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LOW FREQUENCY STABILITY THEORY OF AXISYMMETRIC TOROIDAL PLASMAS

PART 2 Shear stabilisation of the flute instability

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Part 2 Shear stabilisation of the flute instability*

by

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ABSTRACT

The shear required to stabilise the electrostatic flute instability is calculated for a general low β, axisymmetric, toroidal magnetic field geometry. It is found that previous calculations underestimated the necessary shear. New destabilising effects, which are associated with the geodesic curvature, occur for flute modes with frequencies in the intermediate range (i.e. between electron and ion bounce frequencies).

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I. INTRODUCTION

Criteria for the shear stabilisation of the electrostatic flute instability, which occurs when

$$\beta < \frac{m_e}{m_i} (k_\perp a_i)^2 \tag{1}$$

where β is the ratio of plasma pressure to magnetic field pressure, $k_\perp$ is the wavenumber perpendicular to the magnetic field, a_i the ion larmor radius and m_e and m_i are, respectively, the electron and ion masses, have appeared previously in the literature (Mikhailovskaya and Mikhailovskii, 1965; Rutherford et al, 1969; Jamin, 1971). Mikhailovskaya and Mikhailovskii (1965) considered a cylindrical plasma so that the complexities introduced by trapped particles, geodesic drifts and the double periodicity of toroidal plasmas do not appear. In addition, the electron temperature was neglected in this work. The shear required to stabilise the flute instability as calculated in this way is given by the relation

$$\frac{L}{L_s} < \left(\frac{m_e}{m_i} \right)^{\frac{1}{2}} \tag{2}$$

where L_s is the shear length and L a typical distance for variation in the magnetic field strength.

The stability of the flute in toroidal geometry was investigated by Rutherford et al (1969) and Jamin (1971) using a formalism which is only valid for small shear $\sim \frac{a_i}{L}$. This limitation stems from an inconsistency in the choice of perturbation. The perturbation describing the flute mode has short wavelength perpendicular to the magnetic field and long parallel wavelength, but the requirement of double toroidal periodicity in the presence of shear, is in conflict with these properties and care must be exercised. The analysis of

Part I was developed to overcome this difficulty and the present work is based on that formalism. It is clear from formula (2), however, that only a modest amount of shear $\sim\sqrt{\frac{m_e}{m_i}}$ is required to stabilise the flute and the analysis of references 2 and 3 is valid in this limit. The improved analysis of Part I does, however, bring to light the existence of a wider class of flute-like modes in a torus than has been discussed before and we find that considerable modifications to the previously quoted shear criterion are possible. In addition we consider the effect of a very low plasma density n on the shear criterion, namely when $\frac{B^2}{4\pi m_i n} \gg 1$ where B is the magnetic field strength.

The eigen equations for the flute mode are discussed in Section II using the analysis of Part I for the perturbed charge densities. Three regimes of frequency ω for the flute mode must be considered, depending on where ω lies with respect to the characteristic bounce (or transit) frequencies ω_{bi} and ω_{be} of ions and electrons in their motion along the field lines. These we define as follows

A	$\omega_{bi} < \omega_{be} < \omega$	-	high frequency
B	$\omega < \omega_{bi} < \omega_{be}$	-	low frequency
C	$\omega_{bi} < \omega < \omega_{be}$	-	intermediate frequency

In Section III we solve the eigen equations and obtain dispersion relations which are used to discuss the stability of the flute mode in these three regimes. The effect of shear on the residual resonance instability, remaining when the reactive mode has been stabilised, is also considered.

II THE FLUTE EIGENFUNCTION EQUATIONS

It is necessary to first define the toroidal magnetic field geometry. This is an axisymmetric vacuum field with toroidal and poloidal components

$$\vec{B}_T = \frac{I_c}{R}\vec{e}_\theta \; , \; \vec{B}_p = \nabla\chi = \nabla\psi \times \nabla\theta \; , \tag{3}$$

where ψ, χ and θ are orthogonal co-ordinates, ψ being the poloidal flux, χ the potential for the poloidal field and θ the toroidal azimuthal angle, while R is the distance from the axis of symmetry (the major toroidal axis) and I_o is a constant. The inverse rotational transform (or safety factor) q is defined by

$$q = \frac{1}{2\pi}\oint \nu d\chi \quad ; \quad \nu = \frac{B_T}{RB_p^2} \tag{4}$$

Note, that to avoid ambiguity we denoted the safety factor by Q in Part I, but now we return to the more familiar notation q.

The perturbation describing the flute mode has the form

$$\Phi\,(\chi,\,\theta,\,\psi,\,t) = \Phi_o \exp i\left[\frac{\ell}{\epsilon}\left(\theta - \int^{\chi} d\chi\,(\nu + G)\right) + \frac{S(\psi)}{\epsilon} + \omega t\right] \tag{5}$$

which is the most general form satisfying the toroidal periodicities and with parallel wavenumber $k_{\parallel}$ such that $k_{\parallel}\,L \sim 1$. Here Φ_o is a constant and ℓ is a large integer labelled by the small parameter $\epsilon = \frac{a_i}{L}$. $S(\psi)$ is a function of ψ only, and describes the variation of the mode about some rational surface ψ_o. To meet the condition $k_{\parallel}L \sim 1$, G must satisfy the condition

$$G\,(\psi_o,\,\chi) = 0 \tag{6}$$

while the constraint imposed by periodicity

$$\frac{\ell}{2\pi}\oint (\nu + G)\,d\chi = \text{integer}$$

requires

$$\oint (\nu' + G')\,d\chi = 0 \tag{7}$$

where a derivative with respect to ψ is denoted by a prime; otherwise $G\,(\psi,\,\chi)$ may be chosen arbitrarily.

