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LOW FREQUENCY STABILITY THEORY OF AXISYMMETRIC TOROIDAL PLASMAS PART I - GENERAL THEORY

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LOW FREQUENCY STABILITY THEORY OF AXISYMMETRIC TOROIDAL PLASMAS PART I GENERAL THEORY*

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ABSTRACT

A formalism for discussing low frequency electrostatic modes in a general axisymmetric toroidal geometry is developed. Particular emphasis is laid on the conflict between the requirements of long parallel wavelengths and double toroidal periodicity in the presence of shear. Expressions for the perturbed charge density, useful for examining flute, drift and trapped particle modes are presented.

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I. INTRODUCTION

The kinetic theory of low frequency electrostatic modes of short transverse wavelength and long parallel wavelength ($k_\perp a_i \sim 1$, $k_\parallel a_i \sim \epsilon$ where a_i is the ion larmor radius, and ϵ labels the small parameter a_i/L, L being a typical equilibrium scale length) is well understood and comprehensively documented for toroidal plasmas confined by magnetic fields which have closed lines of force. A convenient procedure was established by TAYLOR and HASTIE (1968) and RUTHERFORD and FRIEMAN (1968), and has been applied to a variety of modes (COPPI et al 1968,a; b; RUTHERFORD et al,1969; COPPI, 1969 a;b; HORTON, 1969). Difficulties arise, however, when the equivalent analysis is attempted for configurations in which the magnetic field lines form closed surfaces, and magnetic field shear is present, i.e. for the investigation of flute instabilities, drift waves and short wavelength trapped particle modes in Stellarators, Tokamaks or Levitrons.

In this paper we consider axisymmetric toroidal geometry and for simplicity consider vacuum magnetic fields so that our results apply strictly only to the Levitron. However the method adopted here is equally applicable to finite β. In section II we discuss the problems encountered by previous treatments. These are essentially caused by the conflict between the periodicity constraints implied by the toroidal geometry and the $k_\parallel a_i \sim \epsilon$ condition, in strongly sheared magnetic fields. A resolution of this conflict is proposed and in section III a consistent solution of the Vlasov equation is obtained. In section IV the contribution of each particle species to the perturbed charge density for an inhomogeneous Maxwellian plasma is given in the limits :

$$\left.\begin{array}{l} \omega < \omega_b, \ \omega/k_\parallel > V_T \\ \omega > \omega_b, \ \omega/k_\parallel > V_T \\ \omega < \omega_b, \ \omega/k_\parallel < V_T \end{array}\right\} \qquad (1)$$

where ω_b is the bounce frequency in the toroidal field modulations, and V_T the thermal velocity. The first two limits are of interest for the study of drift and flute modes while the last one pertains to the trapped particle modes. In the second paper (Part II) the results derived here are used in a detailed investigation of flute modes.

II. PERIODICITY IN A TORUS

(a) The Equilibrium

The equilibrium to be considered is that of a low β plasma in an axisymmetric vacuum magnetic field, having toroidal and poloidal components given by

$$\underset{\sim}{B}_T = I/R\ \underset{\sim}{\hat{e}}_\theta \tag{2}$$

$$\underset{\sim}{B}_p = \nabla\chi = \nabla\psi \times \nabla\theta \tag{3}$$

where (ψ, χ, θ) form an orthogonal set of coordinates, ψ being the poloidal magnetic flux, θ the angle about the axis of symmetry, while R is the distance to that axis and I is a constant. No approximation is made regarding the aspect ratio of the torus, or the shape of magnetic surfaces. In the (ψ,χ,θ) coordinate system the gradient operator is given by

$$\nabla = \underset{\sim}{\hat{e}}_\psi\, RB_p \frac{\partial}{\partial\psi} + \underset{\sim}{\hat{e}}_\chi\, B_p \frac{\partial}{\partial\chi} + \frac{\underset{\sim}{\hat{e}}_\theta}{R} \frac{\partial}{\partial\theta} \tag{4}$$

where $\underset{\sim}{\hat{e}}_\psi$, $\underset{\sim}{\hat{e}}_\chi$, $\underset{\sim}{\hat{e}}_\theta$ are unit vectors along the respective coordinate axes. We denote the rotational transform by ι where

$$\iota = (\oint \nu d\chi)^{-1} \text{ with } \nu = \frac{B_T}{RB_p^{\ 2}}\ .$$

The equilibrium distribution function is assumed to be the collisionless Vlasov distribution, whose zero order limit in the m/e expansion is

$$F_o = F_o\ (\mu,\kappa,\psi) \tag{5}$$

where $\kappa = \frac{1}{2} v^2$, the energy per unit mass, and $\mu = \frac{1}{2}\frac{v_\perp^{\ 2}}{B}$, the magnetic moment per unit mass. Velocity space variables $\mu,\kappa,\varphi,\sigma$, will be used in the following analysis, where φ is the phase angle associated with the gyro-motion and $\sigma = v_\parallel/|v_\parallel|$ is the sign of the longitudinal velocity.

