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RADIATIVE TRANSFER OF DOPPLER BROADENED RESONANCE LINES
Part 2

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RADIATIVE TRANSFER OF DOPPLER BROADENED RESONANCE LINES

Part 2

by

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A B S T R A C T

The solution of the equation of transfer of radiation for Doppler broadened resonance lines emitted by a two level atom in a uniform plane parallel plasma has been calculated allowing for the variation of the source function in frequency and angle. This solution is compared with that obtained by assuming the source function is constant in frequency and angle, that is assuming complete redistribution. The line profiles and total line intensities calculated from the constant source function solution agree to within 10% with the profiles and intensities obtained from the exact solution. Thus the constant source function solution may be used with confidence.

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CONTENTS

1. INTRODUCTION

In paper 1 (Hearn 1963) the radiative transfer of doppler broadened resonance lines in a uniform plane parallel plasma was studied. A two level atomic model was used with four processes transferring electrons between the two levels. These were electron excitation and de-excitation, spontaneous emission which was assumed to be isotropic and unpolarised, and photo-excitation. Stimulated emission was neglected and it was assumed that the population of the excited level was very much smaller than the population of the ground level. Natural and pressure broadening were neglected.

The most important assumption was that the ratio of the emission coefficient to the absorption coefficient, the source function, is independent of frequency and direction. This assumption is unconditionally valid at high densities where the excited atoms suffer many collisions with other atoms before emitting a photon. As a result of these collisions the frequency at which a photon is absorbed and the frequency at which it is emitted are completely uncorrelated, and the excited atoms have a Maxwellian velocity distribution.

The density required for collisions to ensure that the source function is independent of frequency is much higher than the density of many plasmas of interest such as that in ZETA and astronomical plasmas such as the solar chromosphere and corona. This paper considers low density plasmas where the excited atoms undergo no collisions before the spontaneous emission of a photon. This calculation is the other extreme and all physical situations must lie between the two. Generally the excited atoms do not have a Maxwellian velocity distribution since the slower atoms are preferentially excited and the velocity distribution may not be spherically symmetrical. In these circumstances the emission coefficient and hence the source function is not only a function of space but also of frequency and direction of emission. The frequency of an emitted photon is completely determined by the frequency of the absorbed photon and the doppler effect of the absorbing atom.

Thomas (1957) has argued that the source function is constant to a factor of four within three doppler half widths of the centre. In order to determine precisely the magnitude of the variation of the source function and its effect on the line profile, the calculations in paper 1 have been repeated making the assumption that the excited atoms undergo no collisions.

The calculations show that the assumption of a source function which is independent of frequency will give line profiles and total line intensities emitted normally to the plasma which are accurate to about 10%, and consequently this assumption may be used with confidence. Calculations of the source function and radiation density, particularly near the edge of the plasma, require a more elaborate calculation for the same accuracy.

2. THE BASIC EQUATIONS

In a low density plasma where the excited atoms do not undergo collisions with other atoms, their velocity distribution is not Maxwellian and the emission coefficient varies not only in frequency but also in direction. The equation of transfer is still valid. Using the definitions and nomenclature of Ambartsumyan (1958, Chapter 2) the equation of transfer is

$$\frac{dI_x}{d\tau_x} = - I_x + \frac{j_x}{\chi_x} \qquad \ldots (1)$$

where I_x is the specific intensity of radiation at a given point and in a given direction at the dimensionless frequency x, which is given by

$$x = \frac{\nu - \nu_0}{\Delta\nu_D} \qquad \ldots (2)$$

where ν_0 is the central frequency of the line and $\Delta\nu_D$ is the e^{-1} frequency of the Gaussian profile. τ_x is the optical depth measured in the same direction, j_x is the corresponding emission coefficient and χ_x is the absorption coefficient.

It is convenient to express the intensity as a fraction of the black body source function f_{bb} . This also equals the black body intensity. Equation (1) divided by f_{bb}, then has the analytical solution of

$$\frac{I_x}{f_{bb}} = \frac{I_x^o}{f_{bb}} \; e^{-\tau_x} \; + e^{-\tau_x} \int_o^{\tau_x} F_x \; e^{\tau_x'} \; d\tau_x' \qquad \ldots (3)$$

where F_x, the fractional source function, is the ratio of the source function j_x/χ_x to the black body source function f_{bb}. I_x^o is the intensity incident on the plasma at the origin of τ_x . This is at the edge of the plasma and so represents the radiation incident on the plasma from outside. In this calculation it is zero.

