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THE EFFECT OF FINITE LARMOR RADIUS ON THE INTERCHANGE INSTABILITY IN A CYLINDRICAL PLASMA

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THE EFFECT OF FINITE LARMOR RADIUS ON THE INTERCHANGE INSTABILITY IN A CYLINDRICAL PLASMA

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ABSTRACT

The eigenvalue equation for the interchange instability is derived from guiding centre equations including finite Larmor radius corrections. Analytic solutions are found which are valid near the $\underline{k}.\underline{B} = 0$ surface and away from this surface, Matching these two solutions over their common range of validity determines the eigenvalue and hence the mode frequency. The growth rate and radial profile of the mode may readily be evaluated explicitly. The analysis is applicable to the high-β pinch, since cylindrical effects dominate over toroidal. For typical parameters, FLR stabilisation is found to increase the marginally stable pressure gradient by a factor two over that given by the Suydam criterion.

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1. INTRODUCTION

The interchange instability may occur in a magnetically confined plasma when the field lines are convex to the plasma. In such a system, if two flux tubes enclosing equal flux are interchanged, together with their contained plasma, the plasma compressional energy is reduced without changing the magnetic energy. Such an interchange is not topologically possible when the magnetic field is sheared. The energetically most advantageous interchange then bends the field lines, and gives rise to an instability only if the energy released by plasma expansion exceeds the increase in magnetic field energy.

The condition for shear stabilisation of the interchange mode in a cylindrical plasma described by the ideal magnetohydrodynamic (MHD) equations, was derived by Suydam [1] using the energy principle. Since this mode is highly localised in the radial direction, it was expected that its stability in a real plasma would be influenced by finite Larmor radius (FLR) effects, neglected in the MHD equations. FLR effects were first studied for the Rayleigh-Taylor instability of an inhomogeneous plasma supported against gravity by a shear-free magnetic field [2,3]. Gravity plays a role analogous to field line curvature. The main effect is to replace ω^2 in the MHD dispersion equation by $\omega(\omega - k_n U_{ni})$, where ω is the angular frequency, k_n is the wavenumber perpendicular to the confining field, and $U_{ni} = (dp_i/dr)/neB$ is the ion diamagnetic velocity. Whereas all wave lengths are unstable in the MHD model, those below a critical value were now found to be stable. Mikhailovskaya and Mikhailovskii [4] considered a cylindrical plasma confined by a sheared magnetic field. The interchange mode is now radially localised, its radial profile being described by a second order differential equation. This equation again differs from that obtained from the MHD model only by a replacement of ω^2 by $\omega(\omega - k_n U_{ni})$. Thus the eigenvalue for ω can be expressed in terms of that for the MHD model. However, since the eigenvalues of the MHD equation were not derived explicitly, it was only possible to estimate the order of magnitude of the FLR stabilisation.

The effect of FLR on the interchange instability in a straight helical field was investigated simultaneously by Kulsrud [5]. He found the same relation, $\omega(\omega - k_n U_{ni}) = \omega_H^2$, between the eigenvalues with and without FLR terms, ω and ω_H respectively. He determined the eigenvalues of his differential equation and hence was able to evaluate explicitly the critical parameters for stability in a stellarator.

The differential equation for the radial profile in a cylindrical plasma is rederived in Sec. 2, using guiding centre equations. The eigenvalues of this differential equation are determined in Sec. 3, using the same treatment as Kulsrud [5]. The factor by which the critical gradient is increased over the Suydam value is plotted as a function of Larmor radius/scale length. The assumptions made, and application to experiments, are discussed in Sec. 4. The eigenfunction is expressed in a convenient form for evaluation, allowing comparison of observed and predicted radial profiles.

2. DERIVATION OF THE EIGENVALUE EQUATION

The differential equation for the radial variation of the interchange mode was derived by Mikhailovskaya and Mikhailovskii [6] by integrating the Vlasov equation along particle orbits. The following derivation from guiding centre equations is given because, although less rigorous, it is much more concise and more physically transparent.

