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R S PECKOVER
I H HUTCHINSON

CULHAM LABORATORY
Abingdon Berkshire

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CONVECTIVE ROLLS DRIVEN BY INTERNAL HEAT SOURCES

R S Peckover and I H Hutchinson*

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ABSTRACT

Two dimensional Rumford convection in a layer of fluid confined between free horizontal boundaries has been studied using numerical experiments. The dependence on the Rumford number of the maximum horizontally averaged temperature and of the fraction of thermal flux emerging from the upper surface has been found for $Ra_H \lesssim 80\ Ra_{HC}$.

* Commonwealth Scholar, Australian National University.

U.K.A.E.A. Research Group,
Culham Laboratory,
Abingdon,
Berks.

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The simplest problem in thermal convection, the classical Rayleigh-Bénard problem, in which a horizontal layer of Boussinesq fluid is heated from below, has been studied in some detail (see for example Rayleigh [1] (1916) Chandrasekhar [2] (1961), Moore and Weiss [3] (1973)). The classical Rumford problem which may be defined as thermal convection in a horizontal layer of Boussinesq fluid containing a uniform distribution of heat sources, which is of interest in geophysics, meteorology and nuclear reactor design, has received less attention[4]. We present here some preliminary results of numerical experiments carried out for classical Rumford convection restricted to flow in two dimensions.

The most convenient independent parameters in terms of which to describe steady Rumford convection are the Prandtl number Pr ($\equiv \nu/\kappa$) and the Rumford number Ra_H. The Rumford number[5] is defined in terms of dimensional quantities to be:

$$Ra_H = \frac{g\alpha\,\gamma\, d^5}{\kappa^2\,\nu} \qquad (1)$$

where α is the volume coefficient of thermal expansion, g the acceleration due to gravity, ν the kinematic viscosity, κ the thermometric conductivity, d the full depth of the layer and γ the rate of temperature rise due to internal heating in the absence of heat transport. (The Rumford number is a form of modified Rayleigh number. The relationship between Ra_H and the equivalent dimensionless numbers of other authors is considered in Peckover and Hutchinson[5] (1973)).

The dependent quantities of most interest are (i) μ, the flux ratio which is the fraction of the heat emerging at the upper surface; and (ii) M, the convective cooling parameter which measures the reduction in maximum mean temperature brought about by the convection.

In our simulation the Prandtl number has been taken to be 8, a value appropriate for warm aqueous solutions used for joule heating experiments. The maximum Rumford number reached which gave credible results was 80 Ra_{HC} where Ra_{HC} is the critical Rumford number of linear instability theory. (The subscript C here and later implies the value at the onset of convection).

The equations describing Rumford convection in a Boussinesq fluid are the familiar ones of thermal convection (Chandrasekhar[2]) with the addition of a heat source term in the heat flow equation. For two-dimensional convection with flow confined to the x-y plane these equations can be expressed in the form:

$$\left.\begin{aligned} \frac{\partial T}{\partial t} + \underset{\sim}{\nabla}\cdot(T\underset{\sim}{u}) &= \kappa \nabla^2 T + \gamma \\ \frac{\partial \zeta}{\partial t} + \underset{\sim}{\nabla}\cdot(\zeta\underset{\sim}{u}) &= \nu \nabla^2 \zeta - \frac{\partial T}{\partial x} \end{aligned}\right\} \qquad (2)$$

where (x, y), ν, κ, T, γ are normalised in such a way that $\alpha g = d = 1$. The velocity $\underset{\sim}{u}$, by virtue of the incompressibility equation, $\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 0$, may be expressed in terms of a stream function, Ψ, by $\underset{\sim}{u} \equiv (-\partial\Psi/\partial y, \partial\Psi/\partial x) \equiv (u, v)$ in cartesian coordinates with y vertically upwards. ζ is the vorticity and we have $\nabla^2 \Psi = -\zeta$. Normalized variables are used hereafter without explicit comment. We have solved the above equations in the region $0 \leq y \leq 1$, $0 \leq x \leq \lambda$, subject to the "free" boundary conditions: $\zeta = 0 = \Psi$

$(x = 0, \lambda$ or $y = 0, 1)$; $T = 0$ $(y = 0, 1)$; and

$\partial T/\partial x = 0$ $(x = 0, \lambda)$.

