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THE WAVE VELOCITIES CAUSED BY FINITE LARMOR RADIUS EFFECTS FOR MHD INSTABILITIES IN A PINCH DISCHARGE

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THE WAVE VELOCITIES CAUSED BY FINITE LARMOR RADIUS EFFECTS FOR MHD INSTABILITIES IN A PINCH DISCHARGE

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ABSTRACT

Consideration is given to the types of MHD instability expected in the diffuse toroidal pinch discharges observed in the Sceptre experiments. In the central region where there is little or no magnetic shear but an appreciable negative pressure gradient, the instability will be either a pure interchange mode, or a quasi interchange mode with properties very similar to the pure interchange. In the outer region where there is some magnetic shear, a kink instability is expected. The wave velocity which will be imparted to the interchange mode due to the finite Larmor radius effects is computed with all the first order effects included. In the few cases where sufficient experimental data is available, there is agreement in sign and magnitude with the observed velocities for fluctuations in the discharge core. There is insufficient experimental data for the outer region, but the wave velocity imparted to the kink instability due to the Hall effect only is shown to be sensitive to the orientation of $\underset{\sim}{k}$ with respect to $\underset{\sim}{B}$. This explains why the observed wave velocity is often of opposite sign in the outer region.

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C O N T E N T S

1. INTRODUCTION

In an earlier paper (WARE, 1961a) it was shown that if the Hall effect is retained in the plasma electric conduction equation and the electron temperature is small compared with the ion temperature, the magnetic field will be trapped to the mean velocity of the electrons and not to the mass velocity of the plasma. Qualitative arguments were put forward to show that this would lead to the propagation of magnetohydrodynamic instabilities as waves and a simple formula for the velocity of such waves was derived. The preceding three papers (ALLEN, 1964; WILLIAMS, 1964; ALLEN et al, 1964) give details of experiments carried out on the Sceptre discharges to detect and study such waves. Preliminary experimental results have already been published (ALLEN, 1960; WILLIAMS, 1961) and it has been shown (WARE, 1963) that the wave velocity formula predicts both the correct direction and the right order of magnitude for most of the cases, as it does for the helical instabilities observed in other pinch discharges.

The Hall effect term is one of several transverse magnetic effects in the plasma equations whose magnitudes relative to the other terms can be shown to contain the factor ρ_i/L, where ρ_i is the ion Larmor radius and L is the characteristic length associated with gradients in the discharge, (provided ωL is of the order of the ion thermal velocity and collision frequencies are negligible). The other terms are :-

(a) the electron pressure gradient e.m.f. in the electric conduction equation;

(b) the viscosity term in the momentum balance equation due to the transport of momentum perpendicular to the magnetic field; and

(c) the heat conduction term in the thermal energy equation due to the transport of heat perpendicular to the magnetic field.

In recent years several workers have derived the stability of plasmas with one or more of these terms included but not all of them. Although in all cases these extra terms lead to a real part of ω, and therefore to wave motion, interest has been directed primarily at the stabilizing effect of this wave motion (ROSENBLUTH et al, 1962; LEHNERT, 1962; TAYLER, 1963; ROBERTS and TAYLOR, 1962; RUDAKOV, 1962; MIKHAILOVSKII, 1962), or to the new instability modes resulting from the extra terms (TSERKOVNIKOV, 1957; KADOMTSEV, 1959; RUDAKOV and SAGDEEV, 1959 and 1962). An exception is the recent paper by DAMM et al (1963), in which the wave velocities predicted for a very low density plasma contained in a mirror magnetic field are shown to be in reasonable agreement with the experimental results.

In this paper we first consider the types of magnetohydrodynamic instability predicted by theory for the diffuse pinch discharges observed in Sceptres III and IV (Section 2). For this purpose we utilize the results of another paper by the author (WARE, to be published) in which approximate growth rates are obtained for the diffuse pinch instabilities, following the procedure adopted by NEWCOMB (1961) for the corresponding gravitational instabilities. The significant feature of these calculations is that the compressibility of the plasma is not neglected and, as a result, a fourth-order dispersion relation is obtained involving two modes of instability.

The experimental results show that the discharge has two regions which often behave independently as far as instabilities are concerned; there is a central core and an outer annular region. In the core, which has little or no magnetic shear but appreciable pressure gradient, the expected instability is either an interchange or a quasi-interchange mode. The observed fluctuations agree with the properties of these modes. The fluctuations observed in the outer region are consistent with a kink instability whose helical pitch coincides with the magnetic pitch in the vicinity of the boundary between the two regions and whose displacement amplitude is zero within this radius. In those cases where the fluctuations are regular in time and of approximately constant amplitude, the results can be explained by a helical equilibrium configuration produced in the outer region by the kink instability.

In Section 3.1 the wave velocity is obtained for an interchange instability in a constant magnetic pitch discharge with all the finite Larmor radius effects included except the electron pressure gradient e.m.f. The latter is neglected because the electron temperature is assumed to be small compared with the ion temperature, as suggested by the experimental results for the Sceptre discharges. In Section 3.2 the wave velocity is obtained for the quasi-interchange and kink instabilities for the case where the Hall effect is the only significant finite Larmor radius effect. Although there are conditions under which this will be the dominant effect, primarily the other terms are omitted for mathematical simplicity, and because the lack of experimental data for the outer region prevents a quantitative comparison between theory and experiment.

In the various cases, the dependence of wave velocity on the discharge parameters is found to be more complex than in the simple formula derived qualitatively from field trapping. In the few cases where the experimental parameters are known, the theoretical wave velocities agree well in both sign and magnitude with the observed velocities (Section 4).

