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GROWTH OF AMBIPOLAR ELECTRIC FIELD IN A TOROIDAL PLASMA

by

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ABSTRACT

In the neoclassical theory of transport process in a toroidal plasma with long mean free path, the value of the poloidal rotation is determined by ambipolarity. This paper investigates the time-dependent phase during which this rotation develops. Diffusion arises from resonant particles, for which $v_{\parallel} \approx E_r/B_\theta$. The time a particle spends in resonance, and hence its radial displacement, is now determined by the rate of change of E_r, rather than collisional scattering. The time for the rotation to develop is found to be short compared with typical experimental durations. The contribution to the poloidal rotation of mass flow parallel to the magnetic field is small compared with that of the perpendicular electric drift.

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1. INTRODUCTION

Neoclassical theory[1-5] studies the transport processes in an equilibrium plasma, with long mean free path, confined by a toroidal magnetic field. This transport results from the charged particle drifts due to the curvature and gradients of the confining field. Quasi-neutrality demands that the net ion and electron fluxes across any magnetic surface be equal. This imposes a condition on the rotation in the poloidal direction. When the equilibrium is assumed to be time-independent, the usual condition is that the poloidal guiding centre rotation velocity $v_E + \Theta \bar{v}_{i\parallel} = - U_{ni}$. Here v_E is the electric rotation perpendicular to the magnetic field, $\bar{v}_{i\parallel}$ is the mean ion parallel flow, $U_{ni} = (dp_i/dr)/neB$ is the ion diamagnetic drift velocity, and $\Theta = B_\theta/B_\varphi$ is the ratio of poloidal and toroidal components of magnetic field.

The initial poloidal rotation in any experiment will depend on the detailed method of plasma production, but generally will not satisfy the ambipolar condition for a time-independent plasma. Thus the plasma equilibrium must pass through a time-dependent phase during which the poloidal flow builds up. Nearly all neoclassical theory is restricted to the final stationary state where these flows are fully developed. The only previous attempt [3] to include the time-dependent phase, by including the polarisation drift but ignoring all other time-dependent effects, suffers from obvious inconsistencies. When predicting the time-evolution of the density and temperature profiles by numerical solution of the time-dependent transport equations, it has been assumed that the transport coefficients at any time are those predicted by neoclassical theory for a stationary discharge, i.e. this assumes that the poloidal rotation adjusts itself virtually instantaneously to remain in equilibrium with the changing profiles.

This paper investigates the time-dependent phase consistently. The growth of diffusion-driven current is not considered, since the inductance associated with the self-field of this current raises problems not inherent in the mass flows. Sec.2 describes the model used and some assumptions. Sec.3 studies the basic problem of ion orbits in a toroidal field when the radial electric field increases with time. These orbit equations are used to find the velocity distribution function in Sec.4, where quasi-neutrality is invoked

to determine the poloidal electric field self-consistently. Sec.5 integrates the ion flux over a magnetic surface to determine the diffusion. Applying the ambipolar condition in Sec.6 gives the rate of growth of electric rotation needed to maintain equal ion and electron fluxes. The growth of the parallel ion flow is evaluated in Sec.7. Sec.8 compares the results with those of earlier analyses, and Sec.9 discusses the implications for typical confinement experiments. The analysis in the main text assumes $\partial v_E/\partial t$ to be positive. The modifications when $\partial v_E/\partial t$ is negative are given in Appendix A. Appendix B discusses limits on the validity of various approximations made in the analysis.

2. THE MODEL

We consider the simplest example of an axisymmetric toroidal magnetic field $\underline{B} = (R_o/R)B_o[0,\Theta(r),1]$. The usual (r,θ,φ) coordinates are used, where φ measures angular distance along the magnetic axis, and (r,θ) are polar coordinates in a plane perpendicular to the magnetic axis and centred on this axis. R_o is the radius of the magnetic axis and $R = R_o(1 + \epsilon \cos \theta)$, where $\epsilon = r/R_o$, is the distance from the axis of symmetry. The magnetic field lines lie on surfaces r = constant. The safety factor $q = \epsilon/\Theta$, which is the inverse of the rotational transform/2π, will be assumed of order unity. Thus $B_\theta = O(\epsilon B_o)$, and the field strength $B = B_o\ [1-\epsilon \cos \theta + O(\epsilon^2)]$.

The electrostatic field may be separated into its mean value over a magnetic surface $\Phi_o(r,t)$ and the variation about this mean value $\tilde{\Phi}(r,\theta,t) = \Phi_c(r,t)\cos \theta + \Phi_s(r,t) \sin \theta + O(\epsilon^2)$. Since the equilibrium is axisymmetric, there is no φ variation. It is the radial variation in mean potential which produces the electric rotation $v_E(t) = (\partial\Phi_o/\partial r)/B$, which is mainly in the poloidal direction.

We will assume the initial velocity distribution $f = f_o(r,v_\parallel,v_\perp{}^2)$ to be independent of θ. When a specific form must be taken, it is assumed to be a displaced Maxwellian with mean velocity $\bar{v}_{\parallel j}$ for the jth species. Its density $n_o(r)$, but not its temperature T, is allowed to vary radially. The initial radial potential profile could be regarded as arbitrary. However, for analytic convenience it will be taken as zero and the potential treated as increasing linearly with time, giving $v_E(r,t) = v_E'(r)t$. As will be seen later, the diffusion is dominated by those particles passing through resonance. Since the

behaviour of particles as they pass through resonance is governed by the instantaneous rate of increase in v_E, and is only weakly dependent on past history, the above approximation is reasonable.

When analysing the single particle motion in such a field, and the time evolution of the velocity distribution function, v_E' will be treated as a parameter as yet unspecified. To obtain the equation determining v_E' we must go to second order in ϵ, where we find the net particle flux across a magnetic surface. If it is assumed that there is no other source of diffusion for either ions or electrons, then v_E' is determined by the ambipolar condition.

3. SINGLE PARTICLE MOTION

The motion of a charged particle in toroidal magnetic field with a static radial electric field has been studied extensively[5]. We will now consider the motion when the radial electric field is time-dependent. The equations for the ion guiding centre motion are well known[3][5]

$$r \frac{d\theta}{dt} = v_E + \Theta v_{\parallel} + v_B \cos\theta \tag{1}$$

$$\frac{dr}{dt} = - v_B \sin\theta - \frac{1}{rB}\frac{\partial\tilde{\Phi}}{\partial\theta} - \frac{1}{\Omega_i}\frac{\partial v_E}{\partial t} \tag{2}$$

$$\begin{aligned}\frac{dv_{\parallel}}{dt} &= - \frac{1}{Mv_{\parallel}} (v.\nabla)(\mu B + e\Phi) \\ &= \frac{1}{R}\left[v_E v_{\parallel} - \frac{v_{\perp}^2\Theta}{2}\right]\sin\theta - \frac{e\Theta}{Mr}\frac{\partial\tilde{\Phi}}{\partial\theta}\end{aligned} \tag{3}$$

where

$$v_B = \frac{v_{\parallel}^2 + v_{\perp}^2/2}{\Omega_i R} \quad , \quad \Omega_i = \frac{eB}{M} .$$

The last term in Eqn.(2) is the inertial correction to the guiding centre drift. The v_B terms are the gradient and curvature drifts resulting from the toroidal nature of the magnetic field.