The steady solutions to the partial differential equations (2) in an infinite layer are expected in the laminar régime to be periodic in the x-direction and in that case the above conditions hold at the cell boundaries. If, therefore, λ is chosen equal to the horizontal periodicity wavelength then two convective rolls of opposite circulation, whose form is the same as that for an infinite layer, will appear in the region $0 \leq x \leq \lambda$. Mirror symmetry about the line $x = \lambda/2$ provides one useful measure of accuracy.

In the absence of convection there is a steady state parabolic temperature profile $T = (\gamma/2\kappa)\, y\, (1-y)$ with maximum temperature $T^* = (\gamma/8\kappa)$. When convection occurs the horizontally averaged temperature, $\overline{T}$, has a maximum, $\overline{T}_{max}$, less than T^* ; the cooling parameter, M, is then defined by: $M = T^*/\overline{T}_{max}$. The horizontally averaged thermometric flux is $F \equiv \overline{vT - \kappa\ (\partial T/\partial y)}$. At the upper and lower surfaces F has the values a and -b respectively. The total thermometric flux generated, γ is thus $a + b$ and the flux ratio, μ , is given by: $\mu = a/(a+b) = a/\gamma$.

Finite difference methods were used to solve the non-linear partial differential equations. These are described elsewhere (Moore, Peckover, Weiss [6] (1973)). The variables are represented on a rectangular grid with spacing $\Delta x = \lambda/N_x$, $\Delta y = 1/N_y$. Equations (2) are advanced by a second order leapfrog scheme centred in space and time (Roberts and Weiss [7] (1966)). The stream function, Ψ , is obtained by solving the discrete form of Poisson's equation using fast fourier analysis in the x-direction and tridiagonal elimination in the y-direction. Calculation was carried out over an entire 'cell' $(0 \leq x \leq \lambda,\ 0 \leq y \leq 1)$ with $N_x = N_y = 24$. The effect of finite N_x, N_y was judged by runs at 32 Ra_{HC} and 2.5 Ra_{HC} on a 48 x 48 mesh. The difference between these and the corresponding 24 x 24 runs was for μ , and M better than 0.7%. The main appreciable error arises in v^2_{max} owing to inadequate representation of the falling plume. This error may be as great as 15%.

We have been concerned with the effect of varying the Rumford number at fixed Prandtl number, Pr = 8. Linear theory (Watson[8], Kulacki[9])predicts the onset of stability to be at $Ra_H = Ra_{HC} = 1.6992 \times 10^4$, with a horizontal wavelength of $\lambda_C = 2.075$. The principle of the exchange of stabilities holds[10]. Consequently in an infinite layer, convective rolls of width 1.0375 would be expected to occur with steady laminar convection for Rumford numbers slightly in excess of critical. We chose to take a region of width 2.0 in which to calculate the flow behaviour and heat transfer, expecting two rolls would be contained in the box. This proved to be the case. Overall properties such as heat fluxes are unlikely to be particularly sensitive to the precise cell size[3]. Fig.1 shows flow and temperature distributions for $Ra_H = 25\ Ra_{HC}$. Convection occurs in the form of narrow falling plumes of cooler fluid and hotter broad rising regions. There are no narrow rising plumes such as occur in Rayleigh-Bénard convection.

The profiles of horizontally averaged temperature in fig.2 show how the parabolic conduction profile is modified by convection, the maximum being displaced systematically upwards as Ra_H increases. The profiles have been normalized so that in each case the conduction profile in the absence of convection would be the uppermost, parabolic, profile. The curves then show how the temperature maximum is depressed by convection and the maximum height of each is M^{-1}, where M is the convective cooling parameter. The increase of vertical temperature gradient at the upper surface also indicates the increase of μ.