Having defined the magnetic field and the flute perturbation we may obtain the eigenfunction equations in the three regimes using the results of Part I.

A. The High Frequency Regime

In the high frequency regime we evaluate both ion and electron charge densities using expression (57) of Part I and insert them into Poisson's equation. This equation may be expanded treating the electron $\left(\frac{k_{\parallel} V_T}{\omega}\right)^2$ term dominant, thus yielding a flute solution Φ_o = constant in lowest order (Hastie and Taylor 1971). In next order we integrate $\oint \frac{d\chi}{B_p^2}$ to annihilate the term containing the next order contribution to Φ. We obtain the eigenfunction equation

$$-\left(\frac{\omega_*}{\omega}\right)^2\left(\frac{n'}{n}\right)^{-1}(1+\tau)\ V_G'' - \left(1-\frac{\omega_*\tau}{\omega}\right)\frac{T_i}{m_e\omega^2}(\ell\psi)^2\oint d\chi\left(\frac{G' B_p}{B}\right)^2$$
$$+\left(1+\frac{\omega_*}{\omega}\right)\oint\frac{b\,d\chi}{B_p^2}+\oint d\,\frac{d\chi}{B_p^2}=0\ . \qquad (8)$$

B. The Low Frequency Regime

In the low frequency regime we use equation (45) of Part I to evaluate the ion and electron charge densities. Again the dominant part of Poisson's equation has a flute solution, and in next order we find the eigenfunction equation

$$-\left(\frac{\omega_*}{\omega}\right)^2\left(\frac{n'}{n}\right)^{-1}(1+\tau)\ V^{++} - \left(1-\frac{\omega_*\tau}{\omega}\right)\frac{T_i}{m_e\omega^2}(2\pi\ell\psi)^2\, Iq'^2$$
$$+\left(1+\frac{\omega_*}{\omega}\right)\frac{m_i T_i}{e^2}S'^2\left\{\oint\frac{d\chi R^2}{B_p^2} - I\left(RB_T\oint\frac{d\chi}{B_p^2}\right)^2\right\}+\oint d\,\frac{d\chi}{B_p^2}=0\ . \qquad (9)$$

C. The Intermediate Frequency Regime

Finally in the intermediate region we use equation (57) of Part I for the ion charge density and equation (45) for the electrons and obtain the eigenfunction equation:

$$-\left(\frac{\omega_*}{\omega}\right)^2\left(\frac{n'}{n}\right)^{-1}(1+\tau)V^{++}-\frac{\omega_*}{\omega}\left(\frac{\omega_*}{\omega}+1\right)\left(\frac{n'}{n}\right)^{-1}(V_G''-V^{++})-$$

$$-\left(1-\frac{\omega_*\tau}{\omega}\right)\frac{T_i}{m_e\omega^2}(2\pi\ell\psi)^2\, Iq'^2+\left(1+\frac{\omega_*}{\omega}\right)\oint\frac{b\,d\chi}{B_p^2}+\oint\frac{d\,d\chi}{B_p^2}=0\,. \tag{10}$$

Let us first define the quantities appearing in these expressions. $\tau = \frac{T_e}{T_i}$ is the temperature ratio while the diamagnetic frequency $\omega_* = \frac{\ell\, T_i n'}{e\;\, n}$. In addition

$$V_G'' = V'' + RB_T\oint\frac{d\chi}{B^2}G' \tag{11}$$

$$V^{++} = V'' - 2\pi RB_T\, q'\, I\oint\frac{d\chi}{B_p^2} \tag{12}$$

with the usual definition

$$V'' = \frac{\partial}{\partial\psi}\oint\frac{d\chi}{B_p} \tag{13}$$

and

$$I = \frac{3}{2}\int_0^{\frac{1}{B_m}}\frac{dy}{\oint\frac{B}{\sqrt{1-yB}}\frac{d\chi}{B_p^2}} \tag{14}$$

where B_m is the maximum value of B along the given field line. Finally

$$b = \frac{m_i T_i}{e^2B^2}k^2 \qquad d = \frac{T_i}{4\pi ne^2}k^2 \tag{15}$$

with

$$k^2 = (RB_p)^2(S' - \ell g)^2 + \left(\frac{\ell B}{RB_p}\right)^2 \tag{16}$$

where

$$g = \int^{\chi}(\nu' + G')\,d\chi\,. \tag{17}$$

The three equations (8 - 10) have a similar structure. The leading term in each case is a curvature term which drives the instability, either V_G'' or V^{++}. In case C however there is an additional part containing a term linear in $\frac{\omega_*}{\omega}$ which we shall see complicates the analysis in that case. After these terms appears a term quadratic in ψ (where ψ is measured from a rational surface). In cases B and C this term arises directly from the shear q', but in Case A it is produced by a $k_{\parallel}^2$ term introduced by G, although we shall be able to relate this term to the shear for the least stable mode. The last term in cases A and C is a FLR term but in case B a little more explanation is necessary. In order to avoid including a number of insignificant terms, which consistency would demand if the complete FLR term were retained, we retain only the S' terms in b, corresponding to $k_\psi \gg \frac{1}{R}$, where k_ψ is the wavenumber in the direction normal to the flux surfaces. In addition at frequencies below ω_{bi}, this term contains additional finite banana width averaging. Finally, at low densities

$$\frac{4\pi\, nm_i}{B^2} \lesssim 1$$

the Debye term becomes important (Mikhailovskaya and Mikhailovskii, 1965) and we have included this term.