For a Maxwellian distribution of ground level atoms with negligible natural and pressure broadening, the absorption coefficient is

$$\chi_x = \frac{n_1 \; h\nu_o \; B_{12} \; e^{-x^2}}{\rho c\sqrt{\pi}} \qquad \ldots (4)$$

where B_{12} is the Einstein absorption coefficient, ρ the mass density of the plasma and n_1 is the number density of atoms in the ground level. The absorption coefficient does not depend on direction.

The emission coefficient is determined partly by photo-excitation from the ground level and partly by electron excitation and de-excitation collisions together with spontaneous emissions. That part of the emission coefficient contributed by photo-excitation may be described in terms of the redistribution function.

If a photon having a frequency in the range x_i to $x_i + dx_i$ travels in the direction of the unit vector $\underline{r}_i$ within a solid angle $d\omega_i$, then the probability that the photon is absorbed and emitted in the frequency range x_e to $x_e + dx_e$ in the direction of $\underline{r}_e$ within a solid angle $d\omega_e$ is

$$R(x_i, \underline{r}_i, x_e, \underline{r}_e) dx_i \; d\omega_i \; dx_e \; d\omega_e \qquad \ldots (5)$$

This probability is normalised so that

$$\iiiint R(x_i, \underline{r}_i, x_e, \underline{r}_e) dx_i \; d\omega_i \; dx_e \; d\omega_e = 1 \qquad \ldots (6)$$

where the integrals are over all frequencies and solid angles.

For a Maxwellian distribution of atoms whose natural width is negligible compared with the width of the doppler profile and when the excited atoms do not undergo an elastic collision before the spontaneous emission of the photon, the redistribution function is

$$R(x_i, \underline{r}_i, x_e, \underline{r}_e) = \frac{e^{-\left[x_e^2 + \left(\frac{x_i - x_e \cos\alpha}{\sin\alpha}\right)^2\right]}}{16\,\pi^3 \sin\alpha} \qquad \ldots (7)$$

where α is the angle between the vectors $\underline{r}_i$ and $\underline{r}_e$. It assumes that the spontaneous emission is isotropic. This redistribution function was first calculated by Thomas (1957) assuming coherent scattering within the rest frame of the atom. It is also given together with other types of redistribution functions for doppler broadening by Hummer (1962) from which it may be shown that the same result is obtained for the redistribution function as the natural width tends to zero assuming complete redistribution within the rest frame of the atom.

The probability of absorption irrespective of the direction and frequency of emission may be obtained by integrating the redistribution function over all solid angles and frequencies of emission (Hummer 1962).

$$\iint R(x_i, \underline{r}_i, x_e, \underline{r}_e)\,d\omega_e\,dx_e = \frac{1}{8\pi}\int \mathrm{Erfc}\,|\overline{x}|\,dx_e = \frac{e^{-x_i^2}}{4\pi\sqrt{\pi}} \qquad \ldots (8)$$

where $|\overline{x}|$ is the larger of the moduli of the two frequencies x_i and x_e and Erfc x is $1 - \mathrm{Erf}\,x$. The probability of absorption in a doppler broadened profile is normalised so that the probability of absorption integrated over all incident frequencies is unity. With this normalisation the probability of absorption is

$$4\pi R(x_i, \underline{r}_i, x_e, \underline{r}_e)\,dx_i\,d\omega_i\,dx_e\,d\omega_e \qquad \ldots (9)$$

and the energy emitted in the frequency range x_e to $x_e + dx_e$ in a direction $\underline{r}_e$ within a solid angle $d\omega_e$ due to photo-excitation caused by light incident at all frequencies and angles is

$$4\pi\, n_1\, B_{12}\,\frac{h\nu_0}{c}\int_{-\infty}^{\infty}\int^{4\pi} I_{x_i}\,R(x_i, \underline{r}_i, x_e, \underline{r}_e)\,d\omega_i\,dx_i\,d\omega_e\,dx_e$$

$$= j_{x_e}(\underline{r}_e)\rho\,d\omega_e\,dx_e \;\;\mathrm{erg\ cm^{-3}\ sec^{-1}} \qquad \ldots (10)$$

This defines the contribution to the emission coefficient $j_{x_e}(\underline{r}_e)$ from photo-excitation.