The variation of M and μ with Ra_H is shown in figs. 3 and 4. Kulacki and Goldstein[11] (1972) report laboratory experiments for

$Ra_H \lesssim 675\ Ra_{HC}$ for non-slip boundaries and obtain results whose trends in μ and M are similar.

Correlations of the following forms are suggested for μ and M. For $3\ Ra_{HC} \lesssim Ra_H \lesssim 80\ Ra_{HC}$

$$M - 1 = 0.023\ (Ra_H - Ra_{HC})^{0.260} \tag{3}$$

while the exponent increases to approximately 1 as Ra_H decreases to $1.1\ Ra_{HC}$. Similarly for $5\ Ra_{HC} \lesssim Ra_H \lesssim 80\ Ra_{HC}$

$$\mu - \tfrac{1}{2} = 0.0113\ (Ra_H - Ra_{HC})^{0.194} \tag{4}$$

with the exponent again increasing to 1 as Ra_H decreases to $1.1\ Ra_{HC}$.

A measure of the kinetic energy in the rolls as convection increases is u_{max}^2 or v_{max}^2 the square of the maximum velocity in the x or y directions respectively. Fig.5 shows steady state values of u_{max}^2, v_{max}^2 which are well represented by the functional forms:

$$u_{max}^2 = A_1 \left[\frac{Ra_H - Ra_{HC}}{Ra_{HC}}\right]^{0.925} \tag{5}$$

$$v_{max}^2 = A_2 \left[\frac{Ra_H - Ra_{HC}}{Ra_{HC}}\right]^{1.161} \tag{6}$$

where $A_1 = 4.83 \times 10^{-3}$, and $A_2 = 5.81 \times 10^{-3}$.

These numerical experiments give a critical Ra_H in agreement with the linear stability theory value to within 2%. This provides a quantitative confirmation of the linear theory for Rumford convection; Kulacki and Goldstein's experiments[11] with fixed boundaries were not entirely unequivocal near the onset of convection.

The form of the convection in falling plumes and broad rising regions may be explained qualitatively on the grounds that the fluid must rise slowly so as to allow the internal heating to raise its temperature and must fall rapidly to keep its temperature low during descent. The stagnation points in the core of the steady rolls are located close to the cold plumes and the steep temperature gradients there enable the stagnant regions to be effectively cooled by conduction alone.

General arguments concerning energy liberation and dissipation predict an approximately linear relation between $\underset{\sim}{u}^2$ and Ra_{HC}. This is verified by the results summarized in equations (5) and (6).

Since these calculations have been done for a single value of the Prandtl number, the dependence on Pr cannot be included. Ra_{HC} and λ_C are known to be independent of Pr [8,9].

No horizontal velocity at $x = 0, \lambda$ is allowed by our boundary conditions. This excludes the possibility of lateral sloshing, or of a non-integral number of rolls in the box. Future work will examine whether these results differ significantly from those for an infinite layer in which the realized roll width is allowed to be a slowly changing function of Ra_H.