III. THE DISPERSION RELATIONS AND SOLUTIONS

Recalling the form (5) for the perturbation we realise that the equations (8 - 10) are second order differential equations for the ψ dependence of the eigenmode $\Phi(\psi, \chi)$. By means of the transformation

$$\Psi(\psi, \chi) = \exp(-i\ell\bar{g}\psi)\, \Phi(\psi, \chi) \tag{18}$$

where $\bar{g}$ is a certain χ average of g, to be defined later, we eliminate the terms linear in S' and obtain a Weber equation for Ψ in each case

$$\frac{d^2}{d\psi^2}\Psi + [A(w) - \sigma(w)\psi^2]\Psi = 0 \tag{19}$$

where we have defined a normalised frequency $w = \frac{\omega}{\omega_*}$. This equation has two types of relevant solutions:

(a) a solution localised about the rational surface ψ_o

(b) an outgoing wave solution - ie. waves which carry energy outwards (Berk and Pearlstein, 1969)

and we will consider both possibilities below.

We shall now obtain the forms for A(w) and $\sigma(w)$ in the three regimes.

A. <u>High Frequency Regime</u>

It is evident from the result (2) that only a small quantity of shear $\left(\sim\sqrt{\frac{m_e}{m_i}}\right)$ is required to stabilise the flute mode and so shear is the dominant effect in determining stability. Thus in the high frequency case, where the $k_\parallel^2$ term plays the role of shear, the most unstable mode will have a form of G' which minimises the integral $\oint d\chi \left(\frac{G' B_p}{B}\right)^2$ subject to the constraint (7). This is achieved by the choice

$$RB_T\ (\nu' + G') = \frac{\partial}{\partial\psi}\left(\frac{B^2}{B_p^{\,2}}\right) - \frac{B^2}{B_p^{\,2}}\ \frac{2\pi RB_T q'}{\oint \frac{d\chi B^2}{B_p^{\,2}}} \tag{20}$$

which on substitution in V_G'' given in equation (11) yields

$$V_G'' \rightarrow V^{**} = \frac{\partial}{\partial\psi}\oint \frac{d\chi}{B_p^{\,2}} - \frac{\oint \frac{d\chi}{B_p^{\,2}}\ 2\pi RB_T q'}{\oint \frac{d\chi B^2}{B_p^{\,2}}} \tag{21}$$

the effective curvature one obtains from finite resistivity theory. (Rutherford et al, 1969; Johnson and Greene, 1967). After evaluating the $k_\parallel^2$ term, which now explicitly depends on shear, and the FLR term, we obtain A and σ:

$$A(w) = \left(\frac{\kappa^{**}(1+\tau)}{w(w+1)+\alpha w^2} - \gamma\right)\left(\frac{R}{a_{i_T}}\right)^2 \frac{1}{(RB_T)^2 \oint \frac{d\chi R^2}{B^2}} \tag{22}$$

$$\sigma(w) = -\frac{w-\tau}{1+w+\alpha w} \frac{S_H}{w^2}\left(\frac{R}{a_{i_T}}\right)^4 \left[\frac{1}{(RB_T)^2 \oint \frac{d\chi R^2}{B^2}}\right]^2 \tag{23}$$

where $a_{i_T} = \frac{\sqrt{m_i T_i}}{eB_T}$ is the ion larmor radius in the toroidal field,

$$\kappa^{**} = \left(\frac{n'}{n}\right)^{-1} V^{**} \tag{24}$$

$$\gamma = \left\{\bar{g}^2 - (\bar{g})^2 + \frac{\oint \frac{d\chi}{R^2 B_p^4}}{\oint \frac{d\chi R^2}{B^2}} \frac{w+1+\alpha_1 w}{w+1+\alpha w}\right\}\left(\frac{\ell a_{i_T}}{R}\right)^2 (RB_T)^2 \oint \frac{d\chi R^2}{B^2} \tag{25}$$

with

$$\alpha = \frac{1}{4\pi n m_i} \frac{\oint d\chi\, R^2}{\oint \frac{d\chi\, R^2}{B^2}} \quad ; \quad \alpha_1 = \frac{1}{4\pi n m_i} \frac{\oint \frac{d\chi\, B^2}{R^2 B_p^4}}{\oint \frac{d\chi}{R^2 B_p^4}} \tag{26}$$

both measuring the relative sizes of Debye and FLR terms, and the bar symbol is defined by

$$\bar{A} = \frac{(w+1)\oint \frac{d\chi\, R^2 A}{B^2} + \frac{w}{4\pi n m_i}\oint d\chi\, R^2 A}{(w+1+\alpha w)\oint \frac{d\chi\, R^2}{B^2}} \tag{27}$$

and finally

$$s_H = \frac{m_i}{m_e}\left(\frac{n'}{n}\right)^{-2} 4\pi^2 q'^2 \left(\oint \frac{d\chi\, B^2}{B_p^2}\right)^{-1} \oint \frac{d\chi R^2}{B^2} \tag{28}$$