(b) The Perturbation

The class of instabilities we wish to investigate has short transverse wavelength ($k_\perp a_i \sim 1$) and long wavelength parallel to the magnetic field direction ($k_\parallel a_i \sim \epsilon$) and will necessarily be localised about a magnetic surface. Previous analyses (RUTHERFORD et al, 1968; JAMIN, 1971)

$$\Phi = \Phi_o(\psi,\chi)e^{i\frac{\ell}{\epsilon}[\theta - \int \nu d\chi] + i\frac{S(\psi)}{\epsilon} + i\omega t} \tag{6}$$

(where ϵ is included to label the large terms) for the perturbed electrostatic potential, since then the phase factor is strictly constant along a field line and $k_{\parallel} a_i \sim \epsilon$ is thus assured. However this form of potential is only valid in the limit of very weak shear. This follows from the requirement that Φ be a periodic function of χ, so that $\Phi_o(\chi)$ must satisfy the condition

$$\Phi_o(\chi + \chi_o) e^{i \frac{\ell}{\epsilon} \oint \nu d\chi} - \Phi_o(\chi) = 0 \tag{7}$$

and hence after differentiation with respect to ψ

$$\left[\frac{1}{\Phi_o} \frac{\partial}{\partial\psi} \Phi_o\right] + i \frac{\ell}{\epsilon} \frac{\partial}{\partial\psi} \oint \nu d\chi = 0 \tag{8}$$

where $[A] \equiv A(\chi + \chi_o) - A(\chi)$ is the jump in $A(\chi)$ and $\chi_o \equiv \oint d\chi$. Since Φ_o varies slowly (i.e. on the equilibrium scale) the periodicity condition (8) can only be satisfied if $\frac{\partial}{\partial\psi} \oint \nu d\chi \sim \epsilon$. Thus periodicity of $\Phi(\chi)$, and the requirement $k_{\parallel} a_i \sim \epsilon$ appear to be strictly compatible only in the limit of very weak shear. This is in fact the case, and it is important when considering <u>finite</u> shear that this conflict be resolved in favour of periodicity (which must be satisfied <u>everywhere</u>) rather than small $k_{\parallel} a_i$. To this end we shall take a slightly different form for the potential Φ, given by

$$\Phi = \Phi_o(\psi,\chi) e^{i \frac{\ell}{\epsilon} [\theta - \int(\nu + G) d\chi] + i \frac{S(\psi)}{\epsilon} + i\omega t} \tag{9}$$

where $G = G(\psi,\chi)$ is an arbitrary function which must meet the constraints imposed by periodicity: namely, taking $\Phi_o(\chi)$ periodic to remove the necessity for jump conditions, we require

$$\frac{\ell}{\epsilon} \oint (\nu + G) d\chi = 2\pi n \tag{10}$$

and consequently

$$\frac{\partial}{\partial\psi} \oint (\nu + G)\, d\chi = 0 \tag{11}$$

both of which conditions play an important role in the following analysis.

Now having ensured periodicity we return to the small $k_{\parallel} a_i$ condition to determine what further constraints must be imposed on $G(\psi,\chi)$. Taking $k_{\parallel} \sim (\underset{\sim}{B}.\underset{\sim}{\nabla}\Phi)/(\Phi B)$ it follows that

$$k_{\parallel} \sim - \frac{\ell}{\epsilon} \frac{B_p^2}{B} G(\psi, \chi) \qquad (12)$$

and the class of small $k_{\parallel}$ modes can only be investigated if

$$G(\psi_o, \chi) = \epsilon \, G_o(\chi) \qquad (13)$$

and the modes are localised around $\psi = \psi_o$ in a neighbourhood proportional to $\epsilon/(\ell \frac{\partial \iota}{\partial \psi})$. The localisation of the mode across the magnetic surfaces is contained in the $S(\psi)/\epsilon$ dependence of the potential.

Turning now to the form of the perturbed distribution function

$$\delta f = f \, e^{i \frac{\ell}{\epsilon} [\theta - \int(\nu + G) d\chi] + i \frac{S(\psi)}{\epsilon} + i\omega t} \qquad (14)$$

it follows from equation (12) that although the longitudinal wavelength of the modes is long, $k_{\parallel}$ is nevertheless a rapidly varying function of ψ, so that since f in equation (14) will be found to be a functional of $k_{\parallel}$ we must take

$$f = f\left(\frac{\psi - \psi_o}{\epsilon}, \chi, \mu, \varphi, \sigma\right) \qquad (15)$$

Physically this rapid variation of $k_{\parallel}(\psi)$ means that there will be finite larmor radius, and finite banana width averaging effects on $k_{\parallel}$ as well as on $k_{\perp}$.