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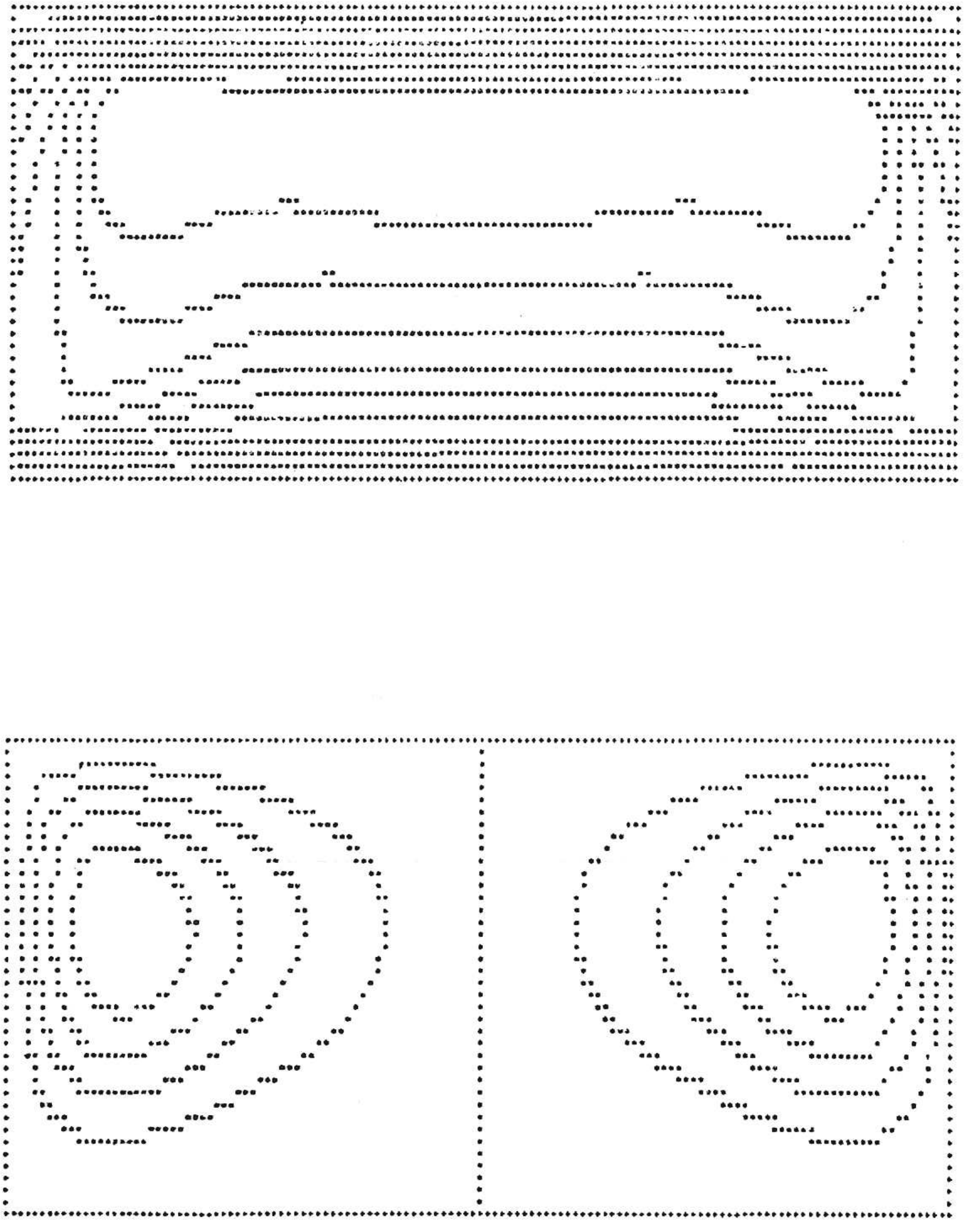

Fig. 1 (a) Isotherms (b) Streamlines for Rumford convection in two dimensions. The horizontal boundaries are stress-free and have the same temperature.