Inelastic electron collisions have two effects on the emission coefficient. Electron excitation collisions increase the emission coefficient by

$$\frac{n_1 \; n_e \; X_{12} \; h\nu_o \; e^{-x_e^2}}{4\pi \; \rho \sqrt{\pi}} \quad \text{erg gm}^{-1} \; \text{sec}^{-1} \qquad \ldots (11)$$

by increasing the rate of excitation, whilst the de-excitation collisions compete with spontaneous transitions so that only a fraction

$$\frac{A_{21}}{A_{21} + n_e \; Y_{21}} \qquad \ldots (12)$$

of the excited atoms emit photons and the total emission coefficient is reduced by this factor. Combining all these processes and dividing by the absorption coefficient gives the source function $f(x_e, \underline{r}_e)$ and this may be expressed as a fraction $F(x_e, \underline{r}_e)$ of the black body source function f_{bb} which is given by

$$f_{bb} = \frac{c \quad A_{21} \; X_{12}}{4\pi \; B_{12} \; Y_{21}} \qquad \ldots (13)$$

so that

$$F(x_e, \underline{r}_e) = \frac{4\pi \sqrt{\pi} \; A_{21} \; e^{x_e^2} \int_{-\infty}^{\infty} \int^{4\pi} \frac{I_{x_i}}{f_{bb}} R(x_i, \underline{r}_i, x_e, \underline{r}_e) d\omega_i \; dx_i + n_e \; Y_{21}}{A_{21} + n_e \; Y_{21}} \qquad \ldots (14)$$

This equation assumes that the excited level population is small compared with the ground level population and that stimulated emission is negligible.

When the plasma is optically thin, the fractional source function is

$$F_o = \frac{n_e \; Y_{21}}{A_{21} + n_e \; Y_{21}} \qquad \ldots (15)$$

This depends only on the electron temperature and density and it is constant in a uniform plasma. Equation (14) may be written in terms of F_o.

$$F(x_e, \underline{r}_e) = (1 - F_o) \; \overline{P}_E \; (x_e, \underline{r}_e) + F_o \quad , \qquad \ldots (16)$$

where

$$\bar{P}_E(x_e, \underline{r}_e) = 4\pi\sqrt{\pi}\, e^{x_e^2} \int_{-\infty}^{\infty} \int^{4\pi} \frac{I_{x_i}}{f_{bb}} R(x_i, \underline{r}_i, x_e, \underline{r}_e) d\omega_i\, dx_i \qquad \dots (17)$$

for a given point in space. The problem is to determine the source function F, as a function of space, frequency and angle of emission, which is consistent with equations (3),(17) and (16) for a uniform plane parallel plasma of optical thickness τ and electron density and temperature defined by F_o. The solution of the equation of transfer using equation (17) to calculate the source function will be called the exact source function solution.

An approximation that has been made in numerical calculations (Hummer 1963) is to ignore the variation of the redistribution function in angle and to assume that it is isotropic, treating the redistribution in frequency only. The isotropic redistribution function is

$$R(x_i, x_e) = \frac{1}{2}\,\mathrm{Erfc}\,|\bar{x}| \qquad \dots (18)$$

It may be obtained by integrating the exact redistribution function over all incident and emitted angles, which has been done by Hummer (1962). It was derived directly by Unno (1952)

The derivation of the equations defining the source function may be repeated using the isotropic redistribution function and only equations (17) and (16) are changed. Equation (17) becomes

$$\bar{P}_I(x_e) = \sqrt{\pi}\, e^{x_e^2} \int_0^{\infty} P_{x_i}\, \mathrm{Erfc}\,|\bar{x}|\, dx_i \qquad \dots (19)$$

where

$$P_{x_i} = \frac{\rho_{x_i}}{\rho_{bb}}, \quad \rho_{x_i} = \frac{1}{c}\int^{4\pi} I_{x_i}\, d\omega_i$$

and

$$\rho_{bb} = \frac{4\pi}{c} f_{bb} \qquad \dots (20)$$

ρ_{x_i} is the radiation density at the frequency x_i, P_{x_i} is the radiation density expressed as a fraction of the black body radiation density ρ_{bb}. ρ_{bb} is effectively constant over the line profile. Equation (16) becomes

$$F\ (x_e) = (1 - F_o)\ \bar{P}_I\ (x_e) + F_o\ . \qquad \ldots (21)$$

The solution of the equation of transfer using equations (19) and (21) to calculate the source function will be called the isotropic source function solution.