2. MAGNETOHYDRODYNAMIC STABILITY

Earlier experimental results (ALLEN and LILEY, 1959) indicated that there are two distinct regions (WARE, 1961b) in the Sceptre discharges. There is an inner central core with diameter roughly half the tube diameter and within which the magnetic pitch is either constant or, varies only slowly with radius. The negative pressure gradient in this region is of the order of but somewhat larger than that for marginal interchange stability (WARE, 1961b) which is

$$-\gamma p B_\theta^2/2\pi r(B^2/4\pi + \gamma P)$$

(an alternative form is $\frac{\gamma}{B_z}\ \frac{dB_z}{dr}$). The second region is the outer annular part of the discharge where the magnetic pitch decreases fairly rapidly with radius to a small value near the wall. (The percentage magnetic field fluctuations are larger in this region (ALLEN et al, 1962) and reliable pressure gradients cannot be deduced.)

The fact that markedly different velocities (often with different sign) are observed for the fluctuations in these two regions is evidence for different instability modes occurring simultaneously in the two regions. Because of this and the different properties of the two regions it is appropriate to consider their stability separately.

2.1 The Discharge Core

In the case of weakly or moderately pinched discharges ($B_\theta < B_z$) with little or no magnetic shear, it can be shown from the energy principle that instability occurs for only small values of the wave vector component $k_\parallel$ ($\equiv \underset{\sim}{k}.\underset{\sim}{B}/B$) such that $|k_\parallel| \ll |k|$. Using this approximation and neglecting finite Larmor radius effects, first order normal mode analysis of the standard MHD equations,

$$nM \frac{d\underset{\sim}{v}}{dt} = \underset{\sim}{j} \times \underset{\sim}{B} - \nabla p \;, \quad \underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B} = 0 \;, \quad \frac{dp}{dt} + \Upsilon p \; \nabla\cdot\underset{\sim}{v} = 0 \qquad \ldots (1)$$

and Maxwell's equations leads to the dispersion relation (WARE, to be published)

$$a\omega^4 + b\omega^2 + c = 0 \qquad \ldots (2)$$

where the symbols in equations (1) have their usual meaning and where

$$\left.\begin{aligned} a &= \rho^2(B^2/4\pi + \Upsilon p) \\ b &= -\frac{2B_\theta^2}{rB^2}\,\rho\left[\left(\frac{B^2}{4\pi} + \Upsilon p\right)\frac{dp}{dr} + \frac{\Upsilon p\, B_\theta^2}{2\pi r} - O\left(\frac{k_\parallel\, B_\theta^2\, B^2}{k\,(4\pi)^2 r}\right)\right] \\ c &= \Upsilon p\,\frac{k_\parallel^2\, B^2}{4\pi r}\left[2\,\frac{B_\theta^2}{B^2}\,\frac{dp}{dr} - O\left(\frac{k_\parallel\, B_\theta^2}{k\, 4\pi r}\right)\right] \equiv \Upsilon p\,\frac{k_\parallel^2\, B^2}{4\pi}\left(\frac{\delta W}{\xi_r^2}\right) \end{aligned}\right\} \qquad \ldots (3)$$

where $k \equiv (k_z^2 + (m^2/r^2))^{1/2}$ and δW is the intergrand of the energy principle after the usual minimisation for ξ_θ and ξ_z. Plasma displacements of the form $\underset{\sim}{\xi}\exp i\,(\omega t - m\theta - k_z z)$ have been assumed and $\nabla\cdot\underset{\sim}{\xi}$ has been retained. The common practice of assuming $\left|\frac{1}{\xi_r}\,\frac{\partial \xi_r}{\partial r}\right| \ll k$ has been followed. This is a reasonable approximation for $m > 1$, since for small or no shear one expects $\left|\xi_r/(\partial\xi_r/\partial r)\right| \lesssim r_o/2$ and since $2\pi m/k_z$ is approximately equal to the magnetic pitch which is observed to be of the order $2r_o$ for the core, (r_o is the tube radius). It is only a rough approximation for $m = 1$. (Electromagnetic units are used throughout this paper.)

Provided there is an appreciable difference between $-\frac{dp}{dr}$ and $\Upsilon p B_\theta^2/2\pi r(B^2/4\pi + \Upsilon p)$ there will be one large and one small root for ω^2, namely

$$\omega_{10}^2 \simeq -\frac{b}{a} = \frac{2\,B_\theta^2}{r\rho B^2}\left[\frac{dp}{dr} + \frac{\gamma p\; B_\theta^2}{2\pi r(B^2/4\pi + \gamma p)} - 0\left(\frac{k_{\parallel}\; B_\theta B_z}{k\; 4\pi r}\right)\right] \qquad \ldots (4)$$

$$\omega_{20}^2 \simeq -\frac{c}{b} = \frac{\gamma p\; k_{\parallel}^2}{\rho}\left(\frac{B^2/4\pi}{B^2/4\pi + \gamma p}\right)\left[\frac{\dfrac{dp}{dr} - 0\left(\dfrac{k_{\parallel}\; B_\theta^2}{k\; 4\pi r}\right)}{\dfrac{dp}{dr} + \dfrac{\gamma p\; B_\theta^2}{2\pi r(B^2/4\pi + \gamma p)}}\right] \qquad \ldots (5)$$

