CLM-P 358

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UKAEA RESEARCH GROUP

Preprint

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1973

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FLUCTUATION LEVELS CONNECTED WITH REACTIVE MARGINAL INSTABILITIES

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(Submitted for publication in Journal of Plasma Physics)

Abstract

The problem is considered of the analytical calculation of the fluctuation level of the collective electric field in the case of a collisionless non-dissipative plasma subject to a reactive marginal instability. The existence of this instability only depends on the negative sign of the real part of the dielectric constant ϵ and is not related to the dispersive or dissipative effects described by the frequency dependence or the imaginary part of ϵ. A typical example is given by the non-resonance two streams instability with a symmetric distribution function. In this situation the methods of the thermodynamics of Vlasov systems can be applied to calculate the fluctuation level connected with the instability, provided the Vlasov system is near enough to the marginally stable state and the effect of mode coupling can be neglected (at least for a certain time). In the linear limit the unstable equilibrium is associated with a minimum of the entropy. When the instability develops new non linear terms appear in the dielectric coefficient ϵ, related to the inhomogenization of the system associated with the growing collective field. These non linear terms counterbalance the negative sign of the linear part of ϵ, leading to a saturation of the instability. A neighbouring inhomogeneous equilibrium with long wavelength is then realised which corresponds to a maximum of the entropy and is stable (under the assumed approximations). The corresponding electrostatic potential is proportional to $u - u_c$, where u is the velocity corresponding to the maxima of the two stream distribution and u_c is the critical value for u for the onset of the instability. The results of the thermodynamic method for the neighbouring equilibria are compared with those obtained by solving the stationary Vlasov equation with conventional analytical methods or with computer calculations.

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September 1973

1. INTRODUCTION

Among the vast class of possible electrostatic instabilities of the collisionless plasma, there is a very particular subclass which is not related to the dispersive or dissipative effects described by the frequency dependence or the imaginary part of the dielectric constant. This is the class of reactive marginal instabilities, whose existence only depends on the fact that the real part of the dielectric constant ϵ is negative. A typical example is given by the non resonance two stream instability with a symmetric distribution function. In this case the dispersion relation admits a marginal mode with $\omega/k = 0$ and $k^2 = -\lambda^{-2}$, where

$$\lambda^{-2} = -4\pi \Sigma_k \frac{q_s^2}{m_s} \int \frac{1}{v_x} \frac{\partial f_s}{\partial v_x} d^3v \; . \qquad (1)$$

At the marginal frequency one has $(\partial\epsilon/\partial\omega)_{\omega=0}$ and the unstable modes are defined by $\epsilon = 1 + (k\lambda)^{-2} < 0$. The interest of considering marginal reactive instabilities lies on the fact that in view of their non-dissipative character, one can apply the methods of the thermodynamics of the Vlasov systems to calculate the fluctuation levels. Moreover, although they constitute a very particular case of instability, there is the theoretical possibility that they occur often in the experiments as a consequence of the fact that a collisionless slightly unstable system has the tendency to evolve towards the marginally stable state, as shown elsewhere (Minardi 1973). The formation of convective cells could be an example of this fact.

Since the stability depends on the negative sign of ϵ, the saturation process should be related to an opposite effect of the non linear terms of ϵ, associated with the growing electric field.

For the purpose of our discussion we observe that the dielectric constant of a Vlasov inhomogeneous equilibrium, in the absence of dissipative or dispersive effects, can be introduced without considering explicitly the time dependent Vlasov equation or the dispersion relation. In fact every inhomogeneous Vlasov equilibrium is characterised by a function $\sigma = \tilde{\sigma}(\varphi)$ relating the charge σ and the potential φ, as is seen from the equality

$$\sigma(x) \equiv 4\pi \Sigma_s q_s \int f_s(W_s) d^3v = \tilde{\sigma}(\varphi) \; , \qquad (2)$$

where $W_s = m_s v^2/2 + q_s \varphi(x)$. If the equilibrium is homogeneous ($\varphi = \sigma = 0$) and associated with a known distribution $f_{os}(v^2)$, the distribution $f_s(W_s)$ of a neighboring inhomogeneous equilibrium is uniquely determined by the limiting condition $\lim_{\varphi \to 0} f_s(W_s) = f_{os}(v^2)$. The relation (2) is then defined also for initially homogeneous equilibria. However it is important to note that, as is known (Bernstein, Greene and Kruskal, 1959) the limit above only exists if trapped particles are not taken into account.

Let us now consider the fluctuations φ_1 and σ_1 of the potential and the charge density (which are related by Poisson equation) around an equilibrium φ_0, σ_0; σ_1 can be split into two parts, a part related to φ_1 while the equilibrium relation $\sigma = \tilde{\sigma}(\varphi)$ is unchanged, and a part σ_{eff} due to the departure of the system from the equilibrium specified by the functional relation $\sigma = \tilde{\sigma}(\varphi)$:

$$\sigma_1 = \tilde{\sigma}(\varphi + \varphi_1) - \tilde{\sigma}(\varphi) + \sigma_{eff} = \sum_{1}^{\infty}{}_n \left(\frac{d\tilde{\sigma}}{d\varphi}\right)^n_{\varphi_o} \frac{\varphi_1^n}{n!} + \sigma_{eff} . \tag{3}$$