First we seek localised eigen-solutions of equation (19)

$$\Psi_n = H_n(\sigma^{\frac{1}{4}}\psi)\exp\left(-\frac{\sigma^{\frac{1}{2}}\psi^2}{2}\right) \tag{29}$$

where H_n is a Hermite function corresponding to the eigenvalue, or dispersion, equation

$$A(w) = (2n+1)\,\sigma^{\frac{1}{2}}(w) \tag{30}$$

which becomes, using results (22) and (23)

$$\left(\frac{\kappa^{**}(1+\tau)}{w(w+1)+\alpha w^2} - \gamma \right)^2 = -(2n+1) \frac{w-\tau}{1+w+\alpha w} \frac{S_H}{w^2} . \tag{31}$$

The localisation condition

$$\mathrm{Re}\,(\sigma^{\frac{1}{2}}) > 0 \tag{32}$$

must also be satisfied. Clearly, if shear is absent, i.e. $S_H = 0$, and we ignore the Debye terms, setting $\alpha = \alpha_1 = 0$, we obtain the familiar flute stability criterion

$$\kappa'' < 0 \tag{33}$$

since $\kappa^{**} \to \kappa''$ as the shear vanishes. In general, however, there are two cases to consider (Coppi et al., 1968)

(i) $$\frac{\kappa^{**}(1+\tau)}{\gamma} \ll 1 \tag{34}$$

corresponding to FLR stabilisation of the flute mode which we shall not discuss further and the long wavelength limit

(ii) $$\frac{\kappa^{**}(1+\tau)}{\gamma} \gg 1 \tag{35}$$

which we now investigate.

It is convenient to make the transformation

$$\left.\begin{aligned} \hat{\tau} &= \tau(1+\alpha) \quad , \quad \hat{w} = w(1+\alpha) \\ \hat{S}_H &= S_H \left[\frac{1+\tau(1+\alpha)}{1+\tau} \right]^2 (1+\alpha)^{-1} \end{aligned}\right\} \tag{36}$$

which simplifies equation (31) in the limit (35) and we find

$$\hat{w} = \frac{1}{2}(\hat{\tau} - 1) \pm \frac{i}{2}(\hat{\tau}+1) \left(\frac{4\kappa^{**2}}{\hat{S}_H} - 1 \right)^{\frac{1}{2}} . \tag{37}$$

Thus we have stability if

$$\hat{S}_H > 4\kappa^{**2} \tag{38}$$

The criterion (32) for a localised mode depends on whether the stability condition (38) is satisfied and becomes, for the eigenfrequencies (37),

$$\left.\begin{array}{ll} \kappa^{**}(\hat{\tau}\hat{S}_H - 2(1+\hat{\tau})\,\kappa^{**2}) > 0 & ;\ \text{if unstable} \\ \kappa^{**}\{\hat{\tau}\,\hat{S}_H - 2(1+\hat{\tau})\,\kappa^{**2} \pm \hat{\tau}\,\hat{S}_H^{\frac{1}{2}}(\hat{S}_H - 4\kappa^{**2})^{\frac{1}{2}}\} > 0 & ;\ \text{if stable} \end{array}\right\} \quad (39)$$

We first consider the case $\hat{\tau} < 1$ when the transition from localised to non-localised modes occurs in the stable region of parameter space. For $\kappa^{**} < 0$ (unstable curvature) the mode is localised for small values of $\hat{S}_H$, becomes stable when (38) is satisfied and becomes a non-localised wave with outgoing energy (as discussed by Berk and Pearlstein, 1969) when (39) is violated. For $\kappa^{**} > 0$, i.e. stable in the absence of shear, the mode is non-localised at small values of $\hat{S}_H$ but corresponds to incoming energy, i.e. no mode satisfying acceptable boundary conditions exists in the unstable region.

When $\hat{\tau} > 1$ the boundary for localisation of the mode appears within the unstable range of the shear parameter, and for $\kappa^{**} < 0$ increasing shear first produces a transition from a localised to a non-localised mode, before stabilising when (38) is satisfied. For $\kappa^{**} > 0$, the non-localised mode occurs at small values of $\hat{S}_H$ but corresponds to an incoming energy flux, so that an unstable mode occurs only in the band defined by

$$\frac{2(1+\hat{\tau})}{\hat{\tau}}\,\kappa^{**2} < \hat{S}_H < 4K^{**2} \quad (40)$$

where it is a localised mode. These results can more readily be assimilated from Figs. 1, 2, 3 where they appear as the asymptotic values (for long wavelength) of the stability and localisation boundaries. These figures are more fully discussed later.

Returning to inequality (38) we observe that the critical shear is a function of the density n, through α, and that there is a critical value of n requiring a maximum amount of shear. If $\tau < 1$, this occurs at $\alpha = (1-\tau)/\tau$, leading to

$$S_H > \frac{(1+\tau)^2}{\tau} \kappa^{**2} \tag{41}$$

and for $\tau > 1$, at $\alpha = 0$, giving

$$S_H > 4\kappa^{**2} \tag{42}$$

for the critical shear.

We have dealt at some length with this case since analogous techniques may be applied in the other two frequency regimes. The significance of this particular regime is not great since the requirement $\omega > \omega_{be}$ can only be satisfied by the solution (37) when $\tau \gtrsim \frac{m_i}{m_e}$ or $\tau \lesssim \frac{m_e}{m_i}$.

B. <u>The Low Frequency Regime</u>

Here we are led to a dispersion relation similar to equation (31), but characterised by V^{++} rather than V^{**}, a modified shear term and a somewhat different definition of γ as a result of finite banana effects and the omission of some FLR terms, as discussed earlier.