With the usual assumption of complete redistribution, equation (17) becomes

$$\bar{P}_C = \frac{2}{\sqrt{\pi}} \int_0^\infty P_{x_i}\ e^{-x_i^2}\ dx_i\ , \qquad \ldots (22)$$

and equation (16)

$$F = (1 - F_o)\ \bar{P}_C + F_o\ . \qquad \ldots (23)$$

The solution obtained using equations (22) and (23) will be called the constant source function solution, or the solution with complete redistribution.

3. THE RESULTS

The distribution of the source function F was calculated for a range of optical thickness measured at the line centre of 0.1 to 1000 using an I.B.M.7030 (Stretch) computer for a uniform plane parallel plasma using the constant, isotropic and exact equations for the source function. The numerical methods used to solve these equations are described elsewhere (Hearn 1964).

The differences between the constant source function solution, where complete redistribution is assumed, and the other two solutions are largest when the electron density and hence F_o is very small so that electron de-excitation collisions are much less than spontaneous emissions. For a finite plane parallel plasma, if F_o is so small that the source function is much less than the black body source function, the shape of the solution for a given optical depth becomes independent of F_o, and the solution becomes simply proportional to F_o. This is not obvious from the integral equations determining the source function but it does become obvious from the equations used in the numerical solution. The results described here are for this low density region. The differences between the three calculations at higher electron densities are always smaller.

The isotropic source function solution assumes that the source function is a function of frequency as well as space, but independent of angle. Fig.1 shows the isotropic source function as a continuous line, calculated at the centre and edge of the plasma as a function of the dimensionless frequency x. The source function is expressed as a ratio of the optically thin source function F_0. The source function is always greater at the centre than at the edge. The horizontal dotted lines are the corresponding constant source function solutions which are independent of frequency.

A comparison between the constant and isotropic solutions shows that at the centre of the frequency profile of the source function the isotropic solution at the centre of the plasma is greater than the constant source function solution, and less at the edge. This comes from the difference between the two redistribution functions. In the isotropic redistribution a photon at the centre in space and frequency finds it more difficult to leave the plasma and so the source function is increased.

Equation (19) shows that for large values of the frequency x where the plasma is optically thin, that part of the source function due to photo-excitation decreases as $e^{x^2}\mathrm{Erfc}\, x$. The source function at the centre of the plasma decreases monotonically with increasing frequency, but the source function at the edge of the plasma rises to a maximum before falling away. This is caused by the self reversal of the line profiles at the edge due to the spatial variation of the source function. This has no effect on the line profile emitted normally to the plasma because the source function at the edge only determines the centre of the line profile. At the frequencies where the source function at the edge is a maximum, the line profile is determined mainly by the source function in the central regions of the plasma, and any effect in the edge regions is overwhelmed. The line profiles emitted normally to the plasma calculated from the isotropic solution of the source function are shown in Fig.2 in continuous lines and the profiles calculated from the constant source function solution in dotted lines. The intensities are expressed as a ratio of the black body intensity and are given in units of F_o.

The differences between these two sets of line profiles are not greater than 10%. The main difference is in the wings of the profiles. At frequencies where the plasma is optically thin, the profiles calculated from the constant source function solution decrease as e^{-x^2} while the profiles calculated from the isotropic solution decrease as Erfc x.

The exact source function solution makes no assumptions about the variation of the emission coefficient with angle. In the middle of a plasma of large optical thickness the intensities at all the significant frequencies at a given point in space are isotropic, and integrating an isotropic intensity distribution over the exact redistribution function gives an isotropic emission coefficient and isotropic source function. At the edge of a plasma of large optical thickness, or in the central regions of a plasma of moderate optical thickness, the intensities incident at any given point will be anisotropic and the differences between the exact and isotropic source function solutions will be largest. The exact source function solution was calculated for an optical thickness of 1 and 10 and the source function for different angles did not vary by more than 2%. This shows that the isotropic source function solution is a good approximation for source function calculations.

The most important assumption made in the calculations in paper 1 is that the source function is independent of frequency and direction. This assumption has been removed for the calculations here so that the restrictions described for the validity of a constant source function are no longer relevant. The other assumptions made in paper 1 and their regions of validity are retained.

4. CONCLUSIONS

These calculations of the constant, isotropic and exact solutions of the equation of transfer show that for a two level atom in a uniform plane parallel plasma, the assumption of a source function which is constant in frequency and angle, that is the assumption of complete redistribution, gives line profiles and total line intensities emitted normally to the plasma which are accurate to 10% or less. Thus the constant source function approximation may be used to calculate them with confidence.