The orbit equations may be solved as a power series in ϵ. The range of validity of this expansion is discussed in Appendix B. To zero order in ϵ, the magnetic drift and the poloidal variation in potential both vanish, giving

$$r^o = r_o - v_E/\Omega_i \tag{4}$$

$$\theta^o = \theta_o + \Theta v_{\parallel} t + v_E' t^2/2 = \theta_1 + \frac{\pi}{2}\xi^2 \tag{5}$$

where $$\xi = \frac{v_E(t) + \Theta v_{\parallel}}{\sqrt{\pi r v_E'}} \quad , \quad \theta_1 = \theta_o + \frac{v_{\parallel}^2 \Theta^2}{2 r v_E'} \quad .$$

It is more meaningful to redefine θ_1 as the poloidal angle of the ion at the instant when $v_E(t) + \Theta v_{\parallel} = 0$, i.e., $\xi = 0$. Correct to first order in ϵ the solution of Eqn.(2) and (3) may be written

$$r = r_o - \frac{v_E}{\Omega_i} - \left(v_B - \frac{\Phi_c}{rB}\right)\left[\frac{\pi r}{v_E'}\right]^{\frac{1}{2}} \left\{[C(\xi) + \tfrac{1}{2}] \sin\theta_1 + [S(\xi) + \tfrac{1}{2}] \cos\theta_1\right\}$$
$$- \frac{\Phi_s}{rB}\left[\frac{\pi r}{v_E'}\right]^{\frac{1}{2}} \left\{[C(\xi) + \tfrac{1}{2}] \cos\theta_1 - [S(\xi) + \tfrac{1}{2}] \sin\theta_1\right\} \qquad (6)$$

$$v_{\parallel} = v_{\parallel o} - \Omega_i \Theta \left[r - r_o + \frac{v_E}{\Omega_i}\right] - \epsilon v_{\parallel} \cos\theta \qquad (7)$$

where
$$C(\xi) = \int_o^{\xi} \cos\left[\frac{\pi}{2}\xi^2\right] d\xi, \quad S(\xi) = \int_o^{\xi} \sin\left[\frac{\pi}{2}\xi^2\right] d\xi$$

are the Fresnel Integrals. The integration constants in Eqn.(6) & (7) are chosen so that r_o and $v_{\parallel o}$ correspond to the radial position and parallel velocity of the particle at early times, i.e. $\xi \to -\infty$. The above equations, and the following analysis, assumes $v_E' > 0$. The modifications when $v_E' < 0$ are discussed in Appendix A. Substituting the asymptotic forms for $C(\xi)$and $S(\xi)$ into Eqn.(6) gives the limiting expression for large (ξ)

$$r(t) - r_o + \frac{v_E(t)}{\Omega_i} = \frac{r}{(v_E + \Theta v_{\parallel})}\left[v_B \cos\theta - \frac{\tilde{\Phi}(\theta)}{rB}\right] \qquad (8)$$
$$- U(\xi)\left[\frac{2\pi r}{v_E'}\right]^{\frac{1}{2}} \left\{\left[v_B - \frac{\Phi_c}{rB}\right] \sin\left[\theta_1 + \frac{\pi}{4}\right] + \frac{\Phi_s}{rB} \cos\left[\theta_1 + \frac{\pi}{4}\right]\right\}$$

where $U(\xi) = 1$ when $\xi > 0$ and $U(\xi) = 0$ when $\xi < 0$, and θ is given in terms of t by Eqn.(5). The orbit of a typical particle whose $v_{\parallel} < 0$ is illustrated in Fig.1. Provided v_E' is not too large, ξ is initially large and negative. The oscillation described by the first term on the right-hand-side of Eqn.(8) is equivalent to the familiar displacement of the particle drift surface from the magnetic surface $r = r_o$. This is superimposed on the slow steady inward drift, $- v_E' / \Omega_i$. As v_E increases with time the displacement from the magnetic surface grows larger. The time during which $|v_E + \Theta v_{\parallel}| < \sqrt{\pi r v_E'}$, i.e., $|\xi| < 1$, will be referred to as the resonant phase for the particle. Its duration is of order $2\sqrt{\pi r / v_E'}$. During this phase the poloidal angle of the particle is localised within a quadrant, and it suffers a radial displacement of order $v_B \sqrt{\pi r / v_E'}$. The sign of the resonant displacement

is a function of $\theta_1 + \pi/4$, i.e., its mean poloidal angle during the resonant phase. After it passes through resonance the particle trajectory may again be described by Eqn.(8). The particle now oscillates around a new magnetic surface which differs from the original surface by the resonant displacement.

We will now briefly examine limitations on the validity of the above solution in real plasmas. In the conventional neoclassical theory of the plateau regime, the time a particle spends in resonance is limited by collisional scattering. Collisional scattering will still occur in the time dependent problem. Resonance occurs as the poloidal rotation velocity, $(v_E + \Theta v_{\parallel})/r$, passes through zero and persists so long as $|\theta - \theta_1| < \pi/2$. The expected increase in $v_E + \Theta v_{\parallel}$ due to small angle collision varies as $C_i \Theta \nu^{\frac{1}{2}} |t-t_1|^{\frac{1}{2}}$, where ν is the ion collision frequency for 90° scattering, c_r is the ion thermal velocity, and t_1 is the time at which $v_E + \Theta v_{\parallel} = 0$. This produces a mean displacement in azimuth of $\theta - \theta_1 = 2\, C_i \Theta \nu^{\frac{1}{2}}\, |t-t_1|^{3/2}/r$. Hence the time for collisions to scatter a particle out of resonance is of order[3]

$$\tau_c = \left[\frac{r^2}{\nu c_i^{\,2}\, \Theta^2}\right]^{\frac{1}{3}} . \tag{9}$$

Collisional scattering during resonance can be neglected if $\tau_c \gg \tau_E$, where $\tau_E = [2\pi r/v_E{}']^{\frac{1}{2}}$ is the time during which a particle passes through resonance due to change in v_E, i.e. if

$$\frac{r v_E'}{C_i^2 \Theta^2} \gg \left(\frac{\nu r}{C_i \Theta}\right)^{\frac{2}{3}} = \left(\frac{Rq}{\lambda_{mfp}}\right)^{\frac{2}{3}} . \tag{10}$$

Here qR/λ_{mfp} is the ratio of connection length to mean free path which is so important in neoclassical theory[1-5]. If the reverse inequality to Eqn.(10) is satisfied, then the change in radial electric field can be neglected.

When $qR/\lambda_{mfp} < \epsilon^{3/2}$, the change in $v_{\parallel}$ due to collisional scattering is less rapid than that caused by variation in magnetic field strength along a field line. In the time-independent equilibrium this gives rise to the banana regime[1,5]. The magnetic field variation is, of course, included in the evaluation of $v_{\parallel} - v_{\parallel o}$ in Eqn.(7). However, constant $v_{\parallel}$ is no longer a valid zero-order approximation to the exact orbit when particle trapping occurs. The change in $v_{\parallel}$ of a resonant particle due to the magnetic mirror effect may be treated as small only if the

particle trapping time, $\tau_T = r/\epsilon^{\frac{1}{2}} C_i\Theta$, is long compared with τ_E, if

$$\frac{r\, v_E'}{C_i^{\,2}\, \Theta^2} > \epsilon \ . \tag{11}$$

When $Rq/\lambda_{mfp} < \epsilon^{3/2}$, and v_E' violates the above inequality but satisfies Eqn.(10), then particle trapping occurs but the time a particle remains trapped is determined by the change in v_E rather than by collisional scattering out of the trapped velocity band. The analysis presented here can readily be modified to include this case.

4. THE VELOCITY DISTRIBUTION FUNCTION

The evolution of the velocity distribution function $f(r,\theta,v_{\parallel},v_{\perp},t)$ may readily be derived from the single particle orbits by invoking the constancy of f along a particle trajectory. Here r,θ are the guiding centre coordinates and $v_{\perp} = [2\mu B/M]^{\frac{1}{2}}$ is the perpendicular velocity relative to the guiding centre motion.

$$f(r,\theta,v_{\parallel},v_{\perp}^2,t) = f_o(r_o,v_{\parallel o},v_{\perp o}^2)$$

$$= f_o(r,v_{\parallel},v_{\perp}^2) - (r-r_o)\frac{\partial f_o}{\partial r} - (v_{\parallel}-v_{\parallel o})\frac{\partial f_o}{\partial v_{\parallel}} - \frac{\partial f_o}{\partial v_{\perp}}(v_{\perp}^2-v_{\perp o}^2) + O(\epsilon^2)$$

$$= f_o(r,v_{\parallel},v_{\perp}^2) + \frac{eB}{T_i}(U_{ni} + \Theta\,\bar{v}_{\parallel i} - \Theta\, v_{\parallel})\left(r - r_o + \frac{v_E}{\Omega_i}\right) f_o$$

$$+ \epsilon\left(v_{\parallel}\frac{\partial f_o}{\partial v_{\parallel}} - \frac{Mv_{\perp}^2}{2T_i} f_o\right)\mathrm{Cos}\ \theta + \frac{v_E}{\Omega_i}\frac{\partial f_o}{\partial r} + O(\epsilon^2) \tag{12}$$

where $U_{ni} = \frac{T_i}{eBn}\frac{dn}{dr}$ is the ion diamagnetic velocity. Use has been made of the relation between $v_{\parallel} - v_{\parallel o}$ and $r - r_o$ given in Eqn.(7). The change in the perpendicular energy of a particle follows immediately from conservation of magnetic moment, μ .

$$v_{\perp}^2 - v_{\perp o}^2 = 2\,\mu(B - B_o)/M = -\, v_{\perp}^2\epsilon(\mathrm{Cos}\ \theta - \mathrm{Cos}\ \theta_o) + O(\epsilon^2).$$

When combined with Eqn.(5) and (6) for $r-r_o+v_E/\Omega_i$, Eqn.(12) gives f explicitly as a function of t, the particle coordinates and velocity, and the poloidal electrostatic potential. v_E' may be regarded as an unspecified parameter whose value will later be determined self-consistently by the ambipolar condition.