The analysis is entirely analagous to the high frequency case with the definitions

$$\kappa^{++} = \left(\frac{n'}{n}\right)^{-1} V^{++} \tag{43}$$

$$S_L = \frac{m_i}{m_e} \left(\frac{n'}{n}\right)^{-2} (2\pi R B_T q')^2 \, I \left[\oint \frac{d\chi}{B_p^{\,2} B_T^{\,2}} - I \left(\oint \frac{d\chi}{B_p^{\,2}} \right)^2 \right] \tag{44}$$

$$\alpha_2 = \frac{1}{4\pi n m_i} \frac{\oint d\chi \, R^2}{\left[\oint \frac{d\chi \, R^2}{B_p^{\,2}} - I \left(R B_T \oint \frac{d\chi}{B_p^{\,2}} \right)^2 \right]} \tag{45}$$

replacing κ^{**}, S_H and α respectively.

C. <u>The Intermediate Frequency Regime</u>

Finally we discuss the intermediate frequency regime and are led to the dispersion relation

$$\left[\frac{\kappa^{++}(1+\tau) + \kappa^G}{w(w+1) + \alpha w^2} + \frac{\kappa^G}{w+1+\alpha} - \gamma \right]^2 = \frac{\tau - w}{1 + w + \alpha w} \cdot \frac{S_I}{w^2} (2n+1) \tag{46}$$

where

$$\kappa^G = \left(\frac{n'}{n}\right)^{-1} (V_G{}'' - V^{++}) \tag{47}$$

and

$$S_I = \frac{m_i}{m_e}\left(\frac{n'}{n}\right)^{-2} (2\pi R\, B_T\, q')^2\, I \oint \frac{d\chi}{B^2 B_T{}^2} \tag{48}$$

The feature peculiar to the intermediate regime is the freedom provided by the quantity κ^G representing the wider class of modes arising from the choice of the phase function $G(\psi, \chi)$. To examine the significance of this we consider first the shear free situation which would be applicable if the rotational transform, or safety factor q, had an extremum at $\psi = \psi_o$. Setting $S_I \to 0$ in equation (46), and solving for w, we obtain the stability criterion

$$(\gamma + \kappa^G)^2 + 4(1+\alpha)(1+\tau)\kappa''\gamma + 4\alpha\kappa^G\gamma > 0 \ . \tag{49}$$

In the high density limit ($\alpha = 0$) it is apparent that a sufficient condition for stability is $\kappa'' > 0$. The charge separation caused by the geodesic drift (the κ^G term) cannot drive an instability, and the most unstable choice of G' would make

$$\gamma + \kappa^G = 0 \tag{50}$$

removing the FLR stabilising term. In general it may not be possible to satisfy equation (50) exactly, but a reduction of FLR stabilisation will occur. For very low densities ($\alpha \gg 1$) the stability criterion (49) becomes

$$\left(\frac{n'}{n}\right)\left\{(1+\tau)\frac{\partial}{\partial\psi}\oint \frac{d\chi}{B_p{}^2} + R B_T \oint \frac{d\chi}{B^2} G'\right\} > 0 \tag{51}$$

so that, with a suitable choice for G', $\kappa'' > 0$ is no longer sufficient for stability and shear becomes necessary whatever the κ'' properties of the equilibrium!

The stability diagram in the presence of shear has been constructed numerically and is shown in Figs. 1 - 3 for the case of a high density plasma ($\alpha = 0$) in the cylindrical limit for $\tau = 0, 1, \infty$ respectively. [This also describes the high frequency case when $S = S_H$ and $\kappa'' = \kappa^{**}$, the low frequency case when $S = S_L$ and $\kappa'' = \kappa^{++}$ and the intermediate case with $\kappa^G = 0$ when $S = S_I$ and $\kappa'' = \kappa^{++}$]. For a cold electron plasma ($\tau = 0$) the results have been given by Mikhailovskaya and Mikhailovskii (1965). Shear progressively stabilises, with maximum shear required for long wavelength modes. For equal ion and electron temperatures, Fig. 2 shows a similar result for a κ'' unstable system, but shows that a small amount of shear may drive a κ'' stable system unstable (also noted by Coppi et al. (1968)). For a hot electron ($\tau = \infty$) plasma in a κ'' unstable system, short wavelength modes are more unstable and increasing shear modifies the localised mode to a non-localised mode with outward energy flow before finally stabilising, while in a κ'' stable system only the localised mode is possible. The typical shear required for stability may be obtained analytically in the long wavelength limit where it is independent of the sign of κ''(or κ^{++}) and is given by

$$S_I > 4(\kappa^{++})^2 \, . \tag{52}$$

Returning to equation (46), we consider the particular case of long wavelength modes, setting $\gamma = 0$ to obtain the following expression for the critical shear.

$$S_I = \frac{4(1+\tau)(\kappa^{++} + \kappa^G)}{[1+\tau+\alpha\tau]^2} \left[(1+\tau)(1+\alpha)\kappa^{++} + \alpha\kappa^G\right] \, . \tag{53}$$

Maximum shear is required for those modes with the largest values of $|\kappa^G|$ which is still to be regarded as arbitrary. To estimate the maximum shear we require an upper bound for $|\kappa^G|$, and the most stringent one arises from the $\frac{kV_{di}}{\omega} < 1$ assumption. In terms of κ^G this inequality is

$$|\kappa^G| < |w| \oint \frac{d\chi}{B_p^{\,2}} \tag{54}$$

From equation (46) we may calculate the value of w at marginal stability, the result being

$$w = \left\{\kappa^G[\kappa^G + (1+\tau)\kappa^{++}] - \frac{1}{2} S_I[(1+\alpha)\tau - 1]\right\} \left\{\kappa^{G^2} + (1+\alpha)S_I\right\}^{-1} \tag{55}$$

with S_I given by (53).