We have dwelt at some length on the question of the general form for the perturbed potential and distribution function because, where our results differ from those of previously published work, the origin of these differences lies in this choice. For a consistent treatment of the flute, drift wave, and trapped particle modes with finite $k_{\perp} a_i$ in configurations with <u>finite shear</u> the ordering and constraints outlined in equations (9) - (15) are essential. In addition in the weak shear limit, where previous analysis is valid, the modes represented by equation (9) constitute a wider class than has been considered hitherto so that more stringent stability criteria may be expected to emerge from our analysis. Indeed it is interesting to note that the arbitrary phase function $G(\psi, \chi)$ introduced into equation (9) also features in the M.H.D. stability analysis of localised modes given by MERCIER, (1960). Indeed the final minimisation of the energy integral, δW, in this reference, which appears unnaturally as a minimisation over the choice of coordinate system is equivalent to a minimisation with respect to $\frac{\partial G}{\partial \psi}$ subject to the constraint (11).

III. SOLUTION OF THE VLASOV EQUATION FOR THE PERTURBED DISTRIBUTION FUNCTION

Substituting the forms (9) and (14) for Φ and δf into the linearised Vlasov equation and expanding f in powers of ϵ,

$$f = f_o + \epsilon f_1 + \ldots \tag{16}$$

the solution for f_o is obtained by following the averaging procedure of TAYLOR and HASTIE, (1968) or RUTHERFORD and FRIEMAN, (1968). The analysis is complicated by the rapid variation of f as a function of ψ, and it is useful to introduce the localised variable x defined by

$$x = (\psi - \psi_o)/\epsilon \tag{17}$$

and expand all quantities about ψ_o. Thus, for example, the phase factor of equations (9) and (14) becomes

$$\frac{\ell}{\epsilon}\,[\theta - \int^{\chi} \nu_o d\chi] + \{x[S_o' - \ell \int^{\chi}(\nu_o' + G_o')d\chi] - \ell \int^{\chi} G_o d\chi\} + O(\epsilon) \tag{18}$$

where the prime denotes differentiation with respect to ψ, and the subscript zero (which will be dropped in the following equations) indicates that all quantities are evaluated at $\psi = \psi_o$.

In lowest order the equation for f_o is

$$\frac{\partial f_o}{\partial \varphi} = \frac{v_\perp RB_p}{\omega_c} \cos\varphi \frac{\partial f_o}{\partial x} + i\left[f_o - \frac{e}{m}\varphi_o\left(\frac{1}{B}\frac{\partial F_o}{\partial \mu} + \frac{\partial F_o}{\partial \kappa}\right)\right]\left(\frac{\partial \alpha}{\partial \varphi}\right) \tag{19}$$

where

$$\alpha = \frac{v_\perp RB_p}{\omega_c}[S' - \ell \int^{\chi}(\nu' + G')d\chi] \sin\varphi + \frac{v_\perp B}{\omega_c RB_p}\,\ell \cos\varphi,$$

$\omega_c = \frac{eB}{mc}$, and $v_\perp = (2\mu B)^{\frac{1}{2}}$. Transforming from x to $\hat{x}$, the value of x at the guiding centre,

$$\hat{x} = x + \frac{v_\perp RB_p}{\omega_c} \sin\varphi \tag{20}$$

the solution of equation (19) is obtained as

$$f_o = \frac{e}{m}\Phi_o\left(\frac{1}{B}\frac{\partial F_o}{\partial \mu} + \frac{\partial F_o}{\partial \kappa}\right) + g_o(\hat{x}, \chi, \mu, \kappa, \sigma)e^{i\alpha} \tag{21}$$

where g_o is an arbitrary function.

In next order the equation involving $\frac{\partial}{\partial\varphi}(f_1 e^{-i\alpha})$ yields, after integration over φ to annihilate the f_1 terms, an equation determining g_o. This equation is

$$L\left(g_o + \frac{e}{m}\Phi_o\, J_o(z)\,\frac{1}{B}\frac{\partial F_o}{\partial\mu}\right) + i\,\frac{e}{m}\Phi_o\, J_o(z)\left(\omega\frac{\partial F_o}{\partial\kappa} - \ell\,\frac{m}{e}\frac{\partial F_o}{\partial\psi}\right) = 0 \tag{22}$$

where J_o is the zero order Bessel function,

$$z = \frac{v_\perp}{\omega_c}\left\{\left(\frac{\ell B}{RB_p}\right)^2 + (RB_p)^2\left[S' - \ell\int^\chi(\nu' + G')\,d\chi\right]^2\right\}^{\frac{1}{2}} \tag{23}$$

and the operator L is

$$L = \sigma q\,\frac{B_p^2}{B}\frac{\partial}{\partial\chi} + RB_T\, q\,\frac{B_p{}^2}{B}\;\frac{\partial}{\partial\chi}\left(\frac{q}{\omega_c}\right)\frac{\partial}{\partial\hat{x}} + i\left\{\omega - \sigma q\,\frac{\ell B_p^2}{B}(G_o + \hat{x}G') + \ell qB\,\frac{\partial}{\partial\psi}\;\frac{q}{\omega_c}\right.$$