The second form of Eqn.(12) assumes the initial ion velocity distribution to be a displaced Maxwellian. In order to reduce the analytic

detail, the mean parallel velocity will be put zero in the following expressions. Its inclusion is quite straightforward. So long as $\bar{v}_{i\parallel} \ll C_i$, its most important effect is to change v_E in most terms into the poloidal velocity $v_E + \Theta \bar{v}_{i\parallel}$.

The poloidal variation in electrostatic potential results from the differing electron and ion density perturbations produced by the magnetic drift. It may be determined self-consistently by applying the quasi-neutrality condition to the poloidal density variation. The ion density variation is obtained by integrating Eqn.(12) over velocity. The dependence of v_B on $v_\perp^2$ and $v_\parallel^2$ must, of course, be taken account of in this integration, as well as the dependence of θ_1 on $v_\parallel$ through ξ. When sin θ, and Cos θ, are expanded using $\theta_1 = \theta - \pi\xi^2/2$, the density variation separates naturally into components proportional to $\sin\theta$ and Cos θ.

$$\frac{n_i}{n_o} - 1 = \frac{\sqrt{\pi\epsilon}}{C_i\Theta}\left[G_+ - F_+\frac{e\Phi_c}{\epsilon T_i} - F_-\frac{e\Phi_s}{\epsilon T_i}\right]\sin\theta + \frac{\sqrt{\pi\epsilon}}{C_i\Theta}\left[G_- - F_-\frac{e\Phi_c}{\epsilon T_i} + F_+\frac{e\Phi_s}{\epsilon T_i}\right]\mathrm{Cos}\,\theta \tag{13}$$

where

$$F_+ = U_{ni}\,[Ss_o + Cc_o + \tfrac{1}{2}(c_o + s_o)] - \sqrt{\pi r v_E'}\,[Ss_1 + Cc_1 + \tfrac{1}{2}(c_1 + s_1)]$$

$$F_- = U_{ni}\,[Sc_o - Cs_o + \tfrac{1}{2}(c_o - s_o)] - \sqrt{\pi r v_E'}\,[Sc_1 - Cs_1 + \tfrac{1}{2}(c_1 - s_1)] \tag{14}$$

$$G_+ = \left(1 - 2\eta\frac{d}{d\eta}\right)F_+(\eta,\alpha),\quad G_- = \left(1 - 2\eta\frac{d}{d\eta}\right)F_-(\eta,\alpha) - \frac{2}{\pi^{\frac{1}{2}}}C_i\Theta \tag{15}$$

$$Ss_n(\eta,\alpha) = \int_{-\infty}^{\infty}(\xi-\alpha)^n e^{-\frac{\pi}{2}\eta(\xi-\alpha)^2}\,S(\xi)\,\sin\left(\frac{\pi}{2}\xi^2\right)d\xi$$

$$Cc_n(\eta,\alpha) = \int_{-\infty}^{\infty}(\xi-\alpha)^n e^{-\frac{\pi}{2}\eta(\xi-\alpha)^2}\,C(\xi)\,\mathrm{Cos}\left(\frac{\pi}{2}\xi^2\right)d\xi$$

$$Sc_n(\eta,\alpha) = \int_{-\infty}^{\infty}(\xi-\alpha)^n e^{-\frac{\pi}{2}\eta(\xi-\alpha)^2}\,S(\xi)\,\mathrm{Cos}\left(\frac{\pi}{2}\xi^2\right)d\xi \tag{16}$$

$$Cs_n(\eta,\alpha) = \int_{-\infty}^{\infty}(\xi-\alpha)^n e^{-\frac{\pi}{2}\eta(\xi-\alpha)^2}\,C(\xi)\sin\left(\frac{\pi}{2}\xi^2\right)d\xi.$$

$$s_n(\eta,\alpha) = \int_{-\infty}^{\infty} (\xi-\alpha)^n e^{-\frac{\pi}{2}\eta(\xi-\alpha)^2} \sin\left(\frac{\pi}{2}\xi^2\right) d\xi$$

$$c_n(\eta,\alpha) = \int_{-\infty}^{\infty} (\xi-\alpha)^n e^{-\frac{\pi}{2}\eta(\xi-\alpha)^2} \operatorname{Cos}\left(\frac{\pi}{2}\xi^2\right) d\xi$$

$$\eta = \frac{2rv_E'}{C_i^2\Theta^2} \quad , \quad \alpha = \frac{v_E}{[\pi r v_E']^{\frac{1}{2}}} .$$

The exponential factor in the integrals comes from the Maxwellian form assumed for f_o, while the powers of $(\xi-\alpha)$ arise from the dependence of f on $v_{\parallel} = (\pi r v_E')^{\frac{1}{2}} (\xi-\alpha)/\Theta$. When performing the η differentiation to obtain $G_{+,-}$ from $F_{+,-}$ the $\sqrt{\pi r v_E'}$ factor which appears explicitly in Eqn.(14) is treated as a constant.

We now need the poloidal variation in electron density. The condition that the resonant motion of electrons be dominated by collisional scattering rather than change in v_E may be obtained by analogy from Eqn.(10) for ions, i.e.,

$$\frac{rv_E'}{C_e^2\Theta^2} \ll \left(\frac{qR}{\lambda_{mfp}}\right)^{\frac{2}{3}} . \tag{17}$$

The electron and ion mean free paths are usually comparable, while the left hand side of the inequality is smaller for the electron by the mass ratio. It will be assumed that inequalities (10) and (17) are both satisfied, i.e., resonant ions are dominated by change in v_E while resonant electrons are dominated by collisional scattering. If Eqn.(17) is not satisfied, the electron behaviour can be derived from the ion results simply by changing the mass and sign of charge, but this situation is not relevant to present experiments.

Since the change in v_E has negligible effect on the resonant electron motion, the electron behaviour is correctly described by the neoclassical analysis for a stationary electric field. If the electrons are in the plateau regime, i.e. $\epsilon^{3/2} < qR/\lambda_{mfp} < 1$, the electron density variation may be obtained from Eqn.(13) of Reference 3,

$$\frac{n_e}{n_o} - 1 = \frac{e}{T_e}\left[\Phi_c \operatorname{Cos}\theta + \Phi_s \sin\theta - \dot{b}_e(\Phi_s \operatorname{Cos}\theta - \Phi_c \sin\theta)\right] + \epsilon b_e \sin\theta \tag{18}$$

where

$$b_j = \frac{\pi^{\frac{1}{2}}}{C_j\Theta}(U_{nj} + v_E) .$$

Provided the radial electric field is not close to the final stationary ambipolar value, $v_E = -(U_{ni} + \Theta\bar{v}_{\parallel i})$, the electron Landau term is negligibly small compared to the ion Landau term. In order to reduce analytic detail, the electron Landau term will be neglected, so that the electron density follows the Boltzmann distribution, $n_e/n_o - 1 = e\tilde{\Phi}/T_e$.