For negative pressure gradients in excess of $\gamma p B_\theta^2/2\pi r(B^2/4\pi + \gamma p)$, as observed experimentally in the discharge core, only the ω_{10} mode is unstable. NEWCOMB, 1961) has given the corresponding gravitational instability mode the name 'quasi-interchange'. This is an appropriate name since the growth rate differs from the growth rate for a pure interchange (constant magnetic pitch and $k_{\parallel} = 0$) only by small terms containing $k_{\parallel}$. Similarly, the values of $\nabla\cdot\underset{\sim}{\xi}$ and the fluctuations in pressure and magnetic field $(\delta p, \delta\underset{\sim}{B})$ are approximately the same as those for a pure interchange. The latter are given by

$$\frac{\delta B_{\parallel}}{B} \simeq \frac{\beta}{2}\frac{\delta p}{p} \simeq -\xi_r(\beta/2)\left[\frac{1dp}{pdr} + \frac{\gamma\; B_\theta^2}{2\pi r(B^2/4\pi + \gamma p)}\right] \qquad \ldots (6)$$

There is some bending of the lines of force but $\delta B_{\perp}/B$ and $\delta B_r/B$ are proportional to $k_{\parallel}$ and are small. From equation (6) $\delta B_{\parallel}/B$ will also be small for either low β or for dp/dr close to the marginal stability value. For the Sceptre experiments with $\beta \sim 0.5$, and the pressure gradient twice that for marginal stability (this is an upper limit, see WARE, 1961b), $\delta B_{\parallel}/B$ for (6) is only $\frac{1}{8}\frac{\xi_r}{p}\frac{dp}{dr}$.

The second mode ω_{20} is unstable only in the range $0 > \frac{dp}{dr} > -\gamma B_\theta^2/2\pi r\left(\frac{B^2}{4\pi} + \gamma p\right)$. As the growth rate tends to zero for this mode, its properties approach those of the kink mode obtained from the energy principle for $k_{\parallel} \neq 0$, namely $\nabla\cdot\underset{\sim}{\xi} \to 0$ and $\xi_{\parallel}$ becomes the dominant component of $\underset{\sim}{\xi}$. In the other direction, as the growth rate increases, this mode merges continuously into the quasi-interchange mode.

Thus the only instabilities predicted by MHD theory for the discharge core with the observed pressure gradient, are pure interchanges, (if the magnetic shear is zero and $k_{\parallel} = 0$), or quasi-interchanges whose properties are very similar to a pure interchange. (An extra condition for the pure interchange to be possible is that the magnetic pitch should be a submultiple of the torus circumference.)

2.2 The Outer Region of the Discharge

In the outer region, since the magnetic pitch varies appreciably with radius, terms involving $k_{\parallel}$ and $\frac{dk_{\parallel}}{dr}$ will no longer be negligible. The terms involving $\partial\xi_r/\partial r$ and $\partial^2\xi_r/\partial r^2$ are no longer small and it is not possible to obtain a dispersion relation without solving for the radial eigen function. The differential equation involved has not been solved for non-zero ω except for a very simple field configuration (TAYLER, 1957) with the added assumption $\nabla\cdot\underset{\sim}{\xi} = 0$. It has however, been solved by several workers for the cases where $\omega = 0$ (where $\nabla\cdot\underset{\sim}{\xi} = 0$ is strictly true) so that a considerable amount of information is known about the form of the plasma perturbations at marginal stability. In the case of diffuse pinch configurations, solutions have been obtained by KADOMTSEV (1963) and WHITEMAN, 1962). Both workers have extended their analysis to include non-linear terms, showing the existence of new neighbouring helical equilibria.

One class of these equilibria (Class A) can be achieved by small displacements of a perfectly conducting plasma from the cylindrical configuration, so that an MHD instability of the type considered in section 2.1 would lead to an equilibrium of this form. The other class (Class B) involves a new topology of the magnetic field and can be achieved by small plasma displacements only if finite conductivity is assumed. The tearing mode type of resistive instability (REBUT, 1962; FURTH et al, 1963) would lead to this type of helical equilibrium. Since the regular magnetic fluctuations observed in the outer discharge region in Sceptre III have both a helical ($m = 1$) character in space and a periodic nature with approximately constant amplitude in time, the results are consistent with a new helical equilibrium as pointed out by KADOMTSEV (1963). The rotation of the helical configuration must be due to a wave motion since mass motion was shown experimentally to be absent in Sceptre IV and in Sceptre III, where mass motion was observed, it was small in the outer region.

Of the two types of helical equilibria, in the case of Class A, if the pitch of the helix exceeds the greatest magnetic pitch ($2\pi r\ B_z/B_\theta$) present in the discharge, a perturbation to the magnetic field will occur at all radii, a change of phase occurring only at the centre. Such fluctuations were observed in a few cases, (see Fig.9 in ALLEN, 1964).

If, on the other hand, the pitch of the helical distortion falls within the range of the magnetic pitch values, the plasma displacement will pass through zero at the radius where the equality occurs. The most unstable mode will often have zero displacement within this radius (WHITEMAN, 1962). In one set of experiments (ALLEN, 1960) the magnetic pitch varied from 26 cm at $r = 10$ cm to 75 cm at $r = 6$ cm and the mean pitch of the fluctuations was 67 cm, so that an equality would often occur near $r = 6$ cm. The absence of fluctuations within this radius is therefore consistent with MHD theory. In the case of the Class B equilibria (Tearing mode instability), the perturbations in the magnetic field occur over the whole tube with a change in phase at the radius where the pitch is the same for the magnetic field and the helical distortion. Such a change in phase was not observed.