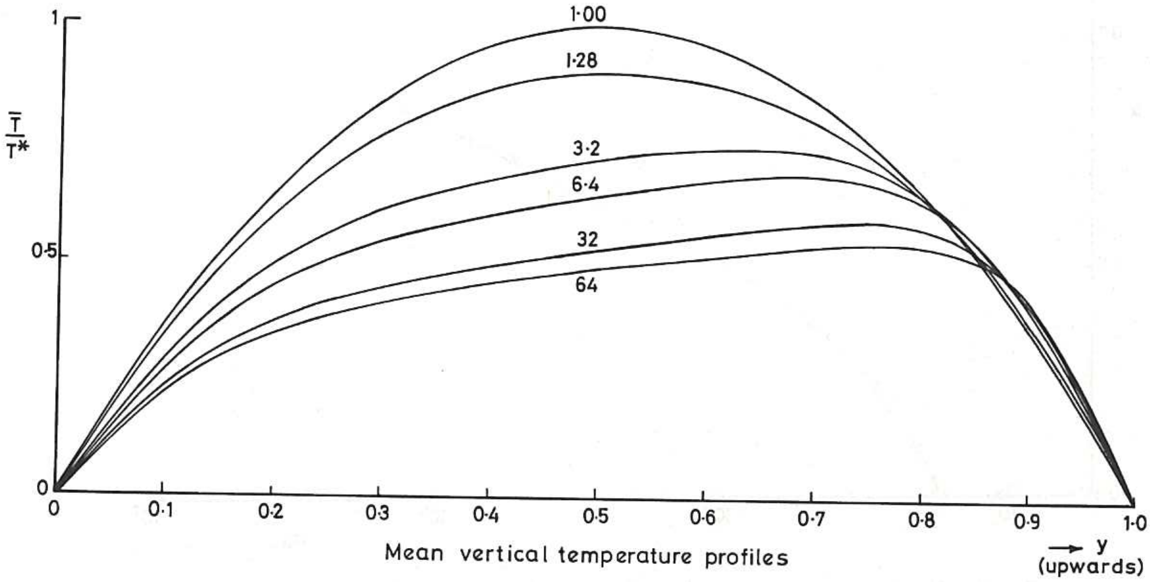

Fig. 2 Horizontally averaged vertical temperature profiles, labelled with a real number n where $n = Ra_H/Ra_{HC}$, and normalized by T^*.

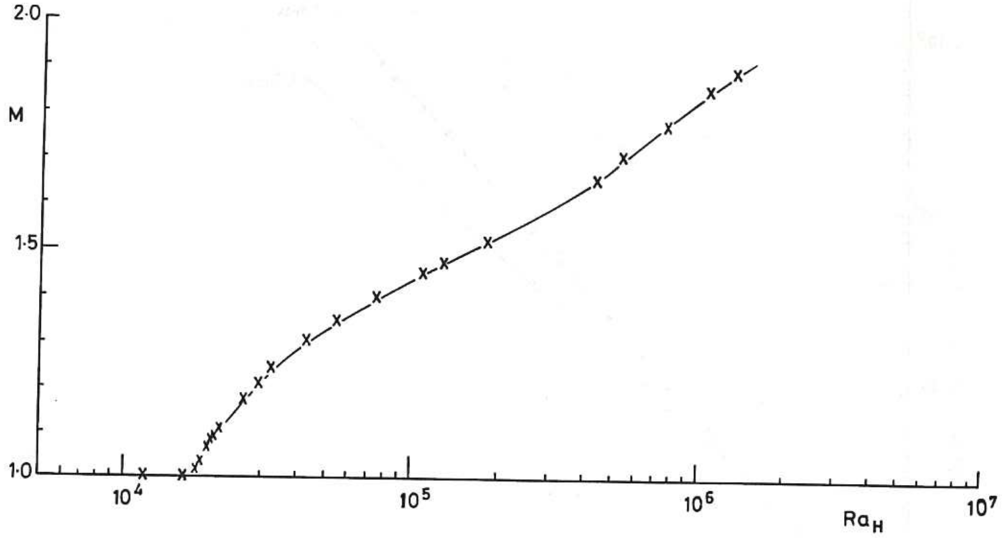

Fig. 3 The convective cooling parameter $M(\equiv \bar{T}_{max}/T^*)$ as a function of Ra_H for $Pr = 8$.