Clearly, if $\sigma_{eff} = 0$, the system does not move from the particular equilibrium defined by $\sigma = \tilde{\sigma}(\varphi)$. So σ_{eff} is the "effective" variation of the charge density in a fluctuation <u>outside</u> equilibrium. Retaining only first order terms in Eq. (3) and passing to the Fourier transforms, one obtains

$$\sigma_{eff,\,k} = \left(1 + \frac{1}{k^2\lambda^2}\right) \sigma_{1,k} \equiv \epsilon_k \, \sigma_{1,k} , \tag{4}$$

where $\lambda^{-2} = -(d\tilde{\sigma}/d\varphi)_{\varphi_0}$ and ϵ_k is, by definition, the linear dielectric constant related to the equilibrium $\sigma = \tilde{\sigma}(\varphi)$ and to the mode k. If however σ_1 is large, higher order terms are important in Eq. (3) and influence significantly σ_{eff} and the corresponding effective energy $W_{eff} = \int \sigma_{eff} \varphi_1 \, dV/8\pi$ of the fluctuation outside equilibrium. The dielectric constant becomes in this case a non-linear operator depending on σ_1 itself. The possibility then exists that a neighboring non linear inhomogeneous equilibrium is realized which is stable as a consequence of the fact that the negative sign of the linear part of ϵ_k is counterbalanced by the non linear terms. In the following the existence of the neighboring equilibrium will be established using thermodynamic methods and the results will be compared with neighboring stationary solutions of

Vlasov equation obtained with conventional analytical methods.

2. THE ENTROPY FUNCTIONAL FOR THE VLASOV EQUILIBRIA

A proper thermodynamic description of the unstable equilibrium and of the subsequent saturation effect considered above would be provided by an entropy functional which is minimum for the initial unstable homogeneous equilibrium. Then when the charge density of the perturbation increases as a consequence of the instability, the entropy functional should reach a maximum associated with a neighboring inhomogeneous stable equilibrium. We will show in Section 4 that this is precisely what happens in the situation we are considering.

The entropy functional associated with any homogeneous or inhomogeneous Vlasov equilibrium was calculated in a preceding paper (Minardi 1973) in the frame of a statistical model of the collective Vlasov equilibria. We will not insist here on the mathematical details of this model but we only recall, for the convenience of the reader, some basic features of the statistical procedure.

One imagines that the collective equilibrium, described by a stationary solution of the Vlasov equation is superimposed on a uniform neutral background of randomly fluctuating ions and electrons, filling a volume V much larger than the volume where the inhomogeneous collective field exists. All electric quantities, as e.g. the charge density σ, can then be split in a random fluctuating part σ_{in} and in a given collective part $\sigma(x)$. From this point of view the total variance of the charge density σ will have the form

$$\overline{(\sigma - \sigma_o)^2} \equiv \overline{\Delta\sigma^2} = \overline{\Delta\sigma_{in}^2} + (\Delta\sigma^2)_c + \frac{1}{V}\int (\sigma(x) - \sigma_o)^2 dV , \qquad (5)$$

where $\Delta\sigma_{in}^2 = \Sigma(4\pi q_s)^2 n_s/\Delta V$ is the variance (averaged in a cell ΔV) of that part of σ which is related to the random fluctuating background of the s-th species of particles of charge q_s and uniform density n_s; $(\Delta\sigma^2)_c$ is a correlation term between the fluctuating part of σ and the given collective part $\sigma(x)$, and the last term of (5) is the variance in space of $\sigma(x)$ with respect to the spatial mean σ_o. The system is characterized by a given value of $\overline{\Delta\sigma^2}$ and also by a given value of the collective energy

$$\Phi = \frac{1}{8\pi}\int \sigma(x) \, [\varphi(x) - \varphi_o] dV , \qquad (6)$$

where φ_o is a suitable reference potential.

Both quantities $\overline{\Delta\sigma^2}$ and Φ should result from the corresponding quantities defined in terms of the fluctuating variable σ, after performing the average over the distribution $p\{\sigma\} \equiv p(\sigma_I, \ldots \sigma_N)$ of σ in the $N = V/\Delta V$ cells in which the system can be arbitrarily subdivided. In fact the two requirements above can be considered as two constraints on $p\{\sigma\}$ itself, namely

$$\int p\{\sigma_j\} \frac{1}{N} \sum_1^N {}_j(\sigma_j - \sigma_o)^2 \, d\Gamma = \overline{\Delta\sigma^2} , \tag{7a}$$

$$\frac{1}{8\pi} \int p\{\sigma_j\} \frac{V}{N} \sum_{1j}^N \sigma_j(\varphi(x_j) - \varphi_o) d\Gamma = \Phi , \tag{7b}$$

where $d\Gamma = d\sigma_1 \ldots d\sigma_N/N!$ is the number of physically distinct states taken by the assembly of the N cells in a volume element of the space of the charge density; $\sigma_j = \sigma_{inj} + \sigma(x_j)$ is the random value of σ in the j^{th} cell. The distribution $p\{\sigma_j\}$ is completely determined by the further requirement that it corresponds to an extremum of the entropy $S = - \int p\{\sigma\} \ell g \, p\{\sigma\} d\Gamma$. One then obtains the following expression for the entropy functional:

$$S(\overline{\Delta\sigma^2}, \tau, \Phi) = N \, \ell g \, \frac{eV}{N} (2\pi\overline{\Delta\sigma^2})^{\frac{1}{2}} + \frac{\Phi}{\tau} . \tag{8}$$