From these arguments, the regular fluctuations can be identified with helical equilibria generated by the $m = 1$ MHD instability. In the case of the irregular fluctuations observed in the outer region, no corresponding attempt can be made to identify their nature because of the lack of information concerning their spatial variation.

3. THE WAVE VELOCITIES CAUSED BY FINITE LARMOR RADIUS EFFECTS

3.1 The Discharge Core

It was seen in the previous section that the types of instability expected for the discharge core are either pure- or quasi-interchanges. Since, to first order, these two modes have similar properties, in order to simplify the algebra, the finite Larmor radius terms will be considered only for the pure interchange mode. Including these higher order terms, but assuming $T_e \ll T_i = T$, the plasma equations (1) are replaced by

$$nM \frac{d\underset{\sim}{v}}{dt} + \underset{\sim}{j} \times \underset{\sim}{B} - \nabla p = \nabla \cdot \overset{o}{\underline{P}} \qquad \ldots (7)$$

$$\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B} = \frac{1}{ne} \underset{\sim}{j} \times \underset{\sim}{B} \qquad \ldots (8)$$

$$\frac{1}{p} \frac{dp}{dt} + \gamma \nabla \cdot \underset{\sim}{\xi} = - \frac{2}{3p} \nabla \cdot \underset{\sim}{q} \qquad \ldots (9)$$

where

$$\underset{\sim}{q} = - \frac{5pk'}{2eB^2} (\nabla T \times \underset{\sim}{B}) \qquad \ldots (10)$$

$\overset{o}{\underline{P}}$ is the non-isotropic part of the pressure tensor and k' is Boltzmann's constant. This is a consistent set of equations in which all the terms of order ρ_i/L have been retained. (See for example HERDAN and LILEY, 1960.)

Considering the same normal modes as in section 2 with $k_{\parallel} = 0$ and constant magnetic pitch, the non-zero components of $\overset{o}{\underline{P}}$ are given by

$$\overset{o}{P}_{rr} \simeq - \frac{p}{2\omega_{ci}} \left(\frac{\partial v_{\perp}}{\partial r} - ikv_r \right)$$

$$\overset{o}{P}_{\perp\perp} \simeq \frac{p}{2\omega_{ci}} \left(\frac{\partial v_{\perp}}{\partial r} - ikv_r \right)$$

$$\overset{o}{P}_{r\perp} \simeq \overset{o}{P}_{\perp r} \simeq \frac{p}{2\omega_{ci}} \left(\frac{\partial v_r}{\partial r} + ikv_{\perp} \right) \qquad \ldots (11)$$

where a right handed set of axes $r, \perp, \parallel$ has been chosen. The symbol $\parallel$ represents the direction of $\underset{\sim}{B}$ and $\perp$ the direction of $\underset{\sim}{B} \times \underset{\sim}{r}$. The wave vector $\underset{\sim}{k}$ is chosen to be in the positive direction of $\perp$ (i.e. for B_θ and B_z positive, m is positive and k_z negative). Strictly these formulae for $\overset{o}{\underline{P}}$ are correct for only cartesian coordinates, but because it is not possible to make accurate quantitative comparisons between theory and experiment, it was not thought worthwhile to evaluate the corrections to these terms due to the curvature of the helical coordinate system used. ω_{ci} is the ion cyclotron frequency, and in equations (10) and (11) it has been assumed large compared with both the ion collision frequency and ω.

Taking the curl of equations (7) and (8), the equations (7) - (11) together with Maxwell's equations reduce to

$$nM\omega^2\,\xi_r - \frac{2B_\theta^2}{4\pi r}\left(\frac{\delta B}{B}\right) + \left(\frac{\omega}{\omega_{ci}}\right) kp\,[(K_p - K_B)\xi_r + \nabla\cdot\underset{\sim}{\xi}] = 0 \qquad \ldots (12)$$

$$\frac{B^2}{4\pi}\left(\frac{\delta B}{B}\right) + \delta p + \left(\frac{\omega}{\omega_{ci}}\right)\frac{kp}{2}\,\xi_r = 0 \qquad \ldots (13)$$

$$\frac{\delta B}{B} + K_{B_z}\xi_r + \nabla\cdot\underset{\sim}{\xi} - \frac{\omega k\,\xi_r}{\omega_{ci}} + \frac{kp}{neB\omega}\,(K_p - \gamma K_n)\,\nabla\cdot\xi = 0 \qquad \ldots (14)$$

$$\frac{\delta p}{p} + K_p\,\xi_r + \gamma\nabla\cdot\underset{\sim}{\xi} + \frac{5}{3}\,\frac{kp}{neB\omega}\left[(K_p - K_{Bz})\left(\frac{\delta p}{p} + K_n\xi_r + \nabla\cdot\underset{\sim}{\xi}\right) - K_T\left(\frac{\delta p}{p} - \frac{\delta B}{B}\right)\right] = 0 \qquad \ldots (15)$$

where $\delta B \equiv \delta B_{\parallel}$ $(\delta B_r = \delta B_{\perp} = 0)$ and where the reciprocal lengths K_α are given by

$$K_\alpha \equiv \frac{1}{\alpha}\frac{d\alpha}{dr}$$

and are related as follows

$$K_p = K_n + K_T \qquad , \qquad K_{B_z} = K_{\frac{B_\theta}{r}}$$

$$\frac{\beta}{2}K_p + K_{B_z} + \frac{2B_\theta^2}{rB^2} = 0 \ , \qquad K_B = \frac{1}{2}K_{B_z} - \frac{\beta}{4}K_p \ .$$