Equating the electron and ion densities gives

$$G_- - F_- \frac{e\Phi_c}{\epsilon T_i} + F_+ \frac{e\Phi_s}{\epsilon T_i} = \frac{C_i\Theta}{\sqrt{\pi}} \frac{e\Phi_c}{\epsilon T_e} \tag{19}$$

$$G_+ - F_+ \frac{e\Phi_c}{\epsilon T_i} - F_- \frac{e\Phi_s}{\epsilon T_i} = \frac{C_i\Theta}{\sqrt{\pi}} \frac{e\Phi_s}{\epsilon T_e} . \tag{20}$$

Hence

$$\frac{e\Phi_c}{\epsilon T_i} = \left[G_-\left(F_- + \frac{C_i\Theta}{\sqrt{\pi\tau}}\right) + G_+ F_+\right]\frac{1}{D^2} \tag{21}$$

$$\frac{e\Phi_s}{\epsilon T_i} = \left[G_+\left(F_- + \frac{C_i\Theta}{\sqrt{\pi\tau}}\right) - G_- F_+\right]\frac{1}{D^2} \tag{22}$$

where

$$D^2 = \left(F_- + \frac{C_i\Theta}{\pi^{\frac{1}{2}}\tau}\right)^2 + F_+^2 \quad , \quad \tau = \frac{T_e}{T_i} .$$

The G & F functions vary with time through their dependence on $\alpha = t[v_E'/\pi r]^{\frac{1}{2}}$. Equations (6), (12), (21) and (22) now completely determine f as a function of time, with v_E' occurring as a parameter as yet undetermined.

5. DIFFUSION FLUX

The diffusion is obtained by integrating the local guiding centre flux over a magnetic surface, r = constant. The surface element is $dS = r d\theta\, R_o(1 + \epsilon \cos\theta) d\varphi$

$$\Gamma_i = -\frac{1}{2\pi r B_o}\int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dv_\parallel \int_0^{\infty} \pi dv_\perp^2 \left[\frac{\partial\tilde{\Phi}}{\partial\theta} + \frac{\epsilon M}{e}\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right)\sin\theta + \frac{rBv_E'}{\Omega_i}\right] f(r,\theta,v,t)\,(1 + \epsilon\cos\theta)^2 \tag{23}$$

$$= -\frac{1}{2\pi r B_o}\int_{-\infty}^{\infty} dv_\parallel \int_0^{\infty} \pi dv_\perp^2 \frac{\epsilon M}{e}(v_\parallel^2 + \tfrac{1}{2}v_\perp^2)\int_0^{2\pi} d\theta\,(f - f_o)\sin\theta \tag{24}$$

$$- \frac{1}{2\pi r B_o} \int_o^{2\pi} d\theta \, \frac{\partial \tilde{\Phi}}{\partial \theta} \, n - \frac{\epsilon n_o}{r B_o} \Phi_s - \frac{n_o v_E'}{\Omega_i} + O(\epsilon^3) .$$

The second factor of $(1 + \epsilon \cos \theta)$ in Eqn.(23) comes from the B^{-1} dependence of the guiding centre velocity. All first order terms vanish when integrated over θ, leaving products of first order terms. The neglect of electron Landau terms, discussed in the preceding section, leads to a Boltzmann density distribution for the electrons, i.e., $n_i = n_e = n_o(1 + e\tilde{\Phi}/T_e)$. Consequently the second integral vanishes. The first integral is evaluated similarly to the density integral in the preceding section, using Eqn.(6) and (12) for $f - f_o$. One readily obtains

$$\Gamma_i = - \frac{\pi^{\frac{1}{2}} \epsilon^2 \rho_i n_o}{2 r \Theta} \left\{ \left[\left(1 - \frac{e\Phi_c}{2\epsilon T_i}\right) \left(1 - 2\eta \frac{d}{d\eta}\right) + 2\eta^2 \frac{d^2}{d\eta^2} \right] F_+ \right.$$

$$\left. - \frac{e\Phi_s}{2\epsilon T_i} \left[\left(1 - 2\eta \frac{d}{d\eta}\right) F_- - \frac{2C_i\Theta}{\pi^{\frac{1}{2}}} \right] \right\} - \frac{n_o v_E'}{\Omega_i} \qquad (25)$$

where $\rho_i = C_i/\Omega_i$ is the ion Larmor radius. This can be simplified somewhat using Eqn.(20) to give

$$\Gamma_i = - \frac{\sqrt{\pi}\, \epsilon^2 \rho_i n_o}{4r\Theta} \left[\left(1 - 2\eta \frac{d}{d\eta} + 4\eta^2 \frac{d^2}{d\eta^2}\right) F_+ + \frac{C_i\Theta}{\pi^{\frac{1}{2}}} \left(2 + \frac{1}{\tau}\right) \frac{e\Phi_s}{\epsilon T_i} \right.$$

$$\left. + 2\eta \frac{e}{\epsilon T_i} \left(\Phi_c \frac{\partial F_+}{\partial \eta} + \Phi_s \frac{\partial F_-}{\partial \eta} \right) + \frac{4\eta\, C_i\, \Theta^3}{\pi^{3/2}\, \epsilon^2} \right] . \qquad (26)$$

Substituting for Φ_s and Φ_c from Equations (21) and (22) gives

$$\Gamma_i = - \frac{\sqrt{\pi}\, \epsilon^2 \rho_i n_o}{4r\Theta} \left[\left(1 + 4\eta^2 \frac{d}{d\eta^2}\right) F_+ + \frac{1}{D^2} \left\{ F_+ H^2 - 4\eta H \frac{\partial F_+}{\partial \eta} \left(F_- + \frac{C_i\Theta}{\sqrt{\pi}\,\tau} \right) \right. \right.$$

$$\left. \left. - 4\eta^2 F_+ \left(\frac{\partial F_+}{\partial \eta} \right)^2 \right\} + \frac{4\eta\, C_i \Theta^3}{\pi^{3/2}\, \epsilon^2} \right] \qquad (27)$$

where

$$H = \frac{C_i\Theta}{\sqrt{\pi}} \left(2 + \frac{1}{\tau}\right) + 2\eta \frac{\partial F_-}{\partial \eta} .$$

We will now evaluate the electron diffusion. As discussed in the preceding section, in experimental conditions of interest the diffusion of resonant electrons is limited by collisional scattering. We can thus use the earlier analysis for a constant radial electric field. Using Equations (18) and (19) of Ref. 3 for the $v_{\parallel}^2$ and $v_{\perp}^2$ moments of the

electron distribution function and Eqn.(18) above, one readily obtains

$$\Gamma_e = \frac{\pi^{\frac{1}{2}} n_o T_e}{eBr} \frac{(v_E + U_{ne} + \Theta \bar{v}_{e\parallel})}{C_e \Theta} \left[\epsilon^2 + \epsilon \frac{e\Phi_c}{T_e} + \frac{e^2}{2T_e^{\,2}} (\Phi_s^2 + \Phi_c^2) \right]. \tag{28}$$

Terms of order $(v_E/C_e\Theta)^2$ have been omitted from Eqn.(28) as negligible. Substituting for Φ_c and Φ_s from Equations (21) and (22) gives

$$\Gamma_e = \frac{\pi^{\frac{1}{2}} \epsilon^2 n_o T_e}{eBr} \frac{(v_E + U_{ne} + \Theta \bar{v}_{\parallel})}{C_e \Theta} \left[1 + \frac{1}{\tau D^2} \left\{ G_-\left(F_- + \frac{C_i \Theta}{\pi^{\frac{1}{2}} \tau} \right) + G_+ F_+ \right\} + \frac{1}{2\tau^2 D^2} (G_+^2 + G_-^2) \right]. \tag{29}$$

The factor outside the square bracket is the plateau diffusion rate for the time independent cases obtained by Galeev and Sagdeev[2]. (The expression given in their original paper[1] is a factor 4 too large). The terms multiplied by D^{-2} arise from the poloidal electric field which is necessary to equalise the poloidal variation in ion and electron densities.

6. THE AMBIPOLAR CONDITION

Quasi-neutrality requires that the electron and ion density be very nearly equal. We have already invoked this in solving the first order equations, when the poloidal variation in ion and electron densities were equated. In order that the mean densities also remain equal, the net diffusion fluxes across any magnetic surface must be equal.