Finally, taking $|\kappa^G| \sim |w| \oint \frac{d\chi}{B_p^{\,2}}$ as the limiting value of κ^G, and $|\kappa^G| \gg |\kappa^{++}|$ we obtain for the shear

$$S_I = 4\alpha(1+\tau) \left\{\frac{1+\tau - \alpha\tau}{(1+\tau+\alpha\tau)(1+\tau + 2\alpha + \alpha\tau)} \oint \frac{d\chi}{B_p^{\,2}}\right\}^2 \tag{56}$$

in the low density limit where $|\alpha\kappa^G| \gg |\kappa^{++}|$, and

$$S_I = 4|\kappa^{++}| \oint \frac{d\chi}{B_p^{\,2}} \tag{57}$$

in the high density limit ($\alpha = 0$).

With $\kappa_G \sim w$, however, a strong ion resonance occurs between the wave frequency ω and the local geodesic drift frequency ω_{dg}. To investigate this we consider values of the shear exceeding the critical values given by equations (56) and (57). In this strong shear limit there is an electron mode with $w \sim \tau$ and an ion mode with $w \sim -1/(1+\alpha)$. To estimate the resonant growth or damping of these modes we write (following Rutherford and Frieman, 1968) the exact eigenvalue equation, ie. Poisson's equation, in the form

$$L(\omega)\varphi + R(\omega)\varphi = 0 \tag{58}$$

where R contains the ion and electron resonance terms, equations (83) and (81) of Rutherford and Frieman, and for the intermediate frequency range L is constructed from equation (45) and (57) of part I, for electrons and ions respectively. Note that ω and R in this paper correspond to $-\omega$ and $-R$ in Rutherford and Frieman. Perturbation analysis then gives the growth rate $\hat{\gamma}$ from

$$\frac{\hat{\gamma}}{\omega}\int d^3x\left(\varphi^* \,\omega\,\frac{\partial L}{\partial \omega}\,\varphi\right) = \mathfrak{Im}\int d^3x\;\varphi^* R\varphi \tag{59}$$

where the d^3x integration covers all space.

The relevant results are, using the form (29) for the eigenfunction

$$\int d^3x\,\varphi^*\,\omega\,\frac{\partial L}{\partial \omega}\varphi = \frac{e^2n}{T_i}\frac{\sqrt{\pi}}{2}\,\sigma^{-\frac{3}{4}}\,\frac{T_i}{m_e}\,I\left(\frac{2\pi\ell q'}{\omega}\right)^2\left\{\frac{w-\tau}{w[(1+\alpha)w+1]}-\left(2-\frac{3\tau}{w}\right)\right\} \tag{60}$$

$$\int d^3x\,\varphi^*\,R_i\,\varphi \simeq -\frac{\omega}{|\omega_{*i}|}\,\sigma^{-\frac{1}{4}}\,\tau\left(1+\frac{1}{w}\right)\oint\frac{d\chi}{B_p^2}\,\frac{e^2n}{T_i} \tag{61}$$

where

$$\sigma = \frac{\tau - w}{1+w(1+\alpha)}\left(\frac{2\pi\ell q'}{\omega}\right)^2\frac{e^2}{m_e m_i}\,\frac{I}{\oint\frac{R^2}{B^2}\,d\chi} \tag{62}$$

and we have used the fact that $\omega \sim -\omega_{*i} \sim -\omega_{dg}$.

In the limit $\alpha < 1$ the resulting growth rate is given by

$$\frac{\hat{\gamma}}{|\omega|} \sim \frac{\tau}{1+\tau}\,\delta w(\alpha + \delta w)\left(\frac{n'}{n}\,\frac{\oint\frac{d\chi}{B_p^2}}{V''}\right) \tag{63}$$

where δw is the shift in w away from $-1/1+\alpha$, i.e.

$$\delta w \sim \frac{m_e}{m_i}\left(\frac{|V''|(1+\tau)}{\oint\frac{d\chi}{B_p^2}}+\frac{n'\alpha}{n}\right)^2\frac{\left(\oint\frac{d\chi}{B_p^2}\right)^2}{I\oint\frac{R^2}{B^2}\,d\chi}\;\frac{1}{(1+\tau)(2\pi q')^2}\,. \tag{64}$$

Thus, for values of the shear parameter just in excess of the critical shear required to stabilise the reactive flute mode, the resonant growth rate decreases as the fourth power of the shear, until $\delta w \sim \alpha$ when the decreases becomes quadratic. In addition electron landau damping further decreases the growth rate, though the effect is only significant at higher frequencies, the electron damping cancelling the ion resonance if

$$\frac{m_e}{m_i}\left(\frac{n'}{2\pi m q'}\right)^2 \lesssim \left(\frac{\omega}{\omega_{be}}\right)^3. \tag{65}$$

CONCLUSION

From the foregoing analysis we conclude that cylindrical analysis leads to a significant underestimate of the shear required to remove the electrostatic flute mode in a toroidal device of the Levitron type. The results for the high and low frequency regimes are treated in somewhat greater detail than in earlier work but do not essentially change the amount of shear required for stability. In the intermediate range, however, we find a significant increase in the necessary shear. Three separate effects combine to bring this about. First, shear is effective only through passing particles (appearing through the factor I in equation (48)). Second, the geodesic curvature of the magnetic field lines results in drifts which can cause destabilising charge separation in addition to that provided by positive V'' and third, a strong resonance becomes possible between the wave frequency and the local geodesic drift frequency of the ions. The effect of the first two features on the stabilising shear for the reactive mode is summarised in equations (56) and (57) for low and high densities respectively. The last effect results in a weak instability with growth rate given in equation (63), whatever the value of the shear parameter, provided only that inequality (65) is not satisfied.