The assumptions which have been made in deriving equations (12) - (15) are that the magnitude of $\partial\,\xi_r/\partial r$ and ξ_r/r are both small compared with $k\xi_r$, and that K_n is of the order $1/r$. By equating the determinant of these equations to zero the dispersion relation is found to be

$$\omega^3 + a_1\omega^2 + a_2\omega + a_3 = 0 \qquad \ldots (16)$$

where

$$a_1 = \frac{kp}{neB}\left[\left(2 + \frac{\beta}{4}\right)K_p - \frac{3}{2}K_{B_z} - \frac{p}{(B^2/4\pi) + \gamma p}(K_p - \gamma K_n)\right]$$

$$a_2 = \frac{pB_\theta^2}{2\pi rp\,[(B^2/4\pi) + \gamma p]}\,[\gamma K_B - K_p]$$

$$a_3 = \frac{kp}{neB}\left[-a_2\,K_p - \frac{\gamma p\,B_\theta^2\,K_{Bz}\,(K_n - K_{Bz})}{2\pi rp\,[(B^2/4\pi) + \gamma p]}\right]$$

where terms of second order in ρ_i/L have been neglected. The terms a_1 and a_3 are of first order in ρ_i/L compared with zero order in a_2.

The assumption is now made that K_p and γK_{B_z} are sufficiently different for $(K_p - \gamma K_{B_z})$ to be of the same order of magnitude as K_{B_z}. In which case the first and third terms in equation (16) are dominant and its approximate factors take the form

$$\omega^2 + A_1\omega + A_2 = 0 \qquad \ldots(17a)$$

$$(\omega + D) = 0 \qquad \ldots(17b)$$

where

$$\left.\begin{aligned} A_1 &= \frac{kT_i}{eB}\left[(1+\frac{\beta}{4})K_p - \frac{p}{(B^2/4\pi)+\gamma p}(K_p - \gamma K_n) + \frac{K_{B_z}(\frac{3}{2}K_p - \frac{5}{3}K_n - \frac{5}{6}K_{B_z})}{(\frac{5}{3}K_{B_z} - K_p)}\right] \\ A_2 &= a_2 \\ D &= \frac{kT_i}{eB}\left[K_p + \frac{\gamma K_{B_z}(K_n - K_{B_z})}{(\gamma K_{B_z} - K_p)}\right] \end{aligned}\right\} \ldots (18)$$

Equation (17a) is the interchange mode and (17b) is the entropy wave of KADOMTSEV (1959). The entropy wave is stable in this case because of the large inequality in T_e and T_i (see WESSON, 1963) and an important feature of this mode is that $\delta p/p$ and $2\delta B/\beta B$ are of the order as $(\rho_i^2/L^2)(\delta T/T) \approx (\rho_i^2/L^2)(\delta n/n)$. Although the observed fluctuations in B in the discharge core are small, nevertheless $2\delta B/\beta B$ is of the same order as the observed $\delta n/n$. Hence the observed fluctuations cannot be identified with entropy waves and this mode will not be considered further.

The wave velocity for the interchange mode in the direction of $\underline{k}$ is given

$$\begin{aligned} V_w &= \mathrm{Re}(\omega)/k \\ &= - A_1/2k \end{aligned}$$

and the apparent wave velocity in the positive z direction is therefore

$$V_w^z = \frac{B_\theta A_1}{2Bk} = \frac{T_i}{eB_\theta}\left[(1+\frac{\beta}{4})K_p - \frac{p}{(B^2/4\pi)+\gamma p}(K_p - \gamma K_n) + K_{Bz}\frac{(\frac{3}{2}K_p - \frac{5}{3}K_n - \frac{5}{6}K_{B_z})}{\frac{5}{3}K_{B_z} - K_p}\right] \quad \ldots (19)$$

If $K_n = xK_p$ and $K_p = yK_{B_z}$ then $V_w^z = \frac{\alpha T_i}{eB_\theta}(-K_p)$ or $\frac{\alpha}{neB_\theta}(-\frac{dp}{dr})$ $\qquad \ldots(19a)$

where α is a function of x, y and β. Table 1 gives α for values of x, y and β in the ranges of interest here. It is seen that α is insensitive to β in the range of 0 to 0.5, but varies rapidly with x and y. (It should be noted that equations (18) and (19) are no longer valid approximations for $y \simeq \frac{5}{3}$.)

Table 1

Example Values of α

$y = \frac{d \ln p}{d \ln B_z}$	$y = 2$		$y = \frac{8}{3}$		$y = \frac{10}{3}$	
$x = \frac{d \ln n}{d \ln p}$ \ β	0	0.5	0	0.5	0	0.5
x = 0	+ 3.25	+ 3.30	+ 0.19	+ 0.24	- 0.23	- 0.20
x = ½	- 0.25	- 0.35	- 0.65	- 0.75	- 0.75	- 0.85
x = 1	- 2.75	- 2.99	- 1.48	- 1.72	- 1.25	- 1.49

For certain conditions α is close to the value -2 deduced qualitatively from the trapping of magnetic fluctuations to the mean electron velocity (WARE, 1961a), but for other conditions α can be markedly different from -2.

The wave velocity predicted by equation (19) will in general be a function of radius. This cannot occur in practice and it is likely that the terms involving the radial gradients of $\underset{\sim}{\xi}$, which have been neglected, will make the real part of ω independent of radius, as they do the imaginary part. The resultant wave velocity will be some average value of the right hand side of (19).