In the time-independent case, the component terms in Γ_i are larger than those in Γ_e by a factor $(MT_i/mT_e)^{\frac{1}{2}}$. Hence ambipolarity requires that the ion terms nearly cancel. The situation is similar in the time-dependent case. For example, for η small, $F_+ \approx U_{ni} + v_E - \eta\, C_i\Theta/\pi^{\frac{1}{2}}$ as may be verified later from Equations (30) - (35). We may thus determine v_E' to accuracy $(m/M)^{\frac{1}{2}}$ from the condition $\Gamma_i = 0$. To progress further we must evaluate the functions $F_{+,-}$ defined in Equation (14). The s_o and c_o integrals can be evaluated by expressing as a complete error integral with complex argument, and s_n and c_n derived by integrating by parts.

$$s_o(\eta,\alpha) = \frac{1}{(1+\eta^2)^{\frac{1}{2}}} \exp\left(\frac{-\alpha^2\eta}{1+\eta^2}\cdot\frac{\pi}{2}\right)\left\{[(1+\eta^2)^{\frac{1}{2}}+\eta]^{\frac{1}{2}} \sin\left(\frac{\pi}{2}\frac{\alpha^2\eta^2}{(1+\eta^2}\right)\right.$$

$$\left. +[(1+\eta^2)^{\frac{1}{2}}-\eta]^{\frac{1}{2}} \cos\left(\frac{\pi}{2}\frac{\alpha^2\eta^2}{(1+\eta^2)}\right)\right\} \tag{30}$$

$$c_o(\eta,\alpha) = \frac{1}{(1+\eta^2)^{\frac{1}{2}}} \exp\left(-\frac{\pi}{2}\frac{\alpha^2\eta}{(1+\eta^2)}\right)\left\{[(1+\eta^2)^{\frac{1}{2}}+\eta]^{\frac{1}{2}} \cos\left(\frac{\pi}{2}\frac{\alpha^2\eta^2}{1+\eta^2}\right)\right.$$

$$\left. -[(1+\eta^2)^{\frac{1}{2}}-\eta]^{\frac{1}{2}} \sin\left(\frac{\pi}{2}\frac{\alpha^2\eta^2}{1+\eta^2}\right)\right\} \tag{31}$$

$$s_1(\eta,\alpha) = -\frac{\alpha}{1+\eta^2}(s_o - \eta\, c_o) \tag{32}$$

$$c_1(\eta,\alpha) = -\frac{\alpha}{1+\eta^2}(c_o + \eta\, s_o). \tag{33}$$

The integrals of Fresnel integrals can be evaluated as a power series in $\alpha^2\eta\pi/2 = (v_E/C_i\Theta)^2$, which we will write as γ^2. The earlier time-independent analysis[1] shows the stationary ambipolar electric rotation to be $v_E = -(U_{ni} + \Theta\bar{V}_{\parallel i})$ when $\rho_{i\theta} < r_n$, where $\rho_{i\theta} = C_i/\Omega_i\Theta$ is the Larmor radius in the poloidal field and r_n the density scale length. For this case $\gamma \lesssim U_{ni}/C_i\Theta = \rho_{i\theta}/2r_n$ is always small. When $\rho_{i\theta} > r_n$, the stationary ambipolar rotation is[3] $v_E \approx (2.5\, C_i - \bar{V}_{i\parallel})\Theta$. In this case the assumption $\gamma < 1$ ceases to be valid during the final phase of the build-up. These integrals are found as follows

$$Ss_o + Cc_o = \int_{-\infty}^{\infty} d\xi \exp\left[-\frac{\pi}{2}\eta(\xi-\alpha)^2\right]\int_o^{\xi} dx \cos\frac{\pi}{2}(\xi^2-x^2)$$

$$= 2\pi\alpha\eta \exp\left(-\frac{\pi}{2}\eta\alpha^2\right)\int_o^{\infty} dx \int_o^{\infty} d\xi \exp\left[-\frac{\pi}{2}\eta\xi^2\right]$$

$$\cdot\, \xi\left[1+\frac{\pi^2}{6}(\alpha\eta\xi)^2 + \quad\right]\cos\frac{\pi}{2}(\xi^2-x^2)$$

$$= 2^{\frac{1}{2}}\frac{\alpha\eta^{3/2}}{(1+\eta^2)}\left[1-\frac{\pi\alpha^2\eta}{6}\left(\frac{1-3\eta^2}{1-\eta^2}\right)+o(\gamma^4)\right]\exp\left(-\frac{\pi}{2}\eta\alpha^2\right) \tag{34}$$

$$Cs_o - Sc_o = 2\pi\alpha\eta \exp\left(-\frac{\pi}{2}\eta\alpha^2\right) \int_o^\infty dx \int_x^\infty d\xi \exp\left(-\frac{\pi}{2}\eta\xi^2\right)\xi$$

$$\left[1 + \frac{\pi^2}{6}(\alpha\eta\xi)^2 + \right] \sin\frac{\pi}{2}(\xi^2 - x^2)$$

$$= \frac{2^{\frac{1}{2}}\alpha\eta^{\frac{1}{2}}}{(1+\eta^2)}\left[1 + \frac{\pi\alpha^2\eta}{6}\left(\frac{1+5\eta^2}{1+\eta^2}\right) + O(\gamma^4)\right]\exp\left(-\frac{\pi}{2}\eta\alpha^2\right). \quad (35)$$

The other integrals can be obtained most readily by differentiation, e.g.,

$$Ss_1 + Cc_1 = \frac{1}{\pi\eta}\frac{d}{d\alpha}(Ss_o + Cc_o), \quad Ss_{n+2} + Cc_{n+2} = -\frac{2}{\pi}\frac{d}{d\eta}(Ss_n + Cc_n).$$

Using these results, the following expressions are obtained for Γ_i in the limits $\eta \ll 1$ and $\eta \gg 1$.

$$\Gamma_i \approx -\frac{\pi^{\frac{1}{2}}\epsilon^2\rho_i n_o}{4r\Theta}\left[U_{ni} + v_E + \frac{2\eta}{\pi^{\frac{1}{2}}}\left(\frac{C_i\Theta^3}{\epsilon^2} + 4U_{ni}\gamma\right)\right.$$

$$+ \left\{(1+\tau)(U_{ni}+v_E) + \eta(1+\tau)(C_i\Theta + 2U_{ni}\gamma)\pi^{-\frac{1}{2}} + \eta b_i(U_{ni} + 3v_E)\right\}$$

$$\left.\times(1+\tau)\left\{(1+\tau)^2 + \tau^2 b_i^2 - 2\eta\tau^2 b_i - n\pi^{\frac{1}{2}}\tau^3 b_i^2(U_{ni} + 3v_E)/C_i\Theta(1+\tau)\right\}^{-1}\right] \quad (36)$$

$$\Gamma_i \sim -\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{\epsilon^2\rho_i n_o}{r\Theta}\left[\frac{U_{ni}}{\eta^{\frac{1}{2}}}\left\{\frac{3}{4} + \frac{(1+\tau)^2}{2X^2 + 2X + 1}\right\} + \frac{\eta}{(2\pi)^{\frac{1}{2}}}\frac{C_i\Theta^3}{\epsilon^2}\right] \quad (37)$$

where $X = \tau(\pi/2\eta)^{\frac{1}{2}} U_{ni}/C_i\Theta$. Terms of order γ^2 have been omitted, since $\gamma < 1$ is assumed in evaluating the integrals. In Eqn.(37), $\eta \gg (C_i\Theta/U_{ni})^2$ has been assumed, as well as $\eta \gg 1$. In the limit $\eta = 0$, Eqn.(36) agrees exactly with the corresponding result for the time-independent analysis, i.e., Eqn.(20) of Ref.(3) with $Z_i^2 = \gamma^2 = 0$.

When $U_{ni}/C_i\Theta = \rho_{i\theta}/2r_n < 1$, then $b_i < 1$ and the ambipolar condition $\Gamma_i \approx 0$ is satisfied by Eqn.(36) when

$$\eta \approx -2\sqrt{\pi}\frac{(U_{ni} + v_E)}{C_i\Theta}\left(1 + \frac{2\Theta^2}{\epsilon^2}\right)^{-1}. \quad (38)$$

Expressed in terms of basic parameters this is

$$\frac{1}{C_i\Theta}\frac{\partial v_E}{\partial t} = \frac{\sqrt{\pi}}{2}\frac{\rho_i C_i}{r\, r_n}\left(1 + \frac{v_E}{U_{ni}}\right)\left(1 + \frac{2}{q^2}\right)^{-1}. \quad (39)$$

When $\rho_{i\theta}/2r_n > 1$, Eqn. (36) has no zeros within its range of validity $\eta < 1$. Eqn.(37) gives the following zero of Γ_i in the range $\eta > 1$

$$\eta = \left[\frac{\rho_{i\theta}}{r_n}\left(\frac{\epsilon}{\Theta}\right)^2\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left\{(1+\tau)^2+\frac{3}{4}\right\}\right]^{2/3} \quad \text{if } \frac{1}{(1+\tau)^2}\left(\frac{\Theta}{\epsilon}\right)^2 < \frac{\rho_{i\theta}}{2r_n} < \frac{\epsilon}{\Theta}\frac{(1+\tau)}{\tau^{3/2}} \tag{40}$$

$$= \left[\frac{3}{4}\frac{\rho_{i\theta}}{r_n}\left(\frac{\epsilon}{\Theta}\right)^2\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\right]^{2/3} \quad \text{if } \frac{\rho_{i\theta}}{2r_n} > \frac{\epsilon}{\Theta}\frac{(1+\tau)}{\tau^{3/2}} . \tag{41}$$

When $(\Theta/\epsilon)^2(1+\tau)^{-2} < \rho_{i\theta}/2r_n < 1$, a zero of Γ_i occurs at $\eta = O(1)$ in addition to the zeros given by Equation (38) and (40).