$$\left. + RB_T\, q\,\frac{B_p^2}{B}\frac{\partial}{\partial\chi}\left(\frac{q}{\omega_c}\right)\left[S' - \ell\int^\chi(\nu' + G')\,d\chi\right]\right\} \tag{24}$$

with $q = [2(\kappa-\mu B)]^{\frac{1}{2}}$ and $\sigma = v_{\parallel}/|v_{\parallel}|$. The leading term in L contains the effect of longitudinal motion along the field line, while the second term involves the geodesic drift which carries particles 'radially' across magnetic surfaces, but whose average over a bounce period is zero in axisymmetric geometry with the result that the familiar banana orbits are closed. The imaginary terms are ω, $k_{\parallel}v_{\parallel}$ and the principle curvature and geodesic curvature contributions to $\underset{\sim}{k}_\perp\cdot\underset{\sim}{v}_d$ respectively, where $\underset{\sim}{v}_d$ is the guiding centre drift velocity.

Returning to equation (22) and changing the guiding centre variable $\hat{x}$ to the new variable $\tilde{x}$ (which is just the angular momentum p_θ in the direction of symmetry)

$$\tilde{x} = \hat{x} - RB_T\,\frac{\sigma q}{\omega_c} \tag{25}$$

the solution may be written explicitly as

$$g_o = -\frac{e}{m}\Phi_o\, J_o\,\frac{1}{B}\frac{\partial F_o}{\partial\mu} - i\sigma\,\frac{e}{m}\left(\omega\frac{\partial F_o}{\partial\kappa} - \ell\,\frac{m}{e}\frac{\partial F_o}{\partial\psi}\right)\int_{\chi_1}^{\chi}\frac{Bd\chi'}{B_p^2 q}\,\Phi_o\, J_o\,\exp[i\sigma\beta(\chi,\chi')]$$

$$+ h_o(\tilde{x},\mu,\kappa,\sigma)\,\exp\,[-i\sigma\beta(\chi_1,\chi)] \tag{26}$$

where χ_1 is an arbitrary end point, h_o an arbitrary function, and

$$\beta(a,b) = \int_a^b\frac{Bd\chi}{B_p^2 q}\left\{\omega - \sigma q\,\frac{\ell B_p^2}{B}\left(G_o + \tilde{x}G' + \frac{\sigma qRB_T}{\omega_c}\,G'\right) + \ell Bq\,\frac{\partial}{\partial\psi}\left(\frac{q}{\omega_c}\right)\right.$$

$$\left. + RB_T\,\frac{qB_p^2}{B}\frac{\partial}{\partial\chi}\left(\frac{q}{\omega_c}\right)[S' - \ell\int^\chi(\nu' + G')].\right\} \tag{27}$$

The arbitrary function h_o can now be determined for both trapped and passing particles merely by applying the appropriate boundary conditions to g_o, and the charge density obtained by integrating δf over velocity space, i.e. by calculating

$$\rho_j = e_j \sum_\sigma \iiint \frac{B}{q}\, d\mu d\kappa d\varphi\ \delta f$$

where the summation over σ and integration must be performed at constant position, i.e. at constant x rather than $\tilde{x}$. After this summation over σ the results for passing and trapped particles are:-

$$\sum_\sigma g_{op} = -2\frac{e}{m}\Phi_o J_o \frac{1}{B}\frac{\partial F_o}{\partial\mu} - \frac{\frac{e}{m}\left(\omega\frac{\partial F_o}{\partial\kappa} - \ell\frac{m}{e}\frac{\partial F_o}{\partial\psi}\right)}{[\cos\beta_o^- - \cos\beta_o^+]}\int_\chi^{\chi+\chi_o}\frac{Bd\chi'}{B_p^2 q}\Phi_o J_o A_p(\chi,\chi') \tag{28}$$

$$\sum_\sigma g_{ot} = -2\frac{e}{m}\Phi_o J_o \frac{1}{B}\frac{\partial F_o}{\partial\mu} + \frac{2\frac{e}{m}\left(\omega\frac{\partial F_o}{\partial\kappa} - \ell\frac{m}{e}\frac{\partial F_o}{\partial\psi}\right)}{\sin\gamma_o}\int_{\chi_1}^{\chi_2}\frac{Bd\chi'}{B_p^2 q}\Phi_o J_o A_t(\chi,\chi') \tag{29}$$

where

$$A_p(\chi,\chi') = \sin\beta^+(\chi,\chi')\exp[-i\beta^-(\chi',\chi+\chi_o)] + \sin\beta^+(\chi',\chi+\chi_o)\exp[i\beta^-(\chi,\chi')] \tag{30}$$