Expressed in cylindrical coordinates, the confining magnetic field is $\underline{B}_o = [0, B_\theta(r), B_z(r)]$. The perturbation is taken in the form $f(r) \exp i(m\theta + kz - \omega t)$. We shall assume it to be localised around the radius r_o where its phase is constant along a field line, i.e., $k_{\parallel}(r_o) = 0$ where $k_{\parallel}(r) = (mB_\theta/r + k\,B_z)/B$. The most dangerous modes in a low-β plasma are those which, to first order, do not change the magnetic field energy, i.e. $\underline{B}_1 . \underline{B}_o = 0$ where subscript 1 denotes the perturbation. For such a perturbation the electric field perpendicular to $\underline{B}_o$ can be written as the gradient of a potential, i.e., $\underline{E}_\perp = - \nabla_\perp \varphi$. This substitution eliminates the compressional Alfven (or fast magnetosonic) wave. We shall assume $E_{\parallel} = 0$, as in the standard MHD analysis. If $E_{\parallel}$ were retained, a further relation between $E_{\parallel}$ and $j_{\parallel}$ must be derived from the kinetic equation, as in reference 7. This introduces the drift/ion-acoustic mode, whose coupling to the interchange mode can then be shown to be small if $\beta \ll 1$.

The plasma will be described by guiding centre equations [8]. This assumes $\omega \ll \Omega_i$, where $\Omega_i = eB/m_i$ is the ion gyration frequency, and $\rho_i \ll L_1$, where $\rho_i = (2m_i T/eB)^{\frac{1}{2}}$ is the mean ion Larmor radius and L_1 is the radial scale length of the perturbation. To minimise the analytic detail, both the temperature and the magnetic field strength will be taken to be uniform when evaluating the FLR corrections.

The average guiding centre motion of an ion perpendicular to the magnetic field is

$$\underline{V}_{i\perp} = \left(1 + \frac{\rho_i^2}{4}\nabla_\perp^2\right)\frac{\underline{E}_1 \times \underline{B}}{B^2} + \underline{V}_{Bi} + \frac{1}{\Omega_i B}\frac{d\underline{E}_\perp}{dt} + 0\left(\frac{\omega^2}{\Omega_i^2}, \frac{\rho_i^4}{L^4}\right)\frac{E_1}{B} \quad (1)$$

where $\underline{V}_{Bi} = \frac{2T_i}{eBR_c}\,\underline{b} \times \hat{\underline{r}}$, $R_c = r\frac{B^2}{B_\theta^2}$ are the mean guiding centre drift due to magnetic curvature and gradients, and the radius of curvature of the helical field lines, respectively. $\hat{\underline{r}}$ and $\underline{b}$ are unit vectors in the radial direction and parallel to $\underline{B}_o$ respectively. FLR corrections need be applied only to the dominant $(\underline{E} \times \underline{B})/B^2$ term, where the effect is to replace the electric field by its mean value averaged over the gyration orbit [8].

The linearised conservation equation for ion guiding centres is

$$\frac{\partial N_{i1}}{\partial t} + (\underline{v}_{io}.\nabla)N_{i1} + (\underline{v}_{i1}.\nabla)N_o + N_o\nabla.\underline{v}_{i1} = 0 \quad (2)$$

where N denotes the guiding centre density. This is related to the particle density n by [8]

$$n_i = \left(1 + \frac{\rho_i^2}{4}\nabla_\perp^2\right)N_i + 0\left(\frac{\rho_i}{L}\right)^4 N_i\,. \quad (3)$$

Because of the localisation assumption, $L_1 \ll r$, FLR corrections to equilibrium quantities are relatively small and we need not distinguish between n_o and N_o. The contribution of the electric and magnetic drifts to $\nabla.\underline{v}$ is negligible because

$$\nabla.\left(\frac{\nabla\varphi \times B_o}{B_o^2}\right) = -\nabla\left(\frac{1}{B_o^2}\right).(\nabla\Phi \times B_o) + \left(\frac{\nabla \times B_o}{B_o^2}\right).\nabla\Phi$$

and the variation in $\underline{B}_o$ is slow compared with that of the strongly localised interchange mode. Thus equation (2) gives

$$i(\omega - \underline{k}.\underline{v}_{Bi})n_{i1} = -\frac{dn_o}{dr}\left(1 + \frac{\rho_i^2}{2}\nabla_\perp^2\right)\frac{ik_n\varphi}{B_o} + ik_\parallel n_o v_{i\parallel} + \frac{i(\omega - \underline{k}.\underline{v}_{Bi})}{\Omega_i B_o}\left\{n_o\nabla_\perp^2\varphi + \frac{dn_o}{dr}\frac{d\varphi}{dr}\right\} \quad (4)$$

where $k_n = (m\,B_z/r - k\,B_\theta)/B$. Since $k_\parallel(r) \approx 0$ is assumed over the localisation region, we can write

$$\underline{k}.\underline{v}_{Bi} = \frac{2T_i}{eBR_c}\left(\frac{m}{r} b_z - kb_\theta\right) = k_n \; v_{Bi} \tag{5}$$