τ is a parameter playing the role of a generalized temperature, which is related to the collective charge and potential by the equation

$$\tau = - \frac{\overline{\Delta\sigma^2_{in}} \, V}{32\pi^2 N} \frac{\int(\varphi - \varphi_o)^2 dV}{\Phi} = - \frac{\overline{\Delta\sigma^2_{in}} \, V}{4\pi N} \frac{\int(\varphi - \varphi_o)^2 dV}{\int\sigma(\varphi - \varphi_o) dV} . \tag{9}$$

The meaning of τ is related to the role played in this model by the infinite homogeneous background; τ is, in fact, a lagrange multiplier determined by the constraint (7b). This constraint implies the absence of a spatial correlation between the randomly fluctuating electric field of the background and the given collective electric field. This is necessary if one is willing to consider collective equilibria with a collective energy defined independently from any fluctuating background, as is the case of the Vlasov equilibria. However one can also consider collective fluctuations in which energy is exchanged between the background and the collective electric field. As we shall see these are precisely the isothermal fluctuations, namely those occuring with τ remaining fixed for any variation of φ and σ ; τ = const. implies in fact that the constraint (7b) is violated, so that a non-zero interaction energy between the background and the collective field appears after the variation. This interaction energy, remembering the r.h.s. of Eq. (7b), is given by the expression

$$\Phi_{int} \equiv \frac{V}{8\pi N}\int p\{\sigma_j\}\ \Sigma_j \sigma_{in\,j}\left(\varphi(y_j) - \varphi_o\right) d\Gamma \equiv \frac{1}{8\pi}\left(\overline{\sigma_{in}(x)}\ (\varphi(x) - \varphi_o)\right) =$$

$$= -\frac{1}{8\pi}\int\left[\sigma(x)\left(\varphi(x) - \varphi_o\right) + \frac{V\overline{\Delta\sigma^2_{in}}}{4\pi\tau N}\left(\varphi(x) - \varphi_o\right)^2\right] dV \ , \qquad (10)$$

where use has been made of the expression for $\overline{\sigma_{in}}$ derived in previous papers. If τ is related to σ and φ by Eq. (8), Φ_{int} vanishes. If however one perturbs σ and φ, while τ is kept fixed, one obtains from Eq. (10) at second order in σ_1 and φ_1 (when the initial equilibrium is homogeneous):

$$\delta^2\Phi_{int} = -\frac{1}{8\pi}\int(\sigma_1\varphi_1 + \lambda^{-2}\varphi_1^2)dV = \frac{1}{8\pi}\sum_k \frac{|\sigma_{ik}|^2}{k^2}\left(1 + \frac{1}{k^2\lambda^2}\right) = -\frac{1}{8\pi}\sum_k \epsilon_k \frac{|\sigma_{ik}|^2}{k^2} \ . \qquad (11)$$

This is just the total electrostatic energy at second order of a non dispersive plasma with the opposite sign. The variation can thus be interpreted as one in which the total electrostatic energy of the fluctuating collective electric field is supplied through the interaction with the background, so that the total energy (interaction energy with the background plus total electrostatic collective energy) is conserved. Under these conditions, the collective field is not strictly at equilibrium, because the constraint (7b) characterising all Vlasov equilibria is violated, but a kind of quasi-equilibrium is realized through the interaction with the background and the collective quantities fluctuate isothermally around a Vlasov equilibrium with zero interaction energy. This situation is quite similar to the case of the isothermal fluctuations in the statistics of Gibbs. The infinite homogeneous background plays the same role as the heat reservoir. The existence of the background allows to simulate, through the interaction between the collective field and the background, the complicated dynamical mechanisms which bring a system out of an originally unstable equilibrium towards a finally stable equilibrium. Since it is not unreasonable to assume that such dynamical mechanisms always exist in practice (in view of the highly complicated set of modes which can be excited in the collisionless plasma) the results about the instability of a collective equilibrium in the presence of the background can be extrapolated to the case of an isolated Vlasov system.

3. LINEAR AND NON-LINEAR ISOTHERMAL FLUCTUATIONS

It is easy to verify that the entropy functional is minimum only with respect to those isothermal perturbations which are just predicted unstable by the linearized treatment of the Vlasov equation. This constitutes an example of a dynamical result which can be found using purely thermodynamic methods.

Expanding the logarithm to the first power of $[(\Delta\sigma^2)_c + \int\sigma^2 dV/V]/\overline{\Delta\sigma^2_{in}}$ and putting for simplicity $\sigma_o = 0$ the expression (7) for the entropy becomes

$$S = N\,\ell g \frac{eV}{N}\,(2\pi\,\overline{\Delta\sigma^2_{in}})^{\frac{1}{2}} + \frac{N}{2\overline{\Delta\sigma^2_{in}}}\left((\Delta\sigma^2)_c + \frac{1}{V}\int\sigma^2\,dV\right) + \frac{\Phi}{\tau}\,. \tag{12}$$