REFERENCES

BERK, H.L. and PEARLSTEIN, D. (1969) Phys. Rev. Letts. 23, 220.

COPPI, B., LAVAL, G. PELLAT, R. and ROSENBLUTH, M.N. (1968) Plasma Phys. 10, 1.

HASTIE, R.J. and TAYLOR, J.B. (1971) Plasma Phys. 13, 265.

JAMIN, E. (1971) Ph.D. Thesis, Princeton University.

JOHNSON, J.L. and GREENE, J.G. (1967) Plasma Phys. 9, 611.

MIKHAILOVSKAYA, L.V. and MIKHAILOVSKII, A.B. (1965) Nucl. Fus. 5, 234.

RUTHERFORD, P.H. and FRIEMAN, E.A. (1968) Phys. Fluids 11, 569.

RUTHERFORD, P.H., ROSENBLUTH, M.N., HORTON, W., FRIEMAN, E.A. and COPPI, B. (1968) Plasma Physics and Controlled Nuclear Fusion Research, (Proc. 3rd. Int. Conf. Novosibirsk) 1, I.A.E.A., Vienna (1969) 367.

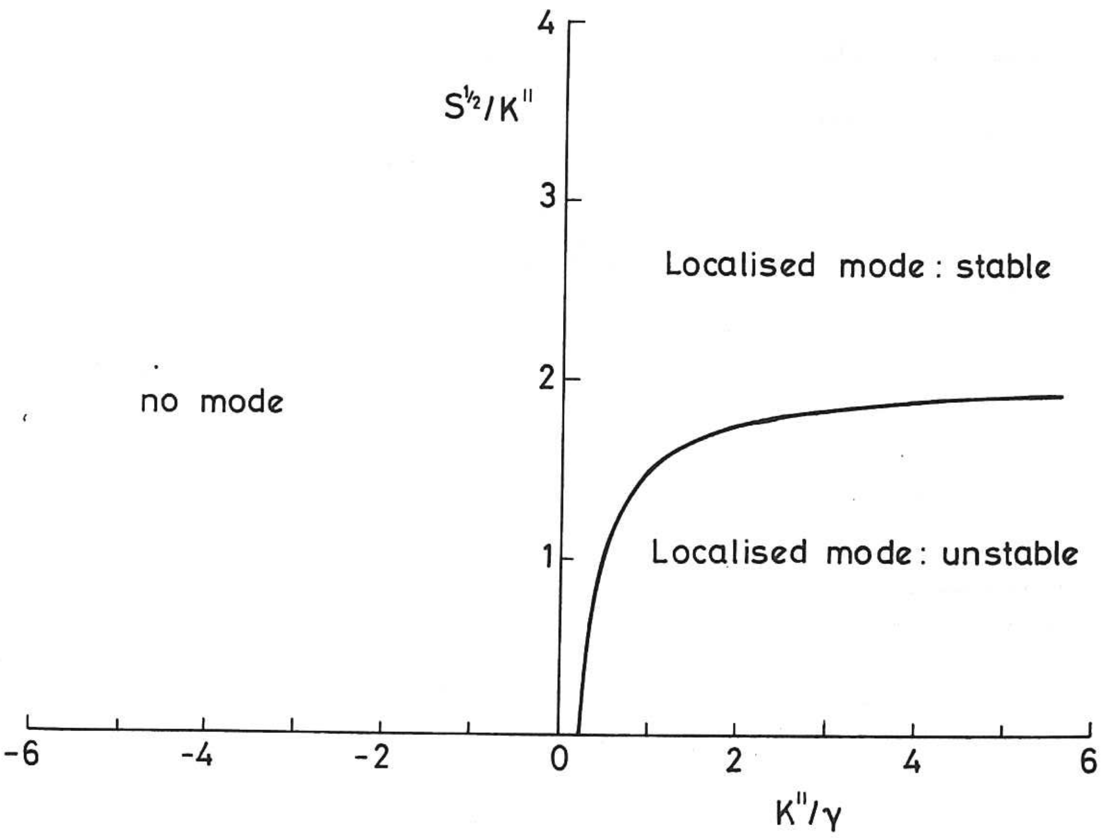

Fig. 1. Stabilising shear as a function of wavelength at high density when $\tau = 0$ for the cylindrical case.