Since the effects of finite electron Larmor radius and inertia are negligible, the electron density perturbation is

$$(\omega - k_n \; v_{Be}) n_{ei} = - \frac{k_n}{B_o} \frac{dn_o}{dr} \varphi + k_{\parallel} n_o v_{e\parallel} \tag{6}$$

where $v_{Be} = - 2T_e / e\,BR_c$. Invoking quasi-neutrality, $n_{i1} = n_{e1}$, gives

$$\nabla_{\perp}\left[n_o(\omega - k_n v_{Be})(\omega - k_n v_{Bi} - k_n U_{ni}) \nabla_{\perp} \varphi\right] + k_n^2 \Omega_i (v_{Be} - v_{Bi}) n_o' \varphi$$

$$+ k_{\parallel} n_o \Omega_i B \left[(\omega - k_n v_{Bi}) v_{i\parallel} - (\omega - k_n v_{Bi}) v_{e\parallel}\right] = 0 \tag{7}$$

where $U_{ni} = (T_i/eB\,n_o)\, dn_o/dr$ is the ion diamagnetic velocity, and $n_o' = dn_o/dr$. The parallel current is carried mainly by the electrons,

i.e., $v_{i\parallel} \ll v_{e\parallel} \approx - j_{\parallel}/n_o e$.

We now relate $j_{\parallel}$ and φ by Maxwell's equations. Since $\underline{E} = - \nabla\varphi + i\, k_{\parallel}\, \varphi \underline{b}$,

$$\begin{aligned} 4\pi i \omega j_{\parallel} &= \underline{b}.\mathrm{curl}(\mathrm{curl}\ \underline{E}) \\ &= i\underline{b}.[\nabla(\nabla.k_{\parallel} \varphi \underline{b}) - \nabla^2 (k_{\parallel} \varphi \underline{b})] \\ &= - i \nabla_{\perp}^2(k_{\parallel} \varphi) - i \nabla(k_{\parallel} \varphi).[\underline{b} \times (\nabla \times \underline{b})] + ik_{\parallel} \varphi \nabla \times (\nabla \times \underline{b}) . \end{aligned} \tag{8}$$

The first term is dominant because of the localisation assumption. (The other two terms become significant only in the less localised kink-mode ordering). Substituting for $v_{e\parallel} - v_{i\parallel}$ in equation (7) leads to

$$\nabla_{\perp}\left[n_o(\omega - k_{\parallel} v_{Be})(\omega - k_n v_{Bi} - k_n U_{ni}) \nabla_{\perp} \varphi\right] - \frac{2\, n_o'(T_i + T_e)}{m_i R_c} k_n^2 \varphi$$

$$- v_A^2 k_{\parallel} n_o \left(\frac{\omega - k_n v_{Be}}{\omega}\right) \nabla_{\perp}^2 (k_{\parallel} \varphi) = 0 \tag{9}$$

where v_A is the Alfven velocity.

To obtain the eigenvalue equation in the same form as in earlier publications [4] we assume $\omega \sim O(k_n U_{ni}) \gg k_n v_{bj}$. The localisation assumption allows the variation of n_o and r to be neglected. Since we do not wish to include the driving term for the kink mode, we neglect $k''_{\parallel}\varphi$ compared with $k'_{\parallel}\varphi'$, equation (9) then takes the form given in reference 4, i.e.

$$\left[\omega(\omega - k_n U_i) - v_A^2 k_{\parallel}^2\right]\left(\frac{d^2\varphi}{dr^2} - k_n^2\varphi\right) - 2v_A^2 k_{\parallel} k'_{\parallel}\frac{d\varphi}{dr} + \frac{2k_n^2 p'\varphi}{n_o m_i R_c} = 0 \tag{10}$$