3.2 The Outer Region

For the outer region of the Sceptre discharges, less is known about the plasma conditions (pressure gradient, density, etc.) than for the core and all that will be attempted is to show that when $k_{\parallel} \neq 0$, terms arise which can change the sign of the wave velocity. It will be assumed that the non-linear terms in the plasma equations are small compared with the linear terms and that the higher order terms cause the observed equilibria only because the difference between the linear destabilising and stabilising terms is small. In which case, only linear terms need be considered to obtain the wave velocity.

For simplicity, of the various finite Larmor radius effects, only the Hall effect will be included. There is a range of magnetic field values for which this is the only important transverse magnetic effect namely $\omega_{ci}\,\tau_i \ll 1 \ll \omega_{ce}\,\tau_e$, where the τ's are the mean collision times but the prime reason is to simplify the problem. The full treatment with $k_{\parallel} \neq 0$ would involve not only the other transverse magnetic effects but also heat conduction and viscosity parallel to the magnetic field. Finally it will be assumed, in addition to $T_e \ll T_i$, that

$$\frac{1}{n}\frac{dn}{dr} = \frac{1}{\gamma p}\frac{dp}{dr} \qquad \text{.. (20)}$$

so that $\nabla \times (\frac{1}{ne}\nabla p_i)$ is zero to first order. Only the contribution to the Hall effect resulting from the plasma acceleration is therefore retained.

With the same large k approximation as before, the equations reduce to

$$\omega^2\rho\left[k^2\,\xi_r + \frac{2m^2}{r^3k^2}\,\nabla\cdot\underset{\sim}{\xi} + \frac{2\,ik_zm}{k^2r^2}\,\xi_{\parallel}\right] =$$

$$-\frac{2k_z^2\,B\,\delta B}{4\pi r} - \frac{2k_{\parallel}\,B}{4\pi r}\left(\frac{m}{r}\,\delta B_\theta - k_z\,\delta B_z\right) - \frac{2\,k_{\parallel}\,B\,\xi_r}{4\pi r}\,\frac{d}{dr}\left(\frac{mB_\theta}{r} + k_zB_z\right)$$

$$-\frac{k_{\parallel}^2\,B^2\,k^2}{4\pi}\,\xi_r + \frac{i\,k_{\parallel}\,k^3\,\omega\,B^2\,\xi_{\parallel}}{\omega_{ci}} \qquad \text{... (21)}$$

$$\delta\underset{\sim}{B} = \nabla \wedge (\underset{\sim}{\xi} \wedge \underset{\sim}{B}) - \frac{i\omega B}{\omega_{ci}}\,\nabla\times\underset{\sim}{\xi} \qquad \text{... (22)}$$

$$\nabla\cdot\underset{\sim}{\xi} = \frac{\xi_r\left(\dfrac{2k_zk_\perp\,B_\theta\,B}{k^24\pi r} - \dfrac{k_\perp\,B^2\omega}{4\pi\,\omega_{ci}}\right)\left(\omega^2\rho + \dfrac{\omega\,k_\perp j_\perp\,B}{\omega_{ci}}\right)}{\gamma p\left(\dfrac{k_{\parallel}^2\,B^2}{4\pi} - \dfrac{\omega\,j_{\parallel}k_{\parallel}\,B}{\omega_{ci}}\right) - \left(\omega^2\rho + \dfrac{\omega\,k_\perp j_\perp\,B}{\omega_{ci}}\right)\left(\dfrac{B^2}{4\pi} + \gamma p\right)} \qquad \text{... (23)}$$

$$\xi_{\parallel}\left(\omega^2\rho + \frac{\omega\,k_\perp j_\perp\,B}{\omega_{ci}}\right) = i\,k_{\parallel}\,\gamma p\,\nabla\cdot\underset{\sim}{\xi} \qquad \text{... (24)}$$

and these lead to the dispersion relation

$$\omega^4 + a_1'\omega^3 + a_2'\omega^2 + a_3'\omega + a_4' = 0 \qquad \ldots (25)$$

where, to the first two orders in $k_{\parallel}$ in each case, the coefficients are given by

$$a_1' = \frac{k_\perp j_\perp}{ne} + \left(\frac{\gamma p}{\frac{B^2}{4\pi} + \gamma p}\right)\left[\frac{k_{\parallel} j_{\parallel}}{ne} - \frac{k_\perp\left(\frac{2B_\theta^2}{4\pi r}\right)}{neB}\right] + \left(\frac{\frac{B^2}{4\pi}}{\frac{B^2}{4\pi} + \gamma p}\right)\left(\frac{4\ m\ k_{\parallel}\ B_\theta}{4\pi r^2 k\ ne}\right)$$

$$a_2' = \frac{b}{a} = -\frac{2k_z^{\ 2}}{\rho k^2 r}\left(\frac{dp}{dr} + \frac{2\gamma p B_\theta^{\ 2}}{4\pi r\left(\frac{B^2}{4\pi} + \gamma p\right)} - 0\left(\frac{k_{\parallel}\ B_\theta\ B_z}{k4\pi r}\right)\right)$$

$$a_3' = a_2'\left(\frac{k_\perp j_\perp}{ne}\right) - \frac{2k_z^2}{rk^2}\left(\frac{\gamma p}{\frac{B^2}{4\pi} + \gamma p}\right)\left(\frac{dp}{dr} + \frac{2B_\theta^2}{4\pi r}\right)\frac{j_{\parallel} k_{\parallel}}{ne}$$

$$a_4' = \frac{\gamma p\ k_{\parallel}^2}{\rho^2}\left(\frac{\frac{B^2}{4\pi}}{\frac{B^2}{4\pi} + \gamma p}\right)\left[\frac{2\ k_z^2}{k^2 r}\frac{dp}{dr} + 0\left(\frac{k_{\parallel} B_\theta^2}{k\ 4\pi r^2}\right)\right]$$