Equations (39) - (41) gives the rate at which the electric rotation builds up towards its stationary ambipolar value. The diffusion rate during this build-up phase is found by substituting the appropriate value for η in Eqn.(29) for Γ_e. When $\rho_{i\theta}/2r_n < 1$ the terms proportional to D^{-2} in Eqn.(29) are of order $(\rho_{i\theta}/r_n)^2$. Hence diffusion starts at $\tau/(1+\tau)$ times the stationary ambipolar rate of earlier analysis, and approaches the stationary rate as $v_E \to -U_{ni}$. When $\rho_{i\theta}/2r_n > (\epsilon/\Theta)\ (1+\tau)/\tau^{3/2}$, the square bracket in Eqn.(29) approximately equals $1 + 2\ (1+\tau)/\tau$. This behaviour is reminiscent of that found in the time-independent analysis[3]. Here the effect of poloidal electric field was also found to be negligible so long as $\rho_{i\theta}/r_n < 1$. For sufficiently large values of $\rho_{i\theta}/r_n$, poloidal electric field produced an enhancement over Galeev-Sagdeev diffusion by a factor $[\tau^2+(2+\tau)^2]/2\tau$.

7. GROWTH OF PARALLEL FLOW

The rate of change of the mean ion flow along the magnetic field is

$$\frac{\partial \bar{v}_{i\parallel}}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{n_o}\oint\frac{d\theta}{2\pi}\int_{-\infty}^{\infty}dv_\parallel\int_0^\infty \pi\, d v_\perp^2\, v_\parallel f\right]$$

$$= \frac{1}{n_o}\oint\frac{d\theta}{2\pi}\int dv_\parallel\int\pi\, d v_\perp^2\left[\frac{dv_\parallel}{dt}f + v_\parallel\frac{df}{dt} - \frac{1}{n_o}\frac{\partial n_o}{\partial t}\ v_\parallel f\right]. \tag{42}$$

The last two terms in the integrand express the convection of momentum associated with the diffusion. As has been pointed out by Kovrizhnykh[6], because the diffusing particles all have the same parallel velocity, $v_\parallel = -v_E/\Theta$, they carry away a parallel momentum $-\Gamma v_E/\Theta$. Since this effect has already been studied[6], we shall now consider only the first term in the integrand.

Evaluation of the parallel ion acceleration due to interaction with the magnetic and electrostatic fields, by substituting for $dv_{\parallel}/dt$ and f from Eqn.(3), (6) and (12) is straightforward. Since the effect of electrostatic field on f is negligible when $\rho_{i\theta} < r_n$ and since, as will be seen later, the acceleration turns out to be relatively unimportant, the poloidal electrostatic field will be ignored for brevity. One then obtains

$$\frac{\partial \bar{v}_{i\parallel}}{\partial t} = \frac{\pi^{\frac{1}{2}}\epsilon^2 C_i}{2r}\left[\left\{U_{ni} - (U_{ni} + v_E)\eta\frac{\partial}{\partial\eta} + 2v_E\,\eta^2\frac{\partial^2}{\partial\eta^2}\right\}\left\{Cc_o + Ss_o + \tfrac{1}{2}(c_o + s_o)\right\}\right.$$

$$\left. - (\pi r v_E')^{\frac{1}{2}}\left\{1 + \frac{U_{ni}\,v_E}{C_i^2\,\Theta^2}\left(1 + \frac{2\,U_{ni}\,v_E}{C_i^2\,\Theta^2}\right)\eta\frac{\partial}{\partial\eta}\right\}\left\{Cc_1 + Ss_1 + \tfrac{1}{2}(c_1 + s_1)\right\}\right]$$

$$\approx \frac{\pi^{\frac{1}{2}}\epsilon^2 C_i}{2r}\left[U_{ni} + v_E - \frac{\eta\, C_i\,\Theta}{2\pi^{\frac{1}{2}}}\right] \qquad \text{if } \eta \ll 1 \tag{43}$$

$$\sim \frac{3}{2}\left(\frac{\pi}{2\eta}\right)^{\frac{1}{2}}\frac{\epsilon^2 C_i}{r}(U_{ni} + v_E) \qquad \text{if } \eta \gg 1\ . \tag{44}$$

The effect of a mean ion parallel velocity was omitted in earlier sections to reduce analytic detail. Is this consistent with the above result which shows the growth of such a velocity? Inclusion of $\bar{v}_{i\parallel}$ in the earlier analysis does not introduce any basically new effect, provided $\bar{v}_{i\parallel} \ll C_i$. Its main effect on the final expressions is to change v_E to $v_E + \Theta\,\bar{v}_{i\parallel}$. We now consider how the growth of $\bar{v}_{i\parallel}$ affects the evolution towards the stationary ambipolar state. In this state $v_E + \Theta\,\bar{v}_{i\parallel}$ must equal $-U_{ni}$ or $2.5\,C_i\Theta$, depending on whether $\rho_{i\theta}/r_n$ is $<$ or $>$ 1. We can now evaluate the relative magnitudes of the two components of the acquired poloidal velocity. Comparing Equations (39) and (43), and Equations (41) and (44) gives

$$\frac{\partial \bar{v}_{i\parallel}}{\partial t}\Big/\frac{\partial v_E}{\partial t} \approx -\epsilon^2 \quad \text{if} \quad \rho_{i\theta} < r_n \tag{45}$$

$$\approx -\Theta^2 \quad \text{if } \rho_{i\theta} > r_n\ . \tag{46}$$

Hence we can neglect the growth of the parallel contribution to the poloidal velocity when considering the build-up towards steady rotation.

We shall now examine whether the same is true of the other source of parallel flow, i.e. due to momentum carried out by resonant particles. For $v_E \approx v_E' t$ and times short compared with the confinement time

$\tau_c = n/(\partial n/\partial t)$, one finds from the expressions given in Reference 6 that the parallel velocity increases as $\bar{v}_{i\parallel} \approx v_E' t^2/2\tau_c\Theta$. During the initial build up of E_r the relative magnitude of the two components of poloidal rotation is $\Theta\,\bar{v}_{i\parallel}/v_E \approx t/2\tau_c$. As we shall see in Sec.9, the build-up time for E_r is much less than τ_c. Hence the growth of $\bar{v}_{i\parallel}$ due to momentum transport is also negligible during the initial growth of rotation.

We now consider what happens after the initial growth of E_r. As pointed out by Kovrizhnykh[6], the momentum transport by diffusing ions produces a steady growth of $\bar{v}_{i\parallel}$ during the life of the plasma. In order to maintain ambipolar rotation, i.e., $v_E + \Theta\,\bar{v}_{i\parallel} = -U_{ni}$ or $2.5\,C_i\Theta$, v_E must decrease with time. Is this change in v_E sufficiently rapid that the time an ion spends in resonance is still determined by the change in v_E, rather than collisions? If diffusion proceeds at the neoclassical rate for the plateau regime, the rate of change of v_E gives $\eta = \epsilon^2(\rho_{i\theta}/r_n)^3(m/M)^{\frac{1}{2}}$. From Eqn.(10) collisions are dominant if $\eta < (Rq/\lambda_{mfp})^{2/3}$. Provided $\rho_{i\theta}/r_n$ is not too large, this is likely to be satisfied.

8. COMPARISON WITH EARLIER ANALYSES

Reference (3) included the polarisation flux $-(n_o/\Omega_i)\,dv_E/dt$ in the ion diffusion. However, the time dependence of v_E was neglected when solving the kinetic equation for f. As was pointed out[3], this is not generally valid for velocities close to resonance, i.e., for the very particles responsible for the diffusion. However, it did provide a time-dependent term to balance the difference between ion and electron diffusion fluxes during build-up. The result deduced for v_E' may be obtained from Eqn.(39) by dropping the unity in the bracket $(1 + 2/q^2)$. Since usually $q \geqslant 3$, the earlier estimate is of an order of magnitude too large.