3. SOLUTION OF THE EIGENVALUE EQUATION

Provided the mode is sufficiently localised around r_o, we can approximate to $k_{\parallel}$ by $(r - r_o)k'_{\parallel}(r_o)$. Equation (10) can then be written in the following non-dimensional form

$$(1 + x^2)\frac{d^2\varphi}{dx^2} + 2x\frac{d\varphi}{dx} + \left[\frac{1}{4}\chi^2 - (1 + x^2)k_n^2\alpha\right]\varphi = 0 \tag{11}$$

where $\alpha = \dfrac{\omega(k_n U_i - \omega)}{v_A^2 k'^2_{\parallel}}$, $\chi^2 = -\dfrac{32\pi p'}{r B_z^2}\left(\dfrac{\mu}{\mu'}\right)^2$

$$x = (r - r_o)/\alpha^{\frac{1}{2}} \quad , \quad \mu = \frac{B_\theta}{r B_z} \quad , \quad k'_{\parallel} = \left(\frac{B_z}{B}\right)^2 k_n r \mu' .$$

μ is the pitch of the field lines, which is related to the safety factor (q) of the analagous toroidal system by $\mu = 1/Rq$.

Equation (11) is formally identical to that solved by Kulsrud [5] in the limit $k_n^2\alpha \ll \chi^2$, when investigating the effect of FLR on the interchange instability in a stellarator. The following derivation of the eigenvalue for χ^2 is the same as Kulsrud's. This section will discuss important details of the solution, most of which are not given in reference (5). Apart from the different application, the method by which the stability condition is determined from the eigenvalue is rather different from that in reference (5). The physical implications of the assumption $k_n^2\alpha \ll \chi^2$ will be discussed in Section 4.

Two approximate solutions to equation (11) will be found, one valid when $x \ll \chi/2k_n\alpha^{\frac{1}{2}}$, and the other for $x \gg 1$. The eigenvalue is determined by matching these two solutions over their common range of validity, $1 \ll x \ll \chi/2k_n \alpha^{\frac{1}{2}}$. We first consider the small x range. The x^2 term can then be neglected in the coefficient of φ, and equation (11) becomes the Legendre equation for imaginary argument, and index ν defined by

$$\nu = -\frac{1}{2} + \frac{iu}{2} \quad , \qquad u = [\chi^2 - 1]^{\frac{1}{2}} \ . \tag{12}$$

The general solution is

$$\varphi = A\,P_\nu(ix) + B[Q_\nu(ix) \pm i(\pi/2)\ P_\nu(ix)] \ . \tag{13}$$

The + sign is taken when $x > 0$, and the - sign when $x < 0$. This term is needed in order to remove the discontinuity in $Q_\nu(ix)$ as the argument crosses the real axis, see reference 9, p.144. Because the equation and boundary conditions are even in x, we choose the solution for which $d\varphi/dx = 0$ at $x = 0$, i.e.,

$$\frac{d\varphi}{dx} = 2\pi^{\frac{1}{2}} \left[\Gamma\left(1 + \frac{\nu}{2}\right)^2\right] \left[\frac{A}{\pi} \sin\left(\frac{\pi\nu}{2}\right) + B \cos\left(\frac{\pi\nu}{2}\right)\right] = 0$$

Hence

$$\varphi = B\left[\frac{\pi}{2}\left(\pm i - \cot\frac{\pi\nu}{2}\right) P_\nu(ix) + Q_\nu(ix)\right] \tag{14}$$

When $x \gg 1$ the unity in the first term of equation (11) can be neglected, and the equation may then be transformed into a Bessel equation. The solution which vanishes at infinity is

$$\varphi(x) = Cx^{-\frac{1}{2}}\, K_{iu/2}\ (k_n \alpha^{\frac{1}{2}} x) \tag{15}$$

where K is the Bessel function of imaginary argument and second kind.

We shall now compare the asymptotic form of equation (14) with the small argument approximation to equation (15). Using the asymptotic forms of the Legendre functions in terms of gamma functions [9], and various relations between the gamma functions, the asymptotic form of equation (14) may be obtained in the form

$$\begin{aligned} \varphi &\approx \frac{B}{i\,u\,a\,x^{\frac{1}{2}}}\left(\xi - \frac{1}{\xi}\right) \\ &= \frac{2B}{u\,a\,x^{\frac{1}{2}}} \sin\left[\frac{u}{2}\,\ell n(8x) + \psi\right] \end{aligned} \tag{16}$$

$$\text{where} \quad \xi = \frac{a}{(\cosh \frac{\pi u}{2})^{\frac{1}{2}}} \cdot \frac{\Gamma(1+\frac{iu}{2})}{\Gamma(1+iu)} (8x)^{\frac{iu}{2}} \exp\left[-i \tan^{-1}\left(e^{-\frac{\pi u}{2}}\right) + \frac{i\pi}{4}\right]$$

$$\psi = 2 \arg \Gamma(1+\frac{iu}{2}) - \arg \Gamma(1+iu) - \tan^{-1}\left[\exp(-\frac{\pi u}{2})\right] + \frac{\pi}{4}$$

and $a^2 = \sinh(\pi u/2)/(\pi u/2)$. To express the first expression in sinusoidal form, one must demonstrate that $|\xi| = 1$. This follows from the relation $|\Gamma(1+iy)| = (\pi y/\sinh \pi y)^{\frac{1}{2}}$.