Here the correlation term $(\Delta\sigma^2)_c$ is given by the equation (see previous papers)

$$(\Delta\sigma^2)_c \equiv \frac{2}{V}\int\overline{\sigma_{in}(x)\sigma(x)}dV = -\frac{2}{V}\left[\int\sigma^2 dV + \frac{V\overline{\Delta\sigma^2_{in}}}{4\pi N\tau}\int\sigma(\varphi-\varphi_o)dV\right]. \tag{13}$$

The isothermal variation of S at second order is then given by the expression

$$\delta^2 S = -\frac{N}{V\overline{\Delta\sigma^2_{in}}}\left[\int\sigma_1^2\,dV + \frac{V\overline{\Delta\sigma^2_{in}}}{4\pi N\tau}\int\sigma_1\varphi_1 dV\right] + \frac{N}{2V\overline{\Delta\sigma^2_{in}}}\int\sigma_1^2\,dV + \frac{1}{8\pi\tau}\int\sigma_1\varphi_1\,dV$$

$$= -\frac{N}{V\overline{\Delta\sigma^2_{in}}}\int\sigma_1\,dV - \frac{1}{4\pi\tau}\int\sigma_1\varphi_1\,dV\,, \tag{14}$$

where τ is given by Eq. (8). Noting that for a nearly homogeneous plasma one has $\varphi = \varphi_o - \lambda^{-2}\sigma$, one immediately obtains from Eq. (9)

$$\tau = \frac{V\overline{\Delta\sigma^2_{in}}\,\lambda^2}{4\pi N} \,. \tag{15}$$

So $\delta^2 S$ takes the form :

$$\delta^2 S = -\frac{N}{2V\overline{\Delta\sigma^2_{in}}}\left[\int \sigma_1^2 dV + \frac{1}{\lambda^2}\int \sigma_1\varphi_1 dV\right] = -\frac{N}{2\overline{\Delta\sigma^2_{in}}\,V}\sum_k\left(1+\frac{1}{\lambda^2 k^2}\right)|\sigma_{1k}|^2 . \tag{16}$$

One then sees that the entropy is minimum ($\delta^2 S > 0$) for $k^2 < -\lambda^{-2}$, which are just the values of k corresponding to the unstable modes in the dispersion relation.

The probability distribution for the collective fluctuations can be calculated using the Einstein relation

$$p\{\sigma_{1k}\} = p_o \exp \delta^2 S = p_o \exp -\frac{N}{2\overline{\Delta\sigma^2_{in}}\,V}\sum_k\left(1+\frac{1}{k^2\lambda^2}\right)|\sigma_{1k}|^2 . \tag{17}$$

Clearly, if one calculates the average of $|\sigma_{1k}|^2$ by means of the distribution above, one finds that it is infinite for the unstable modes $k^2 < -\lambda^{-2}$. However as we know from section 1, higher order terms exist in the effective charge and in the dielectric constant which were not considered in the preceding calculation. It is then natural to investigate whether these non-linear terms can make finite the average value of $|\sigma_{1k}|^2$, also in the unstable case. In fact, as we know from section 1, when the fluctuation σ_1 is large, the effective charge density σ_{eff} associated with a fluctuation outside equilibrium can be significantly affected by the non-linear terms and the same holds for the effective energy $W_{eff} = \int \sigma_{eff}\,\varphi_1\, dV/8\pi$ of a collective fluctuation outside equilibrium. Now this energy, just as in the linear case considered above, should be supplied to the collective field through the interaction with the background. In other words the energy conservation imposes that the variation of the interaction energy Φ_{int}, (defined by Eq. (10)), should be equal to $-W_{eff}$. This condition is easily satisfied if, instead of the variation σ_1 of σ considered in the linear case, one introduces a new variation $\Delta\sigma$ which includes the higher order terms:

$$\Delta\sigma \equiv \sigma_1 - \sum_{2^n}^{\infty}\left(\frac{d^n\tilde{\sigma}}{d\varphi^n}\right)_o \frac{\varphi_1^n}{n!} \,. \tag{18}$$

In fact the corresponding variations of Φ_{int}, when σ is perturbed by $\Delta\sigma$, takes the form (remembering Eq. 10):

$$\Delta\Phi_{int} = -\frac{1}{8\pi}\int\left(\varphi_1\Delta\sigma + \frac{V\overline{\Delta\sigma^2_{in}}}{4\pi\tau N}\varphi_1^2\right)dV = -\frac{1}{8\pi}\int\left[\varphi_1\left(\sigma_1 - \sum_2^\infty\left(\frac{d^n\tilde{\sigma}}{d\varphi^n}\right)_o\frac{\varphi_1^n}{n!}\right)\frac{1}{\lambda^2}\varphi_1^2\right]dV$$

$$= -\frac{1}{8\pi}\int\left[\sigma_1 - \sum_1^\infty\left(\frac{d^n\tilde{\sigma}}{d\varphi^n}\right)_o\frac{\varphi_1^n}{n!}\right]\varphi_1\, dV = -\frac{1}{8\pi}\int\sigma_{eff}\,\varphi_1 dV = -W_{eff}\,. \tag{19}$$

