmuir and two ion acoustic for the oscillating two stream instability. This would seem to be the qualitative difference between the two cases.

The threshold fields for the four interactions considered depend very much on the plasma density and temperature. The Brillouin back scatter and Langmuir-ion acoustic decay instabilities have the same threshold when the Brillouin back scatter occurs near to the electron plasma frequency. Otherwise, the Brillouin process has the lower threshold. In general the Raman scattering and the decay of a transverse wave into two Langmuir waves have higher thresholds. However, at very high densities, the threshold for these effects can be comparable to or even lower than that for Brillouin scattering.

This analysis has been based on simple fluid equations which will give a reasonably accurate representation of the physics provided the phase velocities are high compared with the thermal velocities and the waves are lightly damped i.e. $T_e \gg T_i$. However, if these conditions are not satisfied a kinetic treatment, in which resonant particles are allowed for, should be used.

It is worthwhile noting the strong similarity between the instabilities induced by a steady field (or flow) and an oscillating field. In the former case there are two basic types of instability, the first due to a coupling of two modes of oscillation of the plasma and the second due to an interaction between one mode of the plasma and a group of resonant particles. The resonant particles can cause either wave growth or damping depending on their velocity distribution in the vicinity of the phase velocity of the wave. Similarly, in the case of a plasma driven by oscillating fields, there are instabilities due to the pump coupling two weakly damped waves or the pump coupling one wave with a group of resonant particles - so called non-linear Landau damping. This terminology is confusing because non-linear Landau damping can cause parametric growth or it can act as a damping mechanism. Dysthe (1970) has given general relations for the effect of non-linear Landau damping depending on whether the pump wave and the wave to which it couples have energy of the same or opposite sign and on whether the resonant particles represent an inverted (non-equilibrium) population or not. It has been customary to distinguish the two types of instabilities in steady fields e.g. the ion acoustic and two stream instabilities. However, in the literature on parametric effects these two types of instabilities have not, in general been distinguished - a situation which could cause some confusion.

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