To obtain the small argument form of the Bessel function solution we first express $K_{iu/2}$ in terms of a Whittaker function, which in turn can be expressed in terms of the confluent hypergeometric and gamma functions. Using the small argument expansion for the hypergeometric functions leads to the following form

$$\begin{aligned} \varphi &\approx \frac{C}{i\,u\,a\,x^{\frac{1}{2}}} \left(\eta - \frac{1}{\eta}\right) \\ &= \frac{2C}{u\,a\,x^{\frac{1}{2}}} \sin\left[\frac{u}{2} \ell n\, (k_n \alpha^{\frac{1}{2}} x/2) - \arg \Gamma\left(1+\frac{iu}{2}\right)\right] \end{aligned} \tag{17}$$

$$\text{where} \quad \eta = \left(\frac{k_n \alpha^{\frac{1}{2}} x}{2}\right)^{iu/2} \frac{1}{a\Gamma(1+\frac{iu}{2})} .$$

Equations (16) and (17) must be equal over the common range of validity, $1 \ll x \ll \chi/2k_n\alpha^{\frac{1}{2}}$. This requires $B = C$ and $k_n\alpha^{\frac{1}{2}} = H_n(\chi^2)$, where

$$H_n(\chi^2) = 16 \exp\left[\frac{2}{u}\left\{3 \arg \Gamma(1+\frac{iu}{2}) - \arg \Gamma(1+iu) - \tan^{-1}[\exp(-\pi u/2)] + \pi/4 - n\pi\right\}\right] \tag{18}$$

and $u = (\chi^2 - 1)^{\frac{1}{2}}$. Different n correspond to different eigenfunctions. As shown by Kulsrud [5], the eigenfunction which first goes unstable is given by $n = 1$. That this is the appropriate value of n could also be demonstrated later in the analysis by requiring that the stability criterion shall pass into the Suydam condition as FLR effects tend to zero.

Substituting for α in the eigenvalue equation gives the following quadratic for ω as a function of χ^2.

$$\omega(k_n U_i - \omega) = \frac{v_A^2 k_{\parallel}'^2}{k_n^2} H_1^2(\chi^2) = - \frac{8p'}{rnm_i \chi^2} \left(\frac{B_\theta}{B}\right)^2 H_1^2(\chi^2) \tag{19}$$

i.e.

$$\omega = \frac{1}{2} k_n U_i \left[1 \pm \left\{1 - \left(\frac{8H_1}{\lambda\chi}\right)^2\right\}^{\frac{1}{2}}\right] \tag{20}$$

where

$$\lambda = k_n \rho_i \frac{B}{B_\theta} \left[- \frac{r p_i'}{p_i + p_e}\right]^{\frac{1}{2}}$$

The stability condition is therefore $8H_1(\chi^2) < \lambda\chi$. The function $H_1(\chi^2)$ is plotted in reference (5) and reproduced in Fig. 1. This stability condition can be expressed in the more convenient form, $\chi^2 < \chi_c^2(\lambda)$, where χ_c^2 is plotted as a function of λ in Fig. 2. Since $k_{\parallel} = 0$ at $x = 0$, $k_n = kB/B_\theta = (m/r)(B/B_z)$ provided $B_z \neq 0$. The smallest wavenumber is $m = 1$ though, as we shall see later, this mode may not satisfy the localisation assumption, i.e. $m = 1$ may occur only as a kink-mode. However, if the stability condition is satisfied for $k_n = B/rB_z$, then all localised modes are certainly stable. For the mode centred on the radius where $B_z = 0$, $k_n = k$ and the smallest permissable value is determined by the localisation requirement, discussed later.