It is then seen that, by introducing the variation $\Delta\sigma$, the effect of the inhomogeneity of the equilibrium and of the related higher order terms is properly taken into account in the isothermal energy fluctuations around a given Vlasov equilibrium with zero interaction energy. The isothermal entropy variation corresponding to the variation $\Delta\sigma$ can easily be obtained from Eq. (14) by replacing σ_1 with $\Delta\sigma$:

$$\Delta S_\tau \equiv -\frac{N}{V\overline{\Delta\sigma^2_{in}}}\int\left(\sigma_1 - \sum_2^\infty\left(\frac{d^n\tilde{\sigma}}{d\varphi^n}\right)_o\frac{\varphi_1^n}{n!}\right)^2 dV - \frac{1}{4\pi\tau}\int\left(\sigma_1 - \sum_2^\infty\left(\frac{d^n\tilde{\sigma}}{d\varphi^n}\right)_o\frac{\varphi_1^n}{n!}\right)\varphi_1 dV\,. \tag{20}$$

We are now in a position to see from this expression whether a neighboring inhomogeneous equilibrium exists corresponding to a maximum of ΔS_τ for a non-vanishing value of the perturbed quantities σ_1 and φ_1.

4. THE NEIGHBORING INHOMOGENEOUS STABLE EQUILIBRIUM

For large values of the fluctuations the first term in ΔS is clearly dominating and gives a negative contribution to the entropy variation. If the fluctuations are not excessively large it is sufficient to express ΔS_τ only up to the fourth order in φ_1 and σ_1 by neglecting in $\Delta\sigma$ (see Eq. 18) the terms with $n > 2$:

$$\Delta S_\tau = -\frac{N}{2V\overline{\Delta\sigma^2_{in}}}\int\left(\sigma_1^2 + \frac{1}{\lambda_1^2}\sigma_1\varphi_1 + \frac{1}{\lambda_2^4}\frac{\varphi_1^4}{4}\right)dV \tag{21}$$

where $\lambda_1^{-2} \equiv \lambda^{-2} = -(d\tilde{\sigma}/d\varphi)_o$ and $\lambda_2^{-2} \equiv -(d^2\tilde{\sigma}/d\varphi^2)_o$.

Let us express $\varphi_1(x)$ in terms of a Fourier expansion in V

$$\varphi_1(x) = \frac{1}{V^{\frac{1}{2}}} \Sigma_{\vec{k}} \varphi_{\vec{k}} \exp i \vec{k}\,\vec{x} . \qquad (22)$$

One can then easily derive the following equalities

$$\varphi_1^2(x) = \frac{1}{V} \Sigma_{\vec{q}} p_{\vec{q}} \exp i \vec{q}.\vec{x} , \qquad (23)$$

where

$$p_{\vec{q}} = \frac{1}{V^{\frac{1}{2}}} \Sigma_{\vec{k}} \varphi_{\vec{k}} \varphi_{\vec{q}-\vec{k}} \quad \text{with} \quad p^*_{\vec{q}} = p_{-\vec{q}}$$

and

$$\int \varphi_1^4(x)_o W = \Sigma_{\vec{q}} p_{\vec{q}} p_{-\vec{q}} = \frac{1}{V} \Sigma_{\vec{q}} \left| \Sigma_{\vec{k}} \varphi_{\vec{k}} \varphi_{\vec{q}-\vec{k}} \right|^2 . \qquad (24)$$

The entropy variation ΔS_τ can now be expressed in terms of the new set of variables $\varphi_{\vec{k}}$:

$$\Delta S_\tau \{\varphi_k\} = - \frac{N}{2V\overline{\Delta\sigma^2_{in}}} \left[\Sigma_k \left(1 + \frac{1}{\lambda^2 k^2}\right) k^4 |\varphi_{\vec{k}}|^2 + \frac{1}{4\lambda_2^4 V} \Sigma_{\vec{q}} \left| \Sigma_k \varphi_{\vec{k}} \varphi_{\vec{q}-\vec{k}} \right|^2 \right] . \qquad (25)$$

We shall limit our considerations to the subclass of configurations which are associated with the subset of values of the $\varphi_{\vec{k}}$ such that $p_{\vec{q}} = 0$ for $\vec{q} \neq 0$. In fact, since the fluctuations φ_k are random variables, both the real and imaginary parts of $p_{\vec{q}}$ (with $\vec{q} \neq 0$) fluctuate around the value $p_{\vec{q}} = 0$, which corresponds to the absence of coupling terms involving the product of two φ_k associated with different $\vec{k}$ modes. With this assumption only the term $\vec{q} = 0$ is retained in the $\vec{q}$-summation in the expression for $\Delta S_\tau \{\varphi_k\}$, and one finally obtains

$$\Delta S_\tau \{\varphi_k\} = - \frac{N}{V\overline{\Delta\sigma^2_{in}}} \left[\Sigma_{k>0} \left(1 + \frac{1}{\lambda_1^2 k^2}\right) k^4 \left(\varphi_{1k}^2 + \varphi_{2k}^2\right) + \frac{1}{2\lambda_2^4 V} \left(\Sigma_{k>0} (\varphi_{1k}^2 + \varphi_{2k}^2) \right)^2 \right] \qquad (26)$$

where φ_{1k} and φ_{2k} are the real and imaginary parts of $\varphi_{\vec{k}}$, and the subscript $k > 0$ means that of each pair $k, -k$, only one is included in the sum.