$$\beta^+(a,b) = \int_a^b \frac{Bd\chi'}{B_p^2 q}\left\{\omega + \ell\frac{B_p^2}{B} q\, G' RB_T\left[\frac{q}{\omega_c}(\chi) - \frac{q}{\omega_c}(\chi')\right] + \ell\, Bq\frac{\partial}{\partial\psi}\left(\frac{q}{\omega_c}\right) + \frac{B_p}{B} q\, RB_T\frac{\partial}{\partial\chi'}\left(\frac{q}{\omega_c}\right)\left[S' - \ell\int^{\chi'}(\nu' + G')\right]\right\} \tag{31}$$

$$\beta^-(a,b) = -\ell\int_a^b G_o\, d\chi - \ell\left(x + \frac{v_\perp RB_p}{\omega_c}\sin\varphi\right)\int_a^b G' d\chi \tag{32}$$

$$\beta_o^+ \equiv \beta^+(\chi,\chi+\chi_o); \qquad \beta_o^- \equiv \beta^-(\chi,\chi+\chi_o) \tag{33}$$

$$A_t(\chi,\chi') = -\exp[i\beta^-(\chi,\chi')]\cos[\gamma(\chi_1,\chi') - \Theta(\chi,\chi')\beta^+(\chi,\chi')]\cos[\gamma(\chi',\chi_2) + \Theta(\chi',\chi)\beta^+(\chi,\chi')] \tag{34}$$

$$\Theta(a,b) = \left.\begin{matrix} +1 & b > a \\ 0 & b < a \end{matrix}\right\} \tag{35}$$

$$\left.\begin{aligned}\gamma(a,b) &= \beta^{+}(a,b) - \ell RB_T \frac{q}{\omega_c}\int_a^b G'd\chi \\ \gamma_o &= \gamma(\chi_1,\chi_2)\end{aligned}\right\} \quad (36)$$

and χ_1, χ_2 are now the turning points for trapped particles.

The expressions (28) and (29) complete the solution for δf. In practice one of the two fundamental approximations $\omega \lessgtr \omega_b$ is invariably introduced, so that wave particle resonances may be treated as a perturbation. In the next section we examine these two limits and indicate where our equations differ from those of previous authors.

IV. ASYMPTOTIC EXPRESSIONS FOR THE PERTURBED CHARGE DENSITY

In this section we take F_o to be the Maxwellian distribution

$$F_{oj} = n(\psi)\left(\frac{m_j}{2\pi T_j}\right)^{3/2} e^{-m_j\kappa/T_j} \quad (37)$$

so that the contribution of each species to the perturbed charge density is

$$\rho_j = -\frac{e^2 n}{T_j}\Phi_o + e_j\iiiint \frac{B}{q}\, d\mu d\kappa d\varphi \sum_\sigma e^{i\alpha}\, g_{oj} \quad (38)$$

One of two fundamental approximations is now introduced to simplify the expression involving g_{oj}. We assume that the quantities γ and $\beta^{\pm}$ appearing in equations (28) and (29) are either large or small, corresponding to $\omega/\omega_b \gg 1$ or $\omega/\omega_b \ll 1$.

For $\omega < \omega_b$ the reduction of g_{oj} follows by expanding the exponential and trigonometric functions for small argument, so that only low energy particles are involved in the $\omega/\omega_b = 2N\pi$ ($N \neq 0$) resonances of g_{oj}. These can be shown (RUTHERFORD and FRIEMAN,1968) to yield imaginary contributions of order $(\omega/\omega_b)^3$ and will be ignored here. The N = 0 resonance remains in the simplified g_{oj} and occurs for

$$\left.\begin{aligned}\omega + \langle k_\perp v_d\rangle + \langle\sigma k_\parallel q\rangle &= 0 \qquad \text{Passing Particles} \\ \omega + \langle k_\perp v_d\rangle &= 0 \qquad \text{Trapped Particles}\end{aligned}\right\} \quad (39)$$

where

$$\langle k_\perp v_d\rangle \equiv \ell\frac{\partial}{\partial\psi}\oint \frac{B^2}{B_p^2}\frac{q}{\omega_c}\,d\chi \Big/ \oint \frac{B}{B_p^2 q}\,d\chi \quad (40)$$

with $\oint d\chi$ taken between turning points for trapped particles, and

$$\langle\sigma k_\parallel q\rangle = -\frac{2\pi\sigma\ell}{\oint \frac{B}{B_p^2 q}\,d\chi}\left\{(n/\ell - Q) - \left(\frac{v_\perp}{\omega_c} RB_p \sin\varphi - \frac{\sigma q}{\omega_c} RB_T\right)Q'\right\} \quad (41)$$

with $Q = \frac{1}{2\pi} \oint \nu dx$, the safety factor. In equation (41) the first term on the right hand side may be zero on a rational surface, but the additional Q' terms which arise because of finite larmor radius ($v_\perp$ term) and finite drift (σq term) averaging of $k_\parallel$ are non-vanishing in the presence of shear.