It is assumed that $k_{\parallel}$, although significant is still small compared with k, so that higher order terms in $k_{\parallel}$ can be neglected. Provided $dp/dr \neq -2\gamma p\, B_\theta^2/4\pi r(B^2/4\pi + \gamma p)$ the dispersion relation can be factorised to give the roots corresponding to those obtained in section 2.1.

$$\omega_1^2 + \omega_1\left\{\frac{k_{\parallel}}{ne}\left(\frac{\frac{B^2}{4\pi}}{\frac{B^2}{4\pi}+\gamma p}\right)\left[\frac{4\ B_z B_\theta}{4\pi\ r\ B} - \frac{j_{\parallel}\ \frac{2B_\theta^2}{4\pi r}}{\left(\frac{\frac{B^2}{4\pi}+\gamma p}{\gamma p}\right)\frac{dp}{dr} + \frac{2B_\theta^2}{4\pi r}}\right] - \frac{k_\perp}{ne}\left(\frac{\gamma p}{\frac{B^2}{4\pi} + \gamma p}\right)\frac{2B_\theta^{\ 2}}{4\pi r B}\right\} + \omega_{10}^2 = 0 \qquad \ldots (26)$$

$$\omega_2^2 + \omega_2\left\{\frac{k_{\parallel} j_{\parallel}}{ne}\ \frac{\left(\frac{dp}{dr} + \frac{2B_\theta^2}{4\pi r}\right)}{\left(\frac{\frac{B^2}{4\pi}+\gamma p}{\gamma p}\right)\frac{dp}{dr} + \frac{2B_\theta^2}{4\pi r}} + \frac{k_\perp j_\perp}{ne}\right\} + \omega_{20}^2 = 0 \qquad \ldots (27)$$

where ω_{10}^2 and ω_{20}^2 are given by equations (4) and (5). The apparent wave velocities in the z direction will be $1/(2\,k_z)$ times the coefficients of the ω_1 and ω_2 terms respectively (N.B. k_z is a negative quantity for positive B_θ and B_z).

If $\frac{dp}{dr} > -2\,\gamma p\,B_\theta^2/4\pi r\,(\frac{B^2}{4\pi} + \gamma p)$, (which is likely to be the case in the outer region of the discharge), then the second mode (ω_2) will be the unstable mode. If the pressure gradient is negative, so that $j_\perp < 0$, the wave velocity for this mode will be positive or negative in the z direction depending on whether

$$\left(\frac{k_\parallel}{k_\perp}\right)\left(\frac{j_\parallel}{|j_\perp|}\right)\left(\frac{\frac{2B_\theta^2}{4\pi r} + \frac{dp}{dr}}{\frac{2B_\theta^2}{4\pi r} + \left(\frac{\frac{B^2}{4\pi} + \gamma p}{\gamma p}\right)\frac{dp}{dr}}\right) \gtrless 1 \qquad \ldots (28)$$

Both inequalities are possible. The change over can occur at a value of $k_\parallel$ appreciably less than $k_\perp$, since the other two factors in (28) are likely to be larger than unity. If, however, the pressure gradient is positive ($j_\perp \geqslant 0$) the wave velocity will be always positive for positive $k_\parallel$ and $k_\perp$. In the special case where $\frac{dp}{dr}$ is small compared with both $\frac{2B_\theta^2}{4\pi r}$ and $2\,\gamma p\,B_\theta^2/4\pi r\,(\frac{B^2}{4\pi} + \gamma p)$ the wave velocity reduces to

$$V_w^z = (\underset{\sim}{k} \cdot \underset{\sim}{j})/2\,nek \qquad \ldots (29)$$

This is half the value deduced qualitatively from field trapping arguments (WARE, 1962).

In the case of the quasi-interchange mode (ω_1) the wave velocity will be positive or negative depending on whether

$$\frac{k_\parallel}{k_\perp}\,(\frac{1}{\beta}\left\{2\,\frac{B_z}{B_\theta} - \frac{j_\parallel\,B}{\left(\frac{\frac{B^2}{4\pi} + \gamma p}{\gamma p}\right)\frac{dp}{dr} + \frac{2B_\theta^2}{4\pi r}}\right\} \gtrless 1 \qquad \ldots (30)$$

and again in this case a change of sign can occur for a moderate value of $k_\parallel$.

It must be remembered, however, that the other higher order effects will often be of comparable importance to the acceleration part of the Hall effect, and, in fact, these terms can themselves lead to a change in sign of the wave velocity, as was seen in section 3.1.

4. COMPARISON BETWEEN THEORY AND EXPERIMENT

4.1 The Wave Velocity

In the discharge core, since the expected instability will have either zero or only a small value of $k_{\parallel}$, the appropriate formula for the wave velocity is equation (19) or (19a). It has already been seen (WARE, 1962; WILLIAMS, 1964) that the order of magnitude of $T_i k_p / e B_\theta$ is 2×10^6 cm sec^{-1}. This compares well with the observed range of wave velocities in Sceptre IV namely -2×10^6 to -4.5×10^6 cm sec^{-1}. In Sceptre III the velocity was again negative but only a few $\times 10^5$ cm sec^{-1}. In this case, however, mass velocities in the z direction from $+10^6$ to $+2 \times 10^6$ cm sec^{-1} were observed (HUGHES and KAUFMAN, 1959) so the wave velocity to be compared with equations (19) is -10^6 to -2×10^6 cm sec^{-1}. In both cases therefore there is agreement in magnitude.