We will now look for the extremum points of $\Delta S_\tau \{\varphi_k\}$, which are determined by the conditions

$$\frac{\partial}{\partial \varphi_{jk}} \Delta S_\tau \{\varphi_{jk}\} = 0 \qquad \text{for } j = 1, 2 \text{ and any } k_i , \tag{27}$$

or

$$\left(1 + \frac{1}{\lambda_1^2 k_i^2}\right) k_i^4 \varphi_{j k_i} + \frac{1}{\lambda_2^4} \varphi_{j k_j} \Sigma_{k>0} (\varphi_{1k}^2 + \varphi_{2k}^2) = 0 . \tag{28}$$

This system of equations admits the trivial solution $\varphi_{j k_i} = 0$ for any j and k_i. This solution corresponds to a minimum of ΔS_τ and describes the unstable homogeneous equilibrium. The system (28) then admits only <u>one</u> further solution, associated with all $\varphi_{jk_i} = 0$ equal to zero with the exception of only one, say φ_{jk_n}, which satisfies the equality:

$$\varphi_{1 k_n}^2 + \varphi_{2 k_n}^2 = - \lambda_2^4 V \; k_n^4 \left(1 + \frac{1}{\lambda_1^2 k_n^2} \right) . \tag{29}$$

This equation admits an acceptable solution only for $k_n^2 < - \lambda^{-2}$, namely for a linearly unstable k_n. We observe that, since we are considering a situation near the marginal state of equilibrium, one has $\lambda^{-2} \approx 0$ and only few k_n can satisfy the instability condition $k_n^2 < - \lambda^{-2}$. As one can easily verify, the second solution found above corresponds to a maximum of the entropy. The present theory therefore predicts that a neighboring inhomogeneous <u>stable</u> equilibrium exists which is associated with a <u>single</u> linearly unstable k_n mode. In order to determine this mode we take the maximum value of $(\Delta S)_\tau$ by substituting Eq. (29) into the expression (26) for ΔS_τ and obtaining :

$$(\Delta S_\tau)_M = \frac{N}{2V\overline{\Delta\sigma_{in}^2}} \; k_n^8 \left(1 + \frac{1}{\lambda_1^2 k_n^2} \right)^2 . \tag{30}$$

One easily sees that there is a value of k_n for which $(\Delta S_\tau)_M$ has a sharp maximum which then corresponds to the most probable mode. This value is $|k_n \lambda_1| = 2^{-\frac{1}{2}}$ and it turns out to be very near to the mode $|k\lambda_1| = 3^{-\frac{1}{2}}$ associated with the largest growth rate predicted by the linear theory.

5. THE PROBABILITY DISTRIBUTION OF THE COLLECTIVE FLUCTUATIONS

The probability distribution of the isothermal collective fluctuations can be calculated applying the Einstein relation

$$P = P_o \exp(\Delta S)_T . \tag{31}$$

Using the equality $\sigma_{jk} = k^2 \sigma_{jk}$ one can express P in terms of the Fourier transform of the charge density distribution σ_{jk}. Limiting our consideration to situations with one single k mode (which are the most probable, as we have just seen) one has

$$P\{\sigma_{jk}\} = P_o \exp - \frac{N}{V\overline{\Delta\sigma^2_{in}}}\left[\left(1+\frac{1}{\lambda_1^2 k^2}\right)\left(\sigma_{1k}^2+\sigma_{2k}^2\right)+\frac{1}{2\lambda_2^4 k^8 V}\left(\sigma_{1k}^2+\sigma_{2k}^2\right)^2\right].$$

The average of $\sigma_{1k}^2 + \sigma_{2k}^2$ is then given by the expression

$$\langle(\sigma_{1k}^2 + \sigma_{2k}^2)\rangle = \frac{\int_{-\infty}^{+\infty}(\sigma_{1k}^2+\sigma_{2k}^2)P\{\sigma_{jk}\}d\sigma_{1k}d\sigma_{2k}}{\int_{-\infty}^{+\infty}P\{\sigma_{jk}\}d\sigma_{1k}d\sigma_{2k}} = \frac{\int_0^\infty t\exp(-\alpha t - \gamma t^2)dt}{\int_0^\infty \exp(-\alpha t - \gamma t^2)dt}, \tag{32}$$

where

$$t = \sigma_{1k}^2 + \sigma_{2k}^2 ,$$

$$\alpha = \frac{N}{V\overline{\Delta\sigma^2_{in}}}\left(1 + \frac{1}{\lambda_1^2 k^2}\right), \tag{33}$$

$$2\gamma = \frac{N}{V^2\overline{\Delta\sigma^2_{in}}\,\lambda_2^4 k^8} .$$

The integrals can conveniently be expressed in terms of the parabolic cylinder functions $D_\nu(z)$ of Whittaker (Abramovitz et al. 1969) using the general formula (Erdélyi et al. 1959)

$$\int_0^\infty t^{\nu-1}\exp(-\alpha t - \gamma t^2)dt = \Gamma(\nu)\,(2\gamma)^{-\frac{\nu}{2}}\exp(\alpha^2/8\gamma)D_{-\nu}\left(\frac{\alpha}{(2\gamma)^{\frac{1}{2}}}\right). \tag{34}$$