Three further approximations complete the reduction of g_{oj}. These are:

$$k_\perp a_j < 1 \tag{42}$$

$$\langle k_\perp v_d \rangle / \omega < 1 \tag{43}$$

$$\left.\begin{array}{ll} \langle k_\parallel q \rangle/\omega < 1 & \text{(a)} \\ \text{or} & \\ \langle k_\parallel q \rangle/\omega > 1 & \text{(b)} \end{array}\right\} \tag{44}$$

The first of these permits the expansion of the Bessel functions. A consequence of conditions (43) and (44(a)) is that only high energy particles are involved in the N = 0 resonance which is therefore exponentially weak and is ignored. The denominator is expanded, $\langle k_\perp v_d \rangle/\omega$ and $\langle k_\parallel q \rangle^2/\omega^2$ terms being retained. In this limit one finally obtains for ρ_j

$$\begin{aligned}
\rho_j(\omega/\omega_b < 1) = &-\frac{e^2 n}{T}\Phi_o + \left(1 + \frac{\omega_*}{\omega}\right)\frac{e^2 n}{T}\left\{\frac{1}{2}\int_o^{1/B}\frac{dy}{h}\langle \Phi_o \rangle\right. \\
&+ \frac{3}{2}\frac{T}{m}\left[\frac{2\pi(\ell Q - n)}{\omega}\right]^2 \int_o^{1/B_m}\frac{dy}{h}\frac{\langle \Phi_o \rangle}{t_o^2} \\
&- \frac{3}{8}\frac{T}{m}\int_o^{1/B}\frac{ydy}{h}\left[\frac{B}{\omega_c^2}(RB_p)^2\langle (S' - \ell \int^{\chi'} G')^2 \Phi_o \rangle + \langle \frac{B}{\omega_c^2}(RB_p)^2 (S' - \ell \int^{\chi'} G')^2 \Phi_o \rangle\right] \\
&- \frac{3}{2}\frac{\omega_*}{\omega}\left(\frac{n'}{n}\right)^{-1}\int_o^{1/B}\frac{dy}{h}\frac{\langle \Phi_o \rangle}{t_o}\left(\frac{\partial}{\partial \psi}\oint \frac{B^2}{B_p^2} h d\chi\right) + \frac{3}{2}\frac{\omega_*}{\omega}\left(\frac{n'}{n}\right)^{-1} RB_T Q' \int_o^{1/B_m}\frac{dy}{t_o}\langle \Phi_o \rangle \\
&+ \frac{1}{2}\int_o^{1/B_m}\frac{dy}{h}\frac{1}{t_o}\oint_{\chi}^{\chi+\chi_o}\frac{d\chi'}{B_p^2(\chi')h(\chi')}\Phi_o(\chi')K_p(\chi,\chi') \\
&\left. + \frac{1}{2}\int_{1/B_m}^{1/B}\frac{dy}{h}\frac{1}{t_o}\int_{\chi_1}^{\chi_2}\frac{d\chi'}{B_p^2(\chi')h(\chi')}\Phi_o(\chi')K_t(\chi,\chi')\right\}
\end{aligned} \tag{45}$$

where $h = h(y,\chi) = (1 - yB)^{\frac{1}{2}}/B$ (46)

$$t_o = \oint \frac{d\chi}{B_p^2 h} \qquad \text{Passing particles}$$

$$= \int_{\chi_1}^{\chi_2} \frac{d\chi}{B_p^2 h} \qquad \text{Trapped particles} \qquad (47)$$

$$\omega_* = \frac{T_j}{e_j} \ell \frac{n'}{n} \qquad (48)$$

B_m is the maximum value of $B(\chi)$ and the kernel $K(\chi,\chi')$ is, for passing particles

$$K_p(\chi,\chi') = -i\ell\int_\chi^{\chi'} Gd\chi'' - 2\pi i(\ell Q-n)\left[\frac{t'-t}{t_o} + 3\frac{\omega_*}{\omega}\left(\frac{n}{\ell n'}\right) RB_T[S'-\ell\int^{\chi'} G'][h(\chi')-h(\chi)]\right]$$

$$-\omega^2\frac{m}{T}\left[\frac{t_o^2}{12} - \frac{t_o(t'-t)}{2} + \frac{(t'-t)^2}{2}\right] + \omega RB_T[S'-\ell\int^{\chi'} G'][t_o-2(t'-t)][h(\chi')-h(\chi)]\frac{m}{e}$$

$$-\frac{3}{2}\frac{T}{m}(RB_T)^2\,[S'-\ell\int^{\chi'} G']^2\,[h(\chi')-h(\chi)]^2\left(\frac{m}{e}\right)^2 - \tfrac{1}{2}\left(\int_\chi^{\chi'} Gd\chi''\right)^2 \qquad (49)$$

with

$$t'-t = \int_\chi^{\chi'} \frac{d\chi''}{B_p{}^2 h} \qquad (50)$$

while for trapped particles

$$K_t = -i\ell\int_\chi^{\chi'} Gd\chi'' - \frac{\omega^2 m}{T}\left[\frac{t_o^2}{3} - \frac{t_o}{2}(|t'-t| + t'+t) + \tfrac{1}{2}(t'^2+t^2)\right]$$