The variation of the source function with frequency can be quite large particularly near the edge of the plasma, and although this has no significant effect on the line profile emitted normally it will become increasingly important as the angle to the normal of emission increases. The variation of the source function in frequency will also be reflected in the radiation density in the plasma.

Comparison between the isotropic and exact source function solutions shows that these effects are accurately represented by the isotropic source function solution and that the exact source function solution is quite unnecessary.

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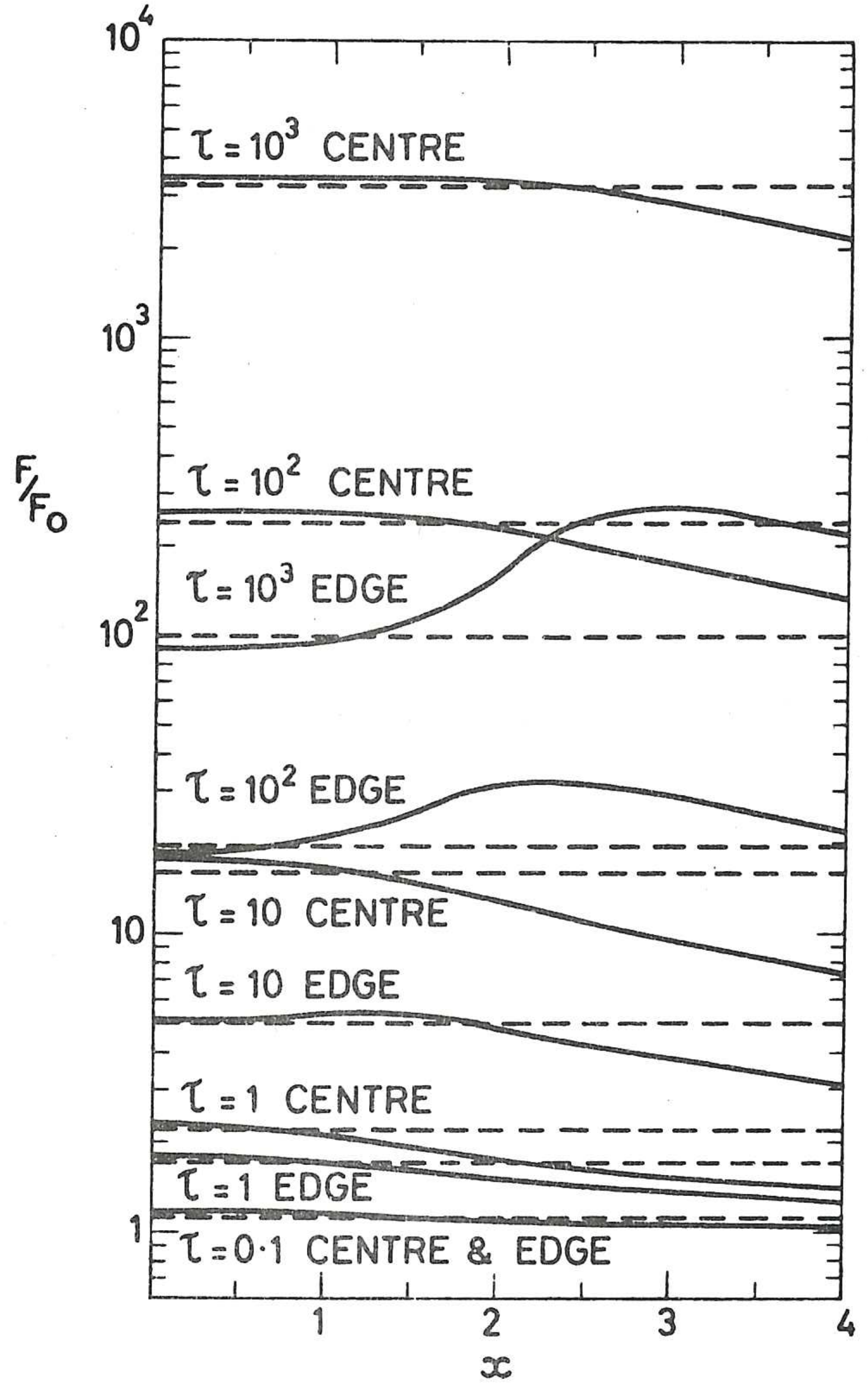