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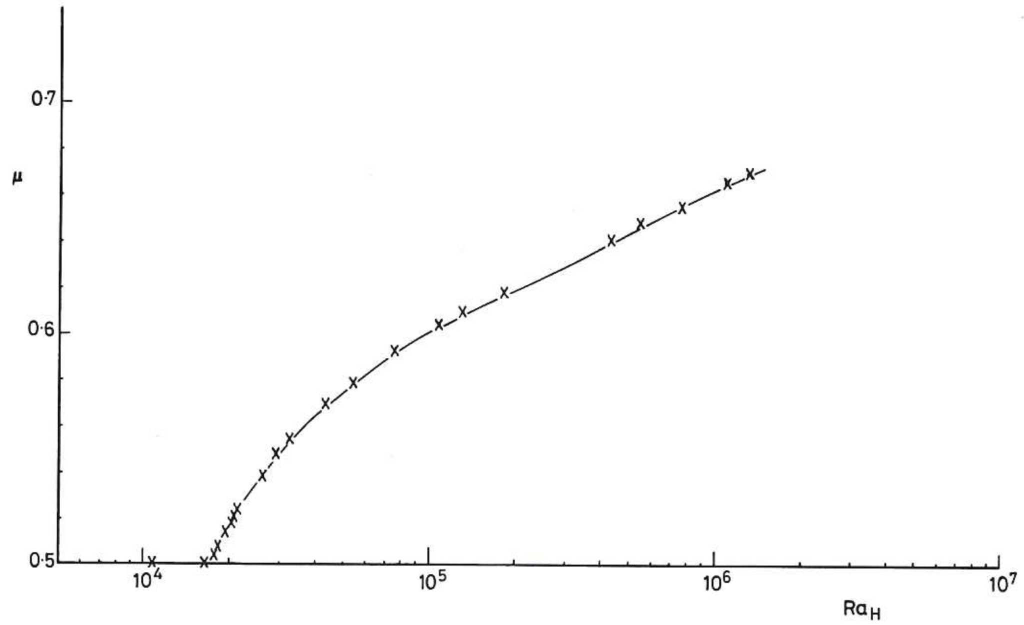

Fig. 4 The flux ratio $\mu(\equiv a/[a+b])$ as a function of Ra_H for $Pr = 8$.

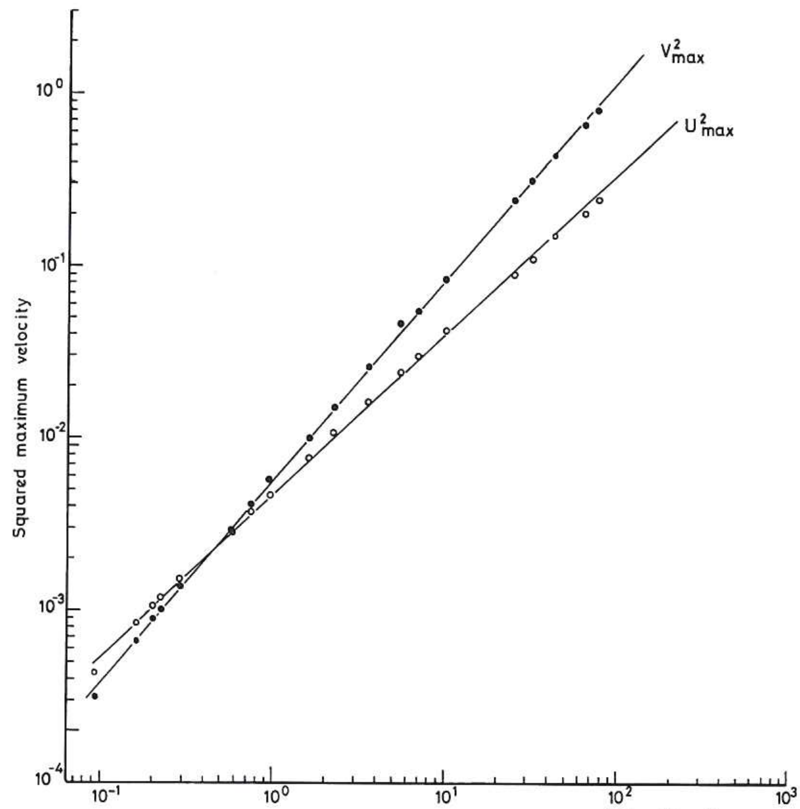

Fig. 5 The squares of the maximum horizontal (u_{max}) and vertical (v_{max}) velocities as functions of Ra_H. The dots are computed values. The straight lines correspond to equations (5) and (6).