The argument

$$\frac{\alpha}{(2\gamma)^{\frac{1}{2}}} = \left(\frac{N}{\Delta\sigma_{in}^2}\right)^{\frac{1}{2}}\left(1 + \frac{1}{\lambda^2 k^2}\right)\left|\lambda_2\right|^2 k^4 \tag{35}$$

is a very large number (because N is large), so that the asymptotic representations of the $D_{-\nu}$ can be used (Abramovitz et al. 1969). In the unstable region $\alpha < 0$, which is of interest to us, one obtains

$$\langle(\sigma_{1k}^2 + \sigma_{2k}^2)\rangle = \frac{\alpha}{2\gamma} = \lambda_2^4 V k^8 \left|1 + \frac{1}{\lambda_1^2 k^2}\right| . \tag{36}$$

It follows that the energy spectrum is given by the expression

$$W_k = \langle|\sigma_k/k|^2\rangle = \lambda_2^4 V k^6 \left|1 + \frac{1}{\lambda_1^2 k^2}\right| \tag{37}$$

and is strongly peaked at $|k\lambda_1| = (2/3)^{\frac{1}{2}}$.

The mean square value of φ is expressed by the relation:

$$\langle\varphi^2\rangle = \frac{1}{V} \langle |\varphi_k|^2\rangle = \lambda_2^4 k^4 \left| 1 + \frac{1}{\lambda_1^2 k^2} \right| \tag{38}$$

and has a maximum for $k|\lambda_1| = \pm 2^{-\frac{1}{2}}$. Summing up with respect to the two signs of k, the maximum value is then given by the equality

$$\langle\varphi_k^2\rangle_M = \frac{1}{2}\left(\frac{\lambda_2}{\lambda_1}\right)^4 . \tag{39}$$

This result can be interpreted in terms of the saturation of the negative part of the dielectric constant, as anticipated in section 1. In fact, one has for the effective charge density up to second order (see Eq. (3))

$$\sigma_{eff} = \sigma_1 + \frac{1}{\lambda_1^2} \varphi_1 + \frac{1}{2\lambda_2^2} \varphi_1^2 . \tag{40}$$

If then the second order term has to compensate, at least partially, the negative contribution to the dielectric constant resulting from the first order term, φ_1^2 should be proportional to $(\lambda_2/\lambda_1)^4$, in agreement with Eq. (39). This equation can also be understood noting that the marginal state is associated with a zero value of $\lambda^{-2}(\varphi)$ (see Eq. (1)). Now expanding $\lambda^{-2}(\varphi)$ up to first order one has

$$\lambda^{-2}(\varphi) = \lambda_1^{-2} + \lambda_2^{-2}\,\varphi\,. \tag{41}$$

Since we are considering a situation near the marginal state, λ_1^{-2} is very small, so that a small fluctuation $\varphi \sim -\lambda_1^{-2}/\lambda_2^{-2}$ is sufficient for reaching the marginal state $\lambda^{-2}(\varphi) = 0$.

Our results can be compared with the computer calculations performed in a similar case by Biskamp and Welter (1972). These authors have shown that when the unstable equilibrium is near enough to the marginal equilibrium, a neighboring stationary inhomogeneous equilibrium exists which is associated with the strongest mode of the linear theory and persists as far as mode coupling is neglected. This agrees with our conclusions. However the present theory predicts that the value of $\langle\varphi^2\rangle$ is proportional to λ_1^{-4} , and then to ϵ^2 , where $\epsilon = (u - u_c)v_{th}$ (v_{th} is the thermal velocity, u the velocity associated with the maxima of the two stream distribution and u_c is the critical value of u for the onset of the instability) provided the unstable equilibrium is near enough to the marginal state.

6. COMPARISON WITH A CLASS OF STATIONARY NON-LINEAR SOLUTIONS OF VLASOV EQUATION

The existence of neighboring non-linear equilibria predicted by the thermodynamic method in the case of the reactive marginal instability considered above can also be recovered by looking at the non linear stationary solutions of the Vlasov equation in the neighborhood of the unstable homogeneous equilibrium. By comparing the results of the two methods one can obtain a clear insight on the reliability and the advantages of the thermodynamic method.

In the case under consideration the Poisson equation, up to second order in φ, takes the form

$$-\frac{d^2\varphi}{dx^2} = \tilde{\sigma}(\varphi) = -\frac{1}{\lambda_1^2}\,\varphi - \frac{1}{\lambda_2^2}\,\frac{\varphi^2}{2}\,, \tag{42}$$

from which follows the first integral

$$\left(\frac{d\varphi}{dx}\right)^2 = \frac{1}{\lambda_1^2}\varphi^2 + \frac{1}{3\lambda_2^2}\varphi^3 + C \equiv F(\varphi) \ , \tag{43}$$

where C is an integration constant. A second integration leads to the expression

$$x - x_1 = \int_{\varphi_1}^{\varphi} \frac{d\varphi}{\sqrt{F(\varphi)}} \ , \tag{44}$$

where for φ_1 we take the lowest zero of $F(\varphi) = 0$. We are looking for periodic solutions $\varphi = \varphi(x - x_1)$ which are regular everywhere in the infinite plasma. These only exist if $F(\varphi) = 0$ admits another finite zero φ_2 such that $F(\varphi) > 0$ for $\varphi_1 < \varphi < \varphi_2$. One can verify that $F(\varphi) = 0$ admits more than one zero only if

$$\left(\frac{d\varphi}{dx}\right)^2_M = C < \frac{4}{3}\frac{\lambda_2^4}{|\lambda_1|^6} \ , \tag{45}$$