A qualitatively similar growth of rotation is predicted for the collision-dominated plasma. Early treatments of the pure-resistive and resistive-viscous plasma[7-8] included the polarisation current while neglecting time dependent terms in the first order equations. This corresponds closely to the early analysis for the kinetic range referred to above. It is interesting to observe that when the resistive case was later treated consistently[9-10], including all time dependent terms, the rotational growth rate when $\rho_{i\theta} < r_n$ was found to be reduced by just the same factor $(1 + q^2/2)$ compared to the earlier estimates. When

$\rho_{i\theta} > 2r_n$ a fundamental change was found[9] in the mechanism determining v_E'. The ion polarisation flux, which is a relatively small term when $\rho_{i\theta} < r_n$, now becomes a dominant one. This behaviour is similar to that found here for the kinetic regime as $\rho_{i\theta}/r_n$ changes from a small to a large quantity.

9. APPLICATION TO EXPERIMENTS

We now consider how long it takes for the rotation to build up to its stationary ambipolar value. When $\rho_{i\theta} < r_n$, the stationary rotation is $v_E \approx - U_{ni}$. Eqn.(39) gives the order of magnitude of this build-up time τ_b to be

$$\frac{1}{\tau_b} \sim - \frac{1}{U_{ni}} \frac{\partial v_E}{\partial t} \sim \pi^{\frac{1}{2}} \frac{C_i \Theta}{r} = \pi^{\frac{1}{2}} \nu_i \left(\frac{\lambda_{mfp}}{qR} \right) . \qquad (47)$$

Outside the collision-dominated range $\lambda_{mfp} > qR$, and so the build-up time is always less than the ion collision time. Thus the rotation should have adequate time to reach its stationary value within the duration of a typical Tokamak discharge.

The time-independent neoclassical diffusion in stellarators has been shown to be approximately the same as in an analogous Tokamak, provided the helical component is not too strong or the collision frequency too low[3]. The same correspondence should apply to the time-dependent phase. The condition typical of stellarators is $\rho_{i\theta} \gg r_n$, and the stationary ambipolar rotation is $v_E \sim 2.5\ C_i \Theta$. Eqn.(41) gives the order of magnitude of the build-up time to be

$$\frac{1}{\tau_b} \sim \frac{1}{2.5\ C_i \Theta} \frac{\partial v_E}{\partial t} \sim \frac{C_i \Theta}{5r} \left(\frac{\rho_{i\theta}}{r_n} \right)^{\frac{2}{3}} q^{4/3} = \frac{\nu_i}{5} \left(\frac{\lambda_{mfp}}{qR} \right) \left(\frac{\rho_{i\theta}}{r_n} \right)^{\frac{2}{3}} q^{4/3} . \qquad (48)$$

Thus the build-up time should again be shorter than an ion collision time.

10. CONCLUSIONS

The diffusion process during the build-up of electric rotation v_E towards its stationary ambipolar value differs significantly from that in the stationary state. The time an ion spends in resonance is now limited by $\partial v_E/\partial t$ rather than collisional scattering. As a consequence the magnitude of the radial displacement during resonance is less and the distortion of the velocity distribution from Maxwellian is quite different. In spite of this, the ion diffusion comes out to

be rather similar to the Galeev-Sagdeev flux for the plateau regime[2], provided $\partial v_E/\partial t$ is not too large. It differs by the addition of terms proportional to $\partial v_E/\partial t$. For larger values of $\partial v_E/\partial t$, the form of the ion diffusion is quite different.

In conditions of experimental interest the electrons are not directly affected by the time dependence of v_E. Their diffusion may be influenced indirectly through the poloidal electrostatic field which couples them to the ions. This effect is negligible when $\rho_{i\theta} \ll r_n$, and even when $\rho_{i\theta} > r_n$ the effect is not large. Equating ion and electron diffusion fluxes determines $\partial v_E/\partial t$. The time taken by the rotation to build-up to its stationary value may be seen to be always less than the ion-ion collision time. Thus the assumption made in the numerical simulation of Tokamaks using neoclassical transport equations, that the rotation maintains itself at the stationary value corresponding to the instantaneous value of the discharge parameters, should not lead to significant errors.

Since the ion diffusion process during build-up is not affected by collisions, most of the foregoing analysis is equally valid at very low collision frequencies. Due to the variation in rotation, ions will not remain in resonance long enough to complete a banana orbit, so the usual distinction between banana and plateau regimes disappears. The expressions obtained for $\partial v_E/\partial t$ are still valid. The only major difference is that the ambipolar diffusion rate during build-up is now given by the electron diffusion rate for the banana regime.

It follows that ambipolar rotation would still develop in the hypothetical case of a completely collisionless plasma, even though the diffusion vanishes. This result is physically plausible. Those ions for which $v_{\parallel} = 0$ initially would certainly suffer a net radial displacement due to their magnetic drift. Quasi-neutrality requires a radial electric field to develop so that the net resonant ion flux is balanced by a small inward polarisation drift of the entire ion distribution. The resonant velocity is thus shifted to $v_{\parallel} = - v_E/\Theta$, and so a new group of ions suffer a net radial displacement. This process would continue until, when v_E reached U_{ni}, the net flux of resonant ions vanishes.

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Appendix A

Analysis for $\partial v_E/\partial t < 0$

The foregoing analysis assumed $\partial v_E/\partial t$ to be positive. The value of $\partial v_E/\partial t$ determined from the ambipolar condition $\Gamma_i = \Gamma_e$ are consistent with this assumption, since U_{ni} has the sign of dn/dr, i.e., negative for a confined plasma. We should, however, check whether the corresponding expressions for negative $\partial v_E/\partial t$ permit other time-dependent ambipolar solutions.

The analysis for $v_E' < 0$ follows identical lines to that for $v_E' > 0$. Only the key results will be quoted. Defining $\xi = [v_E(t) + \Theta v_{\parallel}] [\pi r|v_E'|]^{\frac{1}{2}}$, the particle orbit is given by

$$\theta^{(0)} = \theta_1 - (\pi/2)\xi^2$$

$$r = r_o - \frac{v_E}{\omega_i} + \left(v_B - \frac{\Phi_c}{rB}\right)\left[\frac{\pi r}{|v_E'|}\right]^{\frac{1}{2}}\left\{[C(\xi) - \tfrac{1}{2}]\sin\theta_1 - [S(\xi) - \tfrac{1}{2}]\cos\theta_1\right\}$$

$$+ \frac{\Phi_s}{rB}\left[\frac{\pi r}{|v_E'|}\right]^{\frac{1}{2}}\left\{[C(\xi) - \tfrac{1}{2}]\cos\theta_1 + [S(\xi) - \tfrac{1}{2}]\sin\theta_1\right\}.$$

Equations (25) - (27) for the diffusion are still valid if F_+ and F_- are redefined over negative v_E' as

$$F_+ = U_{ni}\,[\tfrac{1}{2}(c_o + s_o) - (Ss_o + Cc_o)] - [\pi r|v_E'|]^{\frac{1}{2}}\,[\tfrac{1}{2}(c_1 + s_1) - (Ss_1 + Cc_1)]$$

$$F_- = U_{ni}\,[Sc_o - Cs_o - \tfrac{1}{2}(c_o - s_o)] - [\pi r|v_E'|]^{\frac{1}{2}}\,[Sc_1 - Cs_1 - \tfrac{1}{2}(c_1 - s_1)]$$

where s_n, c_n, Ss_n, Cc_n, etc are still defined in terms of α and η by Eq.(15), but now $\alpha = v_E[\pi r|v_E'|]^{-\frac{1}{2}}$ and $\eta = 2r|v_E'|/C_i^2\theta^2$.

The small η expression for Γ_i differs from Eqn.(36) in that the sign of all terms proportional to η is changed. In other words, when η is expressed in terms of v_E', the same expression is valid for $v_E' < 0$ and $v_E' > 0$. The large η expression for Γ_i now differs from Eqn.(37) in that the denominator $2X^2 + 2X + 1$ is replaced by $2X^2 - 2X + 1$ and the sign of the term proportional to η is changed. It may readily be seen that there are no zeros of Γ_i within the range of validity of either expression. We conclude that it is not possible to satisfy the ambipolar condition with v_E' negative.