In the MHD limit, $\lambda \to 0$, we recover the Suydam stability criterion [1], $\chi^2 < 1$. The vertical axis of Fig. 2 gives the factor by which FLR effects increase the limiting pressure gradient over the MHD prediction. We now examine the assumption $\chi^2 \gg 4k_n^2 \alpha = 4H_1^2$, necessary for overlap between the two solutions. It may be seen from Fig. 1 that this is satisfied over a reasonable range of χ^2, e.g., $(2H_1/\chi)^2 = 0.04$ for $\chi^2 = 3$ and $(2H_1/\chi)^2 \approx 0.2$ for $\chi^2 = 5$.

4. DISCUSSION OF THE SOLUTION

Since the magnetic field curvature due to toroidicity is stronger in a Tokamak than that due to rotational transform, the cylindrical model is not adequate. In pinch devices, however, the curvature due to rotational transform is generally dominant, and the cylindrical model should be applicable. For example, a typical set of parameters for the Culham HβTX experiment [10], is $B = 2kg$, $T_e = 30eV$, $T_i = 10eV$, $r = 2cm$, $rB_z/B_\Theta = 10$, and $rp'/p \approx 1$. Then for the $m = 1$ mode, $\lambda \approx 0.2$.

The solution of the eigenvalue equation for $\omega = \omega_R + i\gamma$ is shown as a function of χ^2 with $\lambda = 0.2$ in Fig. 3. The MHD eigenvalue equation may be obtained from equation (19) simply by setting $k_n U_i = 0$ on the left hand side. Its solution, which is a pure imaginary for $\chi^2 > 1$, is shown for comparison by a dashed line. The frequency and growth rates are normalised to $c_s\Theta/r$ where c_s is the sound speed and $\Theta = B_\Theta/B$. It is more natural to normalise the FLR solution to $k_n U_i$, and it may be read off the right hand scale in that form. The ratio between the two scales is given by $r\,k_n U_i/c_s\Theta = 2^{-\frac{1}{2}}\lambda$. We see that the marginal value of χ^2 is almost doubled by FLR effects. As χ^2 increases beyond the marginal value, the growth rate approaches that predicted by the MHD equations. However, the mode retains the real phase velocity, $\omega_R/k_n = U_i/2$, due to FLR effects.

The MHD stability problem is usually treated using the energy principle. This does not give the radial profile of the mode amplitude. The analysis of the preceding section allows the profile to be calculated, for both the MHD and FLR models. Comparison between predicted and observed mode profiles can provide a useful diagnostic. To calculate the profile for a specific case one first evaluates χ^2 and then finds $k_n\alpha^{\frac{1}{2}}$ from $H_1(\chi^2)$. When the Legendre functions are expressed in terms of the hypergeometric function, the solution given in equation (14) becomes

$$\varphi = \frac{\pi B}{2\cosh(\pi u/2)}\left[F\left(-\nu,\ \nu+1,\ 1,\ \frac{1-ix}{2}\right) + F\left(-\nu,\ \nu+1,\ 1,\ \frac{1+ix}{2}\right)\right] \tag{21}$$

$$\approx \frac{\pi B}{\cosh(\pi u/2)}\left[1 + \frac{\chi^2}{4}\left\{\ell n\ 2 - \frac{x^2}{2} + \chi^2\ (0.026 - 0.05x^2) + .\right\}\right] \tag{22}$$

The second form, which is valid for $x < 1$, has been obtained using the hypergeometric series. Because this series converges rather slowly, the coefficients of χ^2 and $x^2\chi^2$ have been evaluated exactly by summing an infinite series of terms. The coefficients of χ^4 and $x^2\chi^4$ are much smaller, and for these the sum of the appropriate terms including $n = 4$ is given. Higher powers of χ^2 and x^2 are negligible if $\chi^2 = O(1)$ and $x < 1$. If the solution were needed for $\chi^2 \gg 1$, these coefficients could be evaluated exactly up to any desired order.

The solution for $x \gg 1$ is given by equation (16) or equation (17), which are identical with $B = C$ and the eigenvalue for $k_n \alpha^{\frac{1}{2}}$. The solutions given by equations (16) and (22) could be joined over the range $x \gtrsim 1$ by retaining higher powers of x in the hypergeometric series. However, it is easy in practice to join the two sections by a smooth curve, as is done in Fig. 4.