where the subscript M indicates the maximum value of $(d\varphi/dx)^2$. One has then a class of equilibria depending on C and corresponding to an electric field limited by the condition (45) and to a potential φ varying between φ_1 and φ_2. These equilibria can easily be expressed in terms of the Weierstrass function $\wp(x, g_2, g_3)$ (Whittaker and Watson, 1959)

$$\varphi - \varphi_1 = \frac{1}{4}F'(\varphi_1)\left\{\wp\left(x - x_1, g_2, g_3\right) - \frac{1}{24}F''(\varphi_1)\right\}^{-1} , \tag{46}$$

where g_2 and g_3 are given by the expressions:

$$g_2 = \lambda_1^{-4}12^{-1}, \quad g_3 = \lambda_1^{-6}6^{-3} - 12^{-2}C\lambda_2^{-4} . \tag{47}$$

The function $\varphi(x - x_1)$ is a periodic function which oscillates around zero with a period depending on the parameter $\eta = C|\lambda_1|^6/\lambda_2^4 > 0$, related to the amplitude of the oscillation. When η (or C) approaches zero the period has a value near 2π. In this case the modes possess small amplitude and their wavelength is in fact near the marginal wavelength $2\pi|\lambda_1|$ of the linear theory. When η increases, the wavelength also increases, corresponding to the unstable modes with $k^2 < -\lambda^{-2}$ of the linear theory and approaches infinity when η approaches the limiting value $\eta = 4/3$.

The limitations on the electric field and the potentials are essentially the same as those expressed by Eqs. (37) or (39) derived with the thermodynamic method. For instance the wavelength $2\pi/k = (2)^{\frac{1}{2}} 2\pi|\lambda_1|$ corresponding to the maximum (39) of $\langle\varphi^2\rangle$ is associated with the value $\eta \approx 1$ in the class of equilibria obtained above. The potential φ then oscillates between $\varphi_2 \sim -2\lambda_2^2|\lambda_1^2$ and $\varphi_1 \sim \lambda_2^2|\lambda_1^2$, so that the average value of φ^2 is just of the same order as given by Eq. (39).

The results of the thermodynamic method are then confirmed by the analysis of the neighboring non-linear stationary solutions of Vlasov equation. Moreover the thermodynamic method allows to reach further conclusions which are not contained in the preceding analysis. In fact no preference arises from this analysis for one of the equilibria labelled by η or C, while the thermodynamic method definitely indicates which mode is the most probable and thus is the most likely to be found in the experiments (in fact it turns out that the most probable mode is very near to the strongest mode of the linear theory). Furthermore the stability of the non-linear equilibria can be easily discussed with the thermodynamic method because it is described by the extremum properties of the entropy. Another advantage of the thermodynamic method is that one does not need to solve explicitly a non-linear differential equation for the neighboring equilibrium or solve linearized time dependent equations for the stability.

7. FINAL REMARKS

In the present paper the thermodynamic method is applied to the study of the behaviour of a plasma outside a marginally unstable equilibrium, when the instability is of the reactive type. The existence is predicted of neighboring inhomogeneous stable equilibria with definite average amplitude which are stable (at least for a certain time). The results of the thermodynamic method are confirmed by a conventional treatment of the neighboring non-linear equilibria of the Vlasov equation.

While the thermodynamics of the Vlasov equilibria developed in preceding papers can be applied to the discussion of the stability of any homogeneous or inhomogeneous collisionless equilibrium, its validity for the description of the behaviour of a system <u>outside</u> equilibrium is limited to the case of non-dispersive, dissipationless marginal instabilities of a nearly homogeneous equilibrium. Indeed in this case the perturbed system is conservative and can be expected to be described by suitably defined potentials. Furthermore, if the system is near enough to the marginal state, the perturbation of the homogeneous equilibrium gives rise to neighboring inhomogeneous "quasi" equilibria so that, the system being approximately time independent, the thermodynamic approach can be applied. In fact it follows from our procedure that the existence of such neighboring non-linear equilibria is a general property of the marginal reactive instabilities.

From the mathematical point of view the reactive marginal instabilities represent a very particular case. However since a collisionless plasma has the tendency to evolve towards marginally stable non dispersive states, which are associated with an absolute maximum of the collisionless entropy (Minardi 1973), one should expect that the situationsdescribed in this paper are often realized in the experiments.

REFERENCES

ABRAMOVITZ, M. and STEGUN, I.A., Handbook of Mathematical Functions (National Bureau of Standards, Washington, D.C., 1969), p. 637-689.

BERNSTEIN, I.B., GREENE, J.M. and KRUSKAL, M.D., 1957 Phys. Rev. 108, 546.

BISKAMP, D. and WELTER, H. 1972, Nuclear Fusion 12, 89.

ERDÉLYI, A., MAGNUS, W., OBERHETTINGER, F. and TRICOMI, F.G., Tables of Integral Transforms, Vol. I (McGraw Hill Book Co. Inc., New York, 1959).

MINARDI, E., 1973 Phys. Fluids 16, 122. Survey Lecture at the Innsbruck Conference 1 - 7 April 1973, Culham Report No. CLM-R 124.

WHITTAKER, E.T. and WATSON, G.N., Modern Analysis (Cambridge University Press, Cambridge, 1959) p. 452.