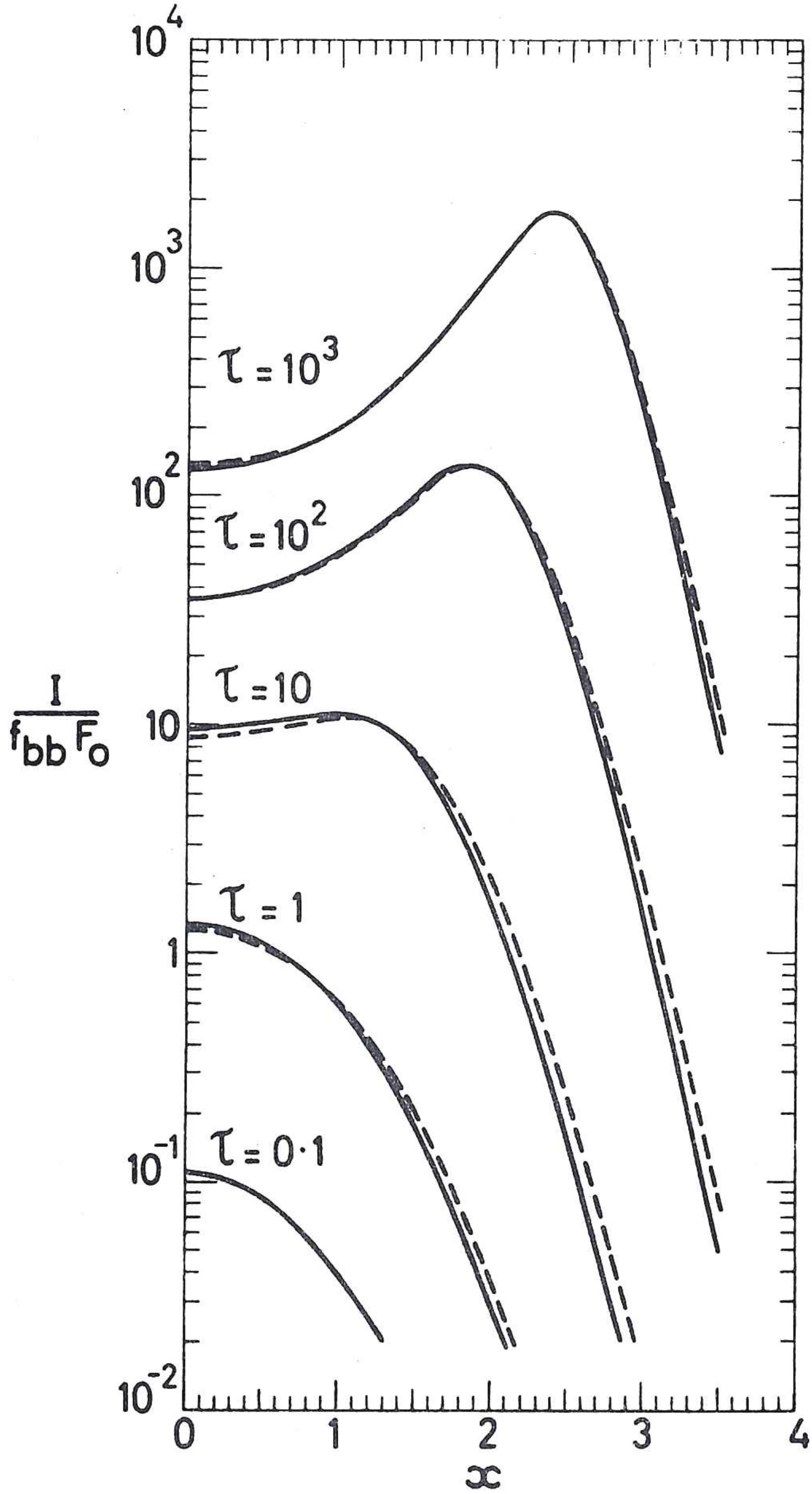


CLM-P 37 Fig. 1

The source function F expressed as a ratio of the optically thin source function F_0 calculated at the edge and centre of the plasma plotted against the dimensionless frequency x for various optical depths. The isotropic source function is shown as a continuous line and the constant source function as a dotted line.



CLM-P 37 Fig. 2

The line profiles emitted normally from the plasma plotted against the dimensionless frequency x for various optical depths. The intensity is expressed as a ratio of the black body intensity in units of F_o. The profiles calculated from the isotropic source function are shown as continuous lines and those from the constant source function as dotted lines.