Appendix B

Limits of Validity

Comparing the effect of collisional scattering and changing v_E in determining the time spent in resonance led to Eqn.(10) as a condition for the neglect of collisions. The distribution function for particles which have already passed through resonance includes rapidly oscillatory function of $v_{\parallel}$. This may be seen from Equations (6) and (12), remembering that $\theta_1 = \theta + (v_E + \Theta v_{\parallel})^2/2rv_E'$. The smoothing effect of collisions on these oscillations may be determined by including a Fokker-Planck collisional term and solving the resulting equation as a power series in the collision frequency. Because the contribution of the oscillations to the diffusion integral is largely self-cancellatory, their smoothing-out does not greatly affect the diffusion. The effect on the collisionless diffusion expression may be shown to be small if

$$\frac{rv_E'}{C_i^2\,\Theta^2} > \left(\frac{R}{t\lambda_{mfp}}\right)^{\frac{1}{3}}\left(\frac{\rho_{i\theta}}{r_n}\right)^{\frac{1}{3}}. \qquad \text{(B1)}$$

We now consider the validity of the ϵ expansion used in solving Equations (1) - (3) for the particle trajectories. The approximation which seems likely to break down first is that $v_{\parallel}$ in the right-hand-sides may be treated as constant when integrating the equations for the first order corrections to $\theta(t)$, r, and $v_{\parallel}$. A non-linear treatment shows that those ions which pass through resonance while in the upper half plane ($0 < \theta_1 < \pi$) spend a slightly longer time in resonance and consequently suffer a larger radial displacement. This is because for these particles $dv_{\parallel}/dt$ is negative during resonance and consequently $v_E + \Theta v_{\parallel}$ changes more slowly. The change in $v_{\parallel}$ during resonance also leads to a smearing out of the trajectories, because $v_{\parallel} - v_{\parallel o}$ depends on θ_1 and so ions with the same $v_{\parallel o}$ can overtake each other. This tends to smooth out the oscillatory part of f mentioned above. The first of the above non-linear effects may be evaluated as follows. Equations (1) and (3) may be integrated including the time dependence of $v_{\parallel}$ to give

$$\frac{r}{2}\left(\frac{d\theta}{dt}\right)^2 = v_E'\,\theta + \frac{\Theta}{r}\left(\frac{v_\perp^2\Theta}{2} - v_E\,v_{\parallel}\right)\cos\theta - \frac{e\theta^2}{Mr}\,\tilde{\Phi} + \text{const.} \qquad \text{(B2)}$$

If the object is to estimate the effect, rather than evaluate it completely, the $\tilde{\Phi}$ term can be omitted. When $\rho_{i\theta} < r_n$ it is expected to have negligible effect anyway. The term $v_E\,v_{\parallel}$ makes a contribution of order $(v_E/C_i\,\Theta)^2$

for particles near resonance, since $v_{\parallel} \approx - v_E/\Theta$, and will be dropped. The exact solution is then an elliptic integral. Since we are primarily interested in the resonance phase, θ will instead be expanded in powers of $\theta - \theta_1$, where θ_1 is the poloidal angle when the ion is exactly in resonance, i.e., $d\theta/dt = 0$. The orbit solution is then

$$\theta = \theta_1 + (v_E'/2r)\ (1 - \beta \sin \theta_1)\ (t - t_1)^2 \qquad \text{(B3)}$$

where $\beta = \epsilon/\eta$.

Eqn. (B3) may now be used instead of Eqn.(5) as the zero order equation for θ when integrating the orbit equations for $r - r_o$ and $v_{\parallel} - v_{\parallel o}$. This approximation becomes less good as the particle moves further from resonance, but the diffusion is dominated by the resonant displacement. The sole effect is to replace v_E' by $(1 - \beta \sin \theta_1)v_E'$. The variable ξ in the orbit equations is now defined in terms of t by

$$\xi = (v_E'/\pi r)^{\frac{1}{2}}\ (1 - \beta \sin \theta_1)^{\frac{1}{2}}\ (t - t_1),$$

while $v_{\parallel}(t)$ is determined in terms of $v_{\parallel o}$ and ξ by the orbit equation. In addition to factors $(1 - \beta \sin \theta_1)^{\frac{1}{2}}$, the diffusion integral now contains additional terms coming from the transformation from $v_{\parallel}$ to ξ. These only introduce corrections of order β^2 to the earlier result for the ion diffusion. Thus the ϵ expansion of the orbit equations is valid as long as $\eta < \epsilon$.

Finally we consider the expansion of the distribution as a Taylor series in $r - r_o$ and $v_{\parallel} - v_{\parallel o}$ in Eqn.(12). The validity of such an expansion requires $r - r_o < r_n$ and $v - v_o < c_i$. As may readily be seen from Eqn.(6) and (7), this is equivalent to $\eta < \epsilon^2(\rho_{i\theta}/r)^2$ and $\eta < \epsilon^2$.

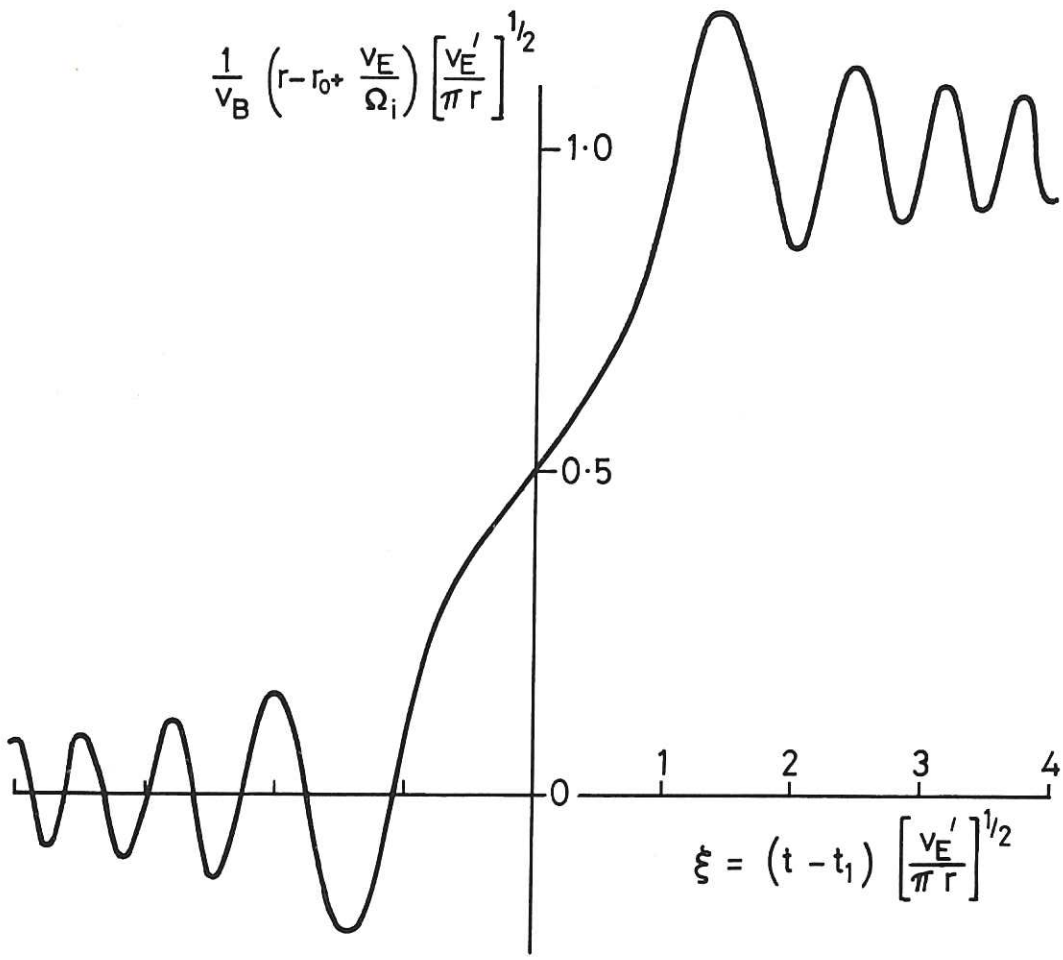

Fig.1. Displacement of the guiding centre from a magnetic surface vs time.

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