$$-\omega RB_T\frac{m}{e}[S'-\ell\int^{\chi'} G']\,[t\,h(\chi) + t'h(\chi') - t_o h(\hat{\chi})]$$

$$-\frac{3}{2}\frac{T}{m}\left(RB_T\frac{m}{e}\right)^2[S'-\ell\int^{\chi'} G']^2\,[h^2(\chi') + h^2(\chi)] - \tfrac{1}{2}\left(\ell\int_\chi^{\chi'} Gd\chi''\right)^2 \qquad (51)$$

with

$$t = \int_{\chi_1}^{\chi'} \frac{d\chi''}{B_p^2 h}\ ; \quad t' = \int_{\chi_1}^{\chi'} \frac{d\chi''}{B_p^2 h}\ ; \quad \hat{\chi} = \max(\chi',\chi) \qquad (52)$$

In this rather complicated expression the energy and phase integrations have been carried out explicitly while the pitch angle integration remains. The first two terms on the right hand side of equation (45) form the dominant contribution to ρ_j. The third term is of order $(k_\parallel v_\parallel/\omega)^2$, involves only passing particles and contains the shear dependence of ρ_j. The fourth term is of order $(k_\psi a_j)^2$ since, as we discuss below, k_ψ is assumed to be the dominant part of $k_\perp$. Term five is of order $\langle k_\perp v_d/\omega\rangle$ and is independent of the geodesic drift since this drift cancels over a bounce period. Term six arises from the finite drift orbit average of $\langle k_{\parallel} v_{\parallel}/\omega \rangle$ and involves passing particles only. This term is responsible for the replacement of V'' by $V^{\prime\prime}$ as the effective curvature in the low frequency analysis of the flute mode (RUTHERFORD et al., 1968; JAMIN, 1971). Finally the last terms contain corrections of order $(\omega + k_{\psi} v_{d\psi})^2/\omega_b^2$, $(k_{\parallel} v_{\parallel}/\omega_b)^2$ and imaginary terms of order $(k_{\parallel} v_{\parallel}/\omega_b)$. The $(k_{\psi} v_{d\psi}/\omega_b)^2$ terms embody the effect of finite banana width, analogous to the finite larmor radius effect (BHADRA & LIU, 1971), but occuring only on time scales longer than the bounce time. Consequently these terms do not appear when $\omega/\omega_b > 1$. Finally we note that

$$\langle \frac{k_{\perp} v_d}{\omega} \rangle^2 \sim k_{\perp}^2 a_j^2 \left(\frac{\omega_{bj}}{\omega}\right)^2 \left(\frac{L_c}{R_c}\right)^2 > k_{\perp}^2 a_j^2 \tag{53}$$

where L_c and R_c are the connection length and field curvature respectively. It follows that it would be inconsistent to retain terms of order $k_{\perp}^2 a_j^2$ while neglecting $\langle k_{\perp} v_d/\omega \rangle^2$. However $\langle k_{\perp} v_d \rangle$ is independent of that part of $k_{\perp}$ involving $(S' - \ell \int^{\chi} G')$ so that if

$$RB_p \, [S' - \ell \int^{\chi} G' d\chi] \gg \frac{\ell}{R} \frac{B}{B_p} \tag{54}$$

then the finite larmor radius and finite banana terms must be retained with $k_{\perp} \equiv RB_p \, [S' - \ell \int^{\chi'} G' d\chi]$ as in equation (45); this corresponds to the assumption $k_{\psi} \gg k_y$, where k_y is the component of $k_{\perp}$ lying on the magnetic surface.

If the flute/drift wave approximation (44a) is replaced by the trapped particle ordering (44b) the contribution of the trapped particles to the perturbed charge density is unchanged, while that of the passing particles is reduced by a factor of order $\omega^2/\langle k_{\parallel} q \rangle^2$. Low energy passing particles provide a resonant term of order $(\omega/k_{\parallel} v_T)^3$ which we neglect. The appropriate expression for ρ_j is

$$\rho_j = -\frac{e^2 n}{T} \Phi_o + \frac{e^2 n}{T} \left(1 + \frac{\omega_*}{\omega}\right) \left\{ \frac{1}{2} \int_{1/B_m}^{1/B} \langle \Phi_o \rangle \frac{dy}{h} \right.$$

$$- \frac{3}{2} \frac{\omega_*}{\omega} \left(\frac{n'}{n}\right)^{-1} \int_{1/B_m}^{1/B} \frac{dy}{h} \frac{\langle \Phi_o \rangle}{t_o} \left(\frac{\partial}{\partial \Psi} \oint \frac{B^2}{B_p^2} h d\chi \right)$$