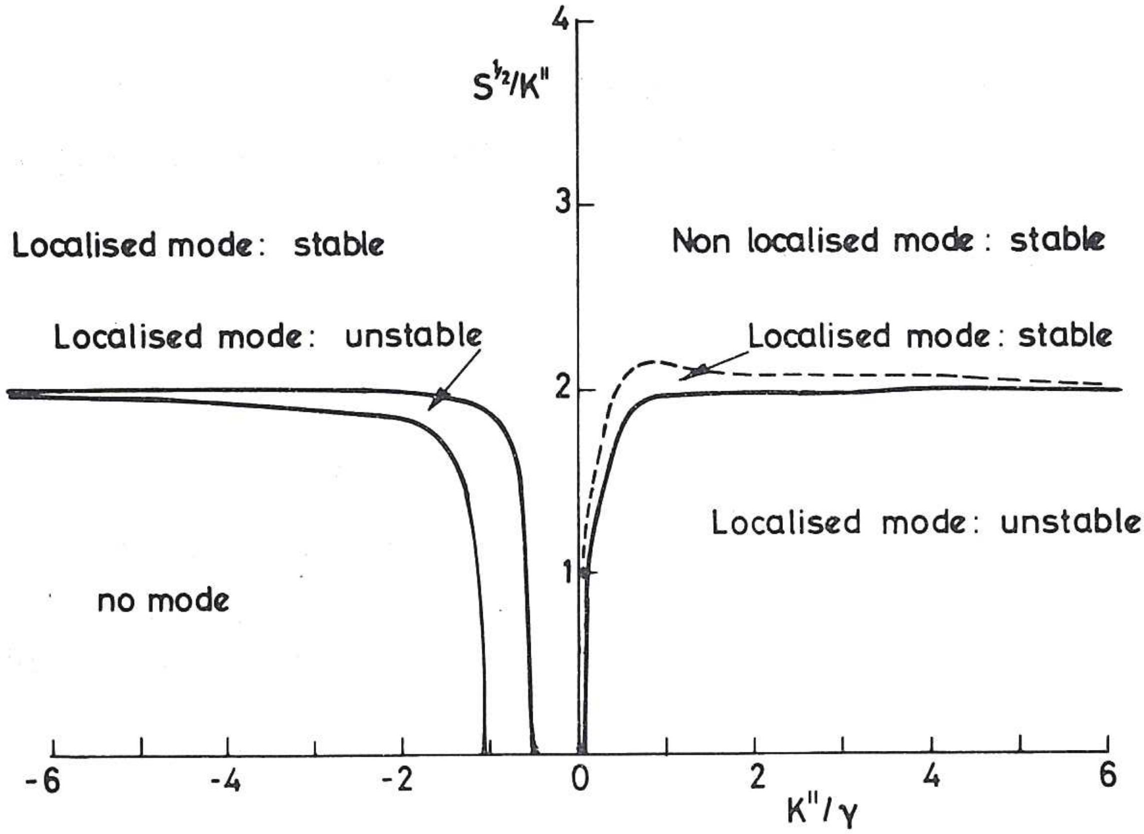

Fig. 2. Stabilising shear as a function of wavelength at high density when $\tau = 1$ for the cylindrical case.

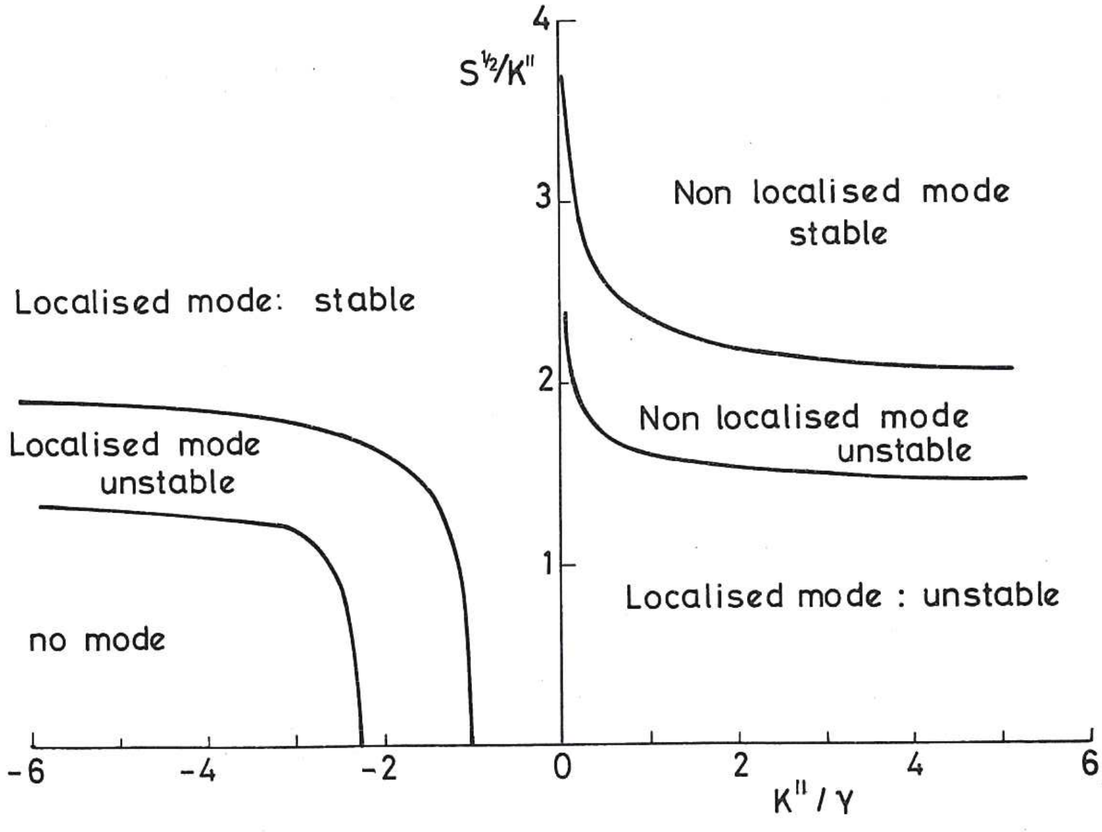

Fig. 3. Stabilising shear as a function of wavelength at high density when $\tau = \infty$ for the cylindrical case.