The next question is the sign and magnitude of α. In the case of Sceptre IV, the Doppler broadening measurements (ALLEN et al, 1962) indicate little or no gradient of ion temperature. Hence in Table 1 the value of x is approximately unity, and if K_p (unknown) is assumed to be in the range given by $2 \leqslant y \leqslant 10/3$, then α will be in the range -1.5 to -2.75. This is the right sign and magnitude to give good agreement with experiment.

In the case of Sceptre III, there is evidence (ALLEN et al, 1962) for a significant ion temperature gradient in the discharge, but there are no actual measurements of either K_T or K_n. The pressure gradient is known in this case (ALLEN and LILEY, 1959) and, despite the uncertainty in the integration constant for p, the value of K_p must lie in the range given by $2 \leqslant y \leqslant 10/3$. From Table 1 it can be seen that most of the possible values of α are negative. The ion temperature gradient would have to account for more than half the pressure gradient to give a positive value of α.

For the outer region no attempt at quantitative comparison between theory and experiment can be made since little is known of the plasma conditions in this region. An example of the range of values of $k_{\parallel}$ for this region can be seen by taking a particular set of experimental results (ALLEN, 1960), where the magnetic pitch was observed to vary from 75 cm on the inside of the region to 26 cm on the outside. the mean pitch of the instability being 67 cm. The value of $k_{\parallel}/k_{\perp}$ will therefore vary from zero at the point near the inner edge where the two pitches coincide to 0.45 at the outside, both $k_{\parallel}$ and $k_{\perp}$ being positive. Since $j_{\parallel}$ is likely to be considerably greater than $j_{\perp}$ in this region, the upper inequality will hold in (28) for the outer part of the region corresponding to a positive wave velocity. Towards the inner part of the region the lower inequality in (28) would probably apply, but there is some evidence (ALLEN and LILEY, 1959) that $j_{\perp}$ is positive ($dp/dr > 0$) over the inner part of the region in Sceptre III. Average current densities are unreliable for this region because of the large magnetic fluctuations but if $j_{\perp} > 0$, this would lead to a positive wave velocity independent of condition (28). Thus the observed positive wave velocity for the outer region in Sceptre III is consistent with the theory.

The equally probably positive and negative velocities observed for this region in Sceptre IV would be consistent with theory if there is a negative pressure gradient and if the average value of $k_{\parallel}$ for the region varies from one discharge to another (due to a different k) so that both inequalities are equally probable in (28).

4.2 The Electron Temperature Fluctuations

It was observed by WILLIAMS, (1964) that for the instability fluctuation in the core $\delta T_e/T_e \ll \delta n/n$. That this result is consistent with an interchange instability in a discharge with the observed electron temperature gradient can be seen as follows.

Equation (14) can be rewritten in the form

$$\frac{\delta B_{\parallel}}{B} = - \xi_{er} K_{B_z} - \nabla\cdot\underset{\sim}{\xi}_e \qquad \ldots (31)$$

where $\underset{\sim}{\xi}_e$ is the displacement experienced by the electron component of the plasma. Since it has already been seen from equation (6) that $\delta B_{\parallel}/B \ll \xi_r K_p$ and since $\xi_r K_p \sim \xi_{er} K_{B_z}$, the term involving $\delta B_{\parallel}/B$ in (31) can be neglected and

$$\nabla\cdot\xi_e \simeq \xi_{er} K_{B_z} .$$

Hence in the electron thermal energy relation corresponding to equation (9)

$$\frac{\delta T_e}{T_e} = - \xi_{er} K_{T_e} - \frac{2}{3} \nabla\cdot\underset{\sim}{\xi}_e \simeq - \xi_{er} \left\{K_{T_e} - \frac{2}{3} K_{B_z}\right\} \qquad \ldots (32)$$

where the $\underset{\sim}{q}_e$ term can be neglected since $T_e \ll T_i$.

The magnetic pitch in the discharge core is observed to be either constant or approximately constant with respect to radius (WARE, 1961b). Since the currents parallel to be $\underset{\sim}{B}$ are produced by a constant E_z with $E_\theta = 0$, the approximately constant pitch requires (WARE, 1963) $K_{\sigma_{\parallel}} \simeq K_{B_z}$ where $\sigma_{\parallel}$ is the electrical conductivity parallel to $\underset{\sim}{B}$, so that provided the amount of impurity in the discharge is small and $\sigma_{\parallel} \sim T_e^{\frac{3}{2}}$

$$K_{T_e} \simeq \frac{2}{3} K_{B_z}$$

Hence from (32)

$$\left|\frac{\delta T_e}{T_e}\right| \ll \xi_r K_{B_z}$$

and is therefore small compared with $\delta p/p$ or $\delta n/n$.

4.3 The Wave Velocities in Hydrogen and Deuterium

The theoretical wave velocity for the discharge core (equation 19) does not depend explicitly on the ion mass of the gas used. The observation of a higher velocity in hydrogen compared with deuterium is consistent with theory only if either the pressure gradient is greater in hydrogen or if the value of x (Table 1) is greater. The latter is unlikely since for Sceptre IV K_{T_i} is already small for deuterium. An attempt to compare the two pressure gradients using magnetic probes was inconclusive because the magnetic fluctuations are large (~ 20%) even in the discharge core in the case of Sceptre IV (ALLEN, et al, 1962). (Sceptre III was dismantled by then.)

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