$$- \frac{3}{8} \frac{T}{m} \int_{1/B_m}^{1/B} \frac{y dy}{h} \left[\frac{B}{\omega_c^2} (RB_p)^2 \langle \Phi_o (S' - \ell \int^{\chi'} G')^2 \rangle + \langle \frac{B}{\omega_c^2} (RB_p)^2 \Phi_o (S' - \ell \int^{\chi'} G')^2 \rangle \right]$$

$$+\frac{1}{2}\int_{1/B_m}^{1/B}\frac{dy}{h}\frac{1}{t_o}\int_{\chi_1}^{\chi_2}\frac{d\chi'}{B_p^2(\chi')h(\chi')}\,\Phi_o(\chi')K_t(\chi,\chi')$$

$$+\frac{m}{2T}\int_o^{1/B_m}\frac{dy}{h}\langle\Phi_o\rangle\frac{\omega^2 t_o^{\,2}}{[2\pi(\ell Q_o-n)+2\pi\ell Q'x]^2}\Bigg\} \tag{55}$$

In this expression we have retained only the leading order contribution from the passing particles, which includes the effect of shear. We note that although B, B_p, n etc are all functions of ψ this implies that they are functions of (ϵx) and essentially constant. Thus the correct radially non-local treatment of the perturbation involves only the x dependence of Q as shown in (55). This differs from the analysis of BHADRA and LIU, (1971) who include other spatial variations which are formally small in this analysis.

In the high frequency limit $\omega/\omega_b \gg 1$ we assume that the inequalities (42), (43) and

$$\frac{k_{\parallel}q}{\omega} < 1 \tag{56}$$

are all satisfied. Provided that $\Phi_o(\chi)$ is not strongly localised in χ (i.e. not within a range of order ω_b/ω) the dominant contribution to the trajectory integrals in equations (28) and (29) is from the end points, and ρ_j reduces to

$$\rho_j \simeq +\frac{e^2n}{T}\frac{\omega_*}{\omega}\Phi_o-\frac{e^2n}{T}\left(1+\frac{\omega_*}{\omega}\right)\Bigg\{b+\frac{T}{e}\frac{\ell B_p^2}{\omega}\left[\frac{B^2}{B_p^2}\frac{\partial}{\partial\psi}\left(\frac{1}{B^2}\right)-RB_T\frac{\partial}{\partial\chi}\left(\frac{1}{B^2}\right)\int^{\chi}(\nu'+G')\right]$$

$$+\frac{T}{e}\frac{B_p^2}{\omega}RB_T\,S'\frac{\partial}{\partial\chi}\left(\frac{1}{B^2}\right)+\frac{T}{m\omega^2}B_p^2\left[\frac{\partial}{\partial\chi}-i\ell(G_o+xG')\right]\frac{B_p^2}{B^2}\left[\frac{\partial}{\partial\chi}-i\ell(G_o+xG')\right]\Bigg\}\Phi_o \tag{57}$$

where

$$b=\left(\frac{T}{m\omega_c^2}\right)\left\{(RB_p)^2\,[S'-\ell\int^{\chi}(\nu'+G')]^2+\left(\frac{\ell B}{RB_p}\right)^2\right\} \tag{58}$$

In equation (57) the first term is largest, succeeding terms being small of order $k_\perp^2a_j^2$, $k_\perp v_d/\omega$, $(\omega_b/\omega)^2$ and $(k_{\parallel}v_T/\omega)^2$. Our result differs from previous expressions in the appearance of the arbitrary function $G'(\chi)$ coupled to the geodesic curvature ($RB_T\,\partial/\partial\chi\,(1/B^2)$ term). Because of the freedom available in choosing G', the possibility arises that more unstable forms of curvature driven modes might exist where the geodesic curvature is comparable to the principle curvature.

V. CONCLUSIONS

We have developed a general theory of low frequency electrostatic modes in an axisymmetric toroidal plasma possessing shear. Reconciling the twin requirements of double toroidal periodicity and parallel wavelengths comparable with the characteristic lengths of the torus, in the presence of shear and perpendicular wavelengths comparable to the ion larmor radius, led to the introduction of an arbitrary phase function $G(\psi, \chi)$ into the form of the perturbation for the electrostatic potential. The resulting perturbation in the charge density has been calculated in a number of limits suitable for discussing various drift, flute and trapped particle modes. For cases in which $\omega > \omega_b$ the function G' may be chosen to couple with geodesic curvature and provide a driving mechanism in addition to the normal curvature. This role of G' will become apparent in a subsequent paper which discusses the flute mode. It is also worth remarking that our formalism, when applied to the trapped electron drift mode (KADOMTSEV and POGUTSE, 1969) clearly shows how the role of shear can be nullified by choosing a mode localised in χ (ADAM, LAVAL and PELLAT, 1971; 1973). Then by choosing $G' = 0$ in the region of localisation G' disappears from equation (57) for ρ_i. This is in disagreement with, for example, YOSHIKAWA and OKABAYASHI, (1973) who essentially assume the constraint $\oint (\nu' + G')d\chi = 0$ implies $G' = -\nu'$ leading to the possibility of shear stabilisation.

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