When $k_n \alpha^{\frac{1}{2}} x \gg 1$, the asymptotic expression may be used for the modified Bessel function in equation (15), giving

$$\varphi(x) \approx \frac{B}{x}\left[\frac{\pi}{2k_n \alpha^{\frac{1}{2}}}\right]^{\frac{1}{2}} \exp\left[-k_n \alpha^{\frac{1}{2}} x\right]\left[1 - \frac{(1+u^2)}{8k_n \alpha^{\frac{1}{2}} x} + \frac{(1+u^2)(9+u^2)}{128k_n^{\,2} \alpha x^2} + \ldots\right] \tag{23}$$

Exact evaluation of the modified Bessel function may again be avoided by joining the curves given by equation (16) and (23) by a smooth curve over the range $k_n \alpha^{\frac{1}{2}} x = O(1)$.

As an example, Fig. 4 shows the radial profile for $\chi^2 = 2$, $k_n \alpha^{\frac{1}{2}} = 0.04$. The sine function approximation of equation (16) is shown by a dashed line. The approximate expressions of equation (22) and (23) merge very smoothly with this curve at the two ends of the overlap range.

We now examine the localisation assumption. From Fig. 4 it may be seen that the mode amplitude is typically down to about 50% of maximum when $k_n(r - r_o) \sim 0.16$, and about 10% when $k_n(r - r_o) \sim 0.6$. The half-width in this example is $|r - r_o| \approx 0.16 \; r_o B_z/mB$ if $B_z \neq 0$, or $0.16k$ if $B_z = 0$. If the change in μ' over this localisation region is comparable to $\mu'(r_o)$, this analysis is not valid. This imposes a lower limit on the m number for which a localised interchange mode can occur. For lower modes, the kink destabilising terms, neglected in this paper, will be important. This limitation equally affects the Suydam criterion for MHD stability.

The relation $\omega(\omega - k_n U_{ni}) = \omega_H^2 = -\gamma_H^2$ between the eigenvalues of the FLR and MHD equations should still apply when kink terms are important. If the ideal MHD equations have been solved numerically, the corresponding solution including FLR effects may be obtained immediately from this relation.

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Fig.1. Variation of $H_1(\chi^2)$ with χ^2.

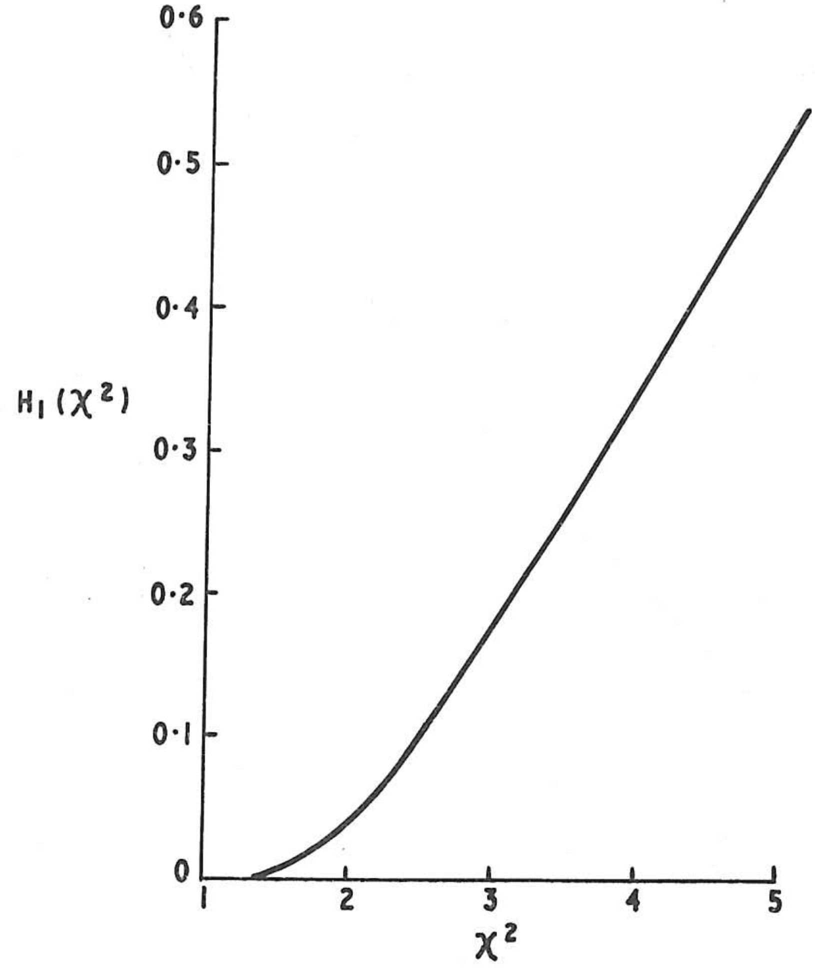

Fig.2. The value of χ^2 for marginal stability vs λ.

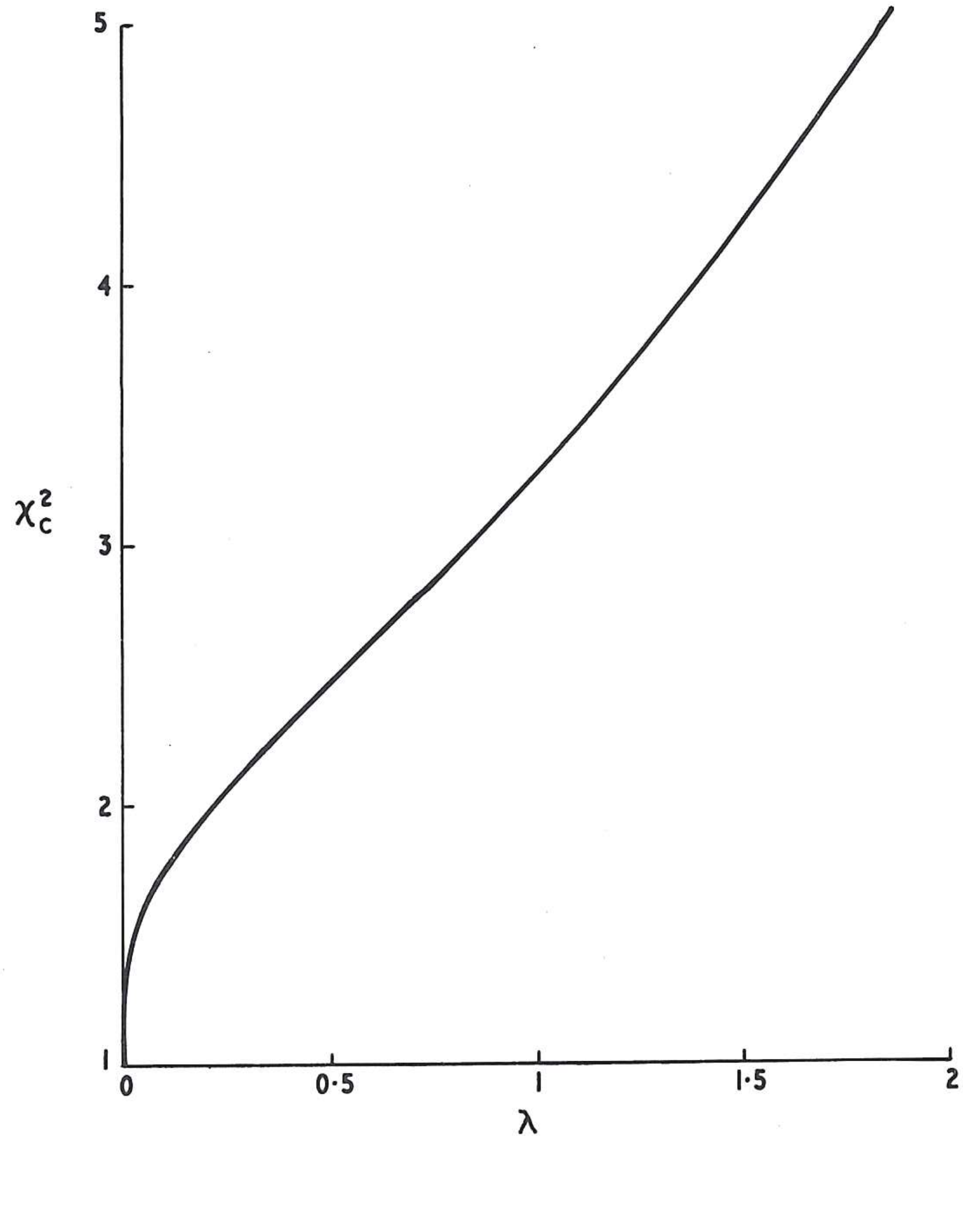

Fig.3. $\omega = \omega_R + i\gamma$ as a function of χ^2 for $\lambda = 0.2$.

γ_H is the growth rate for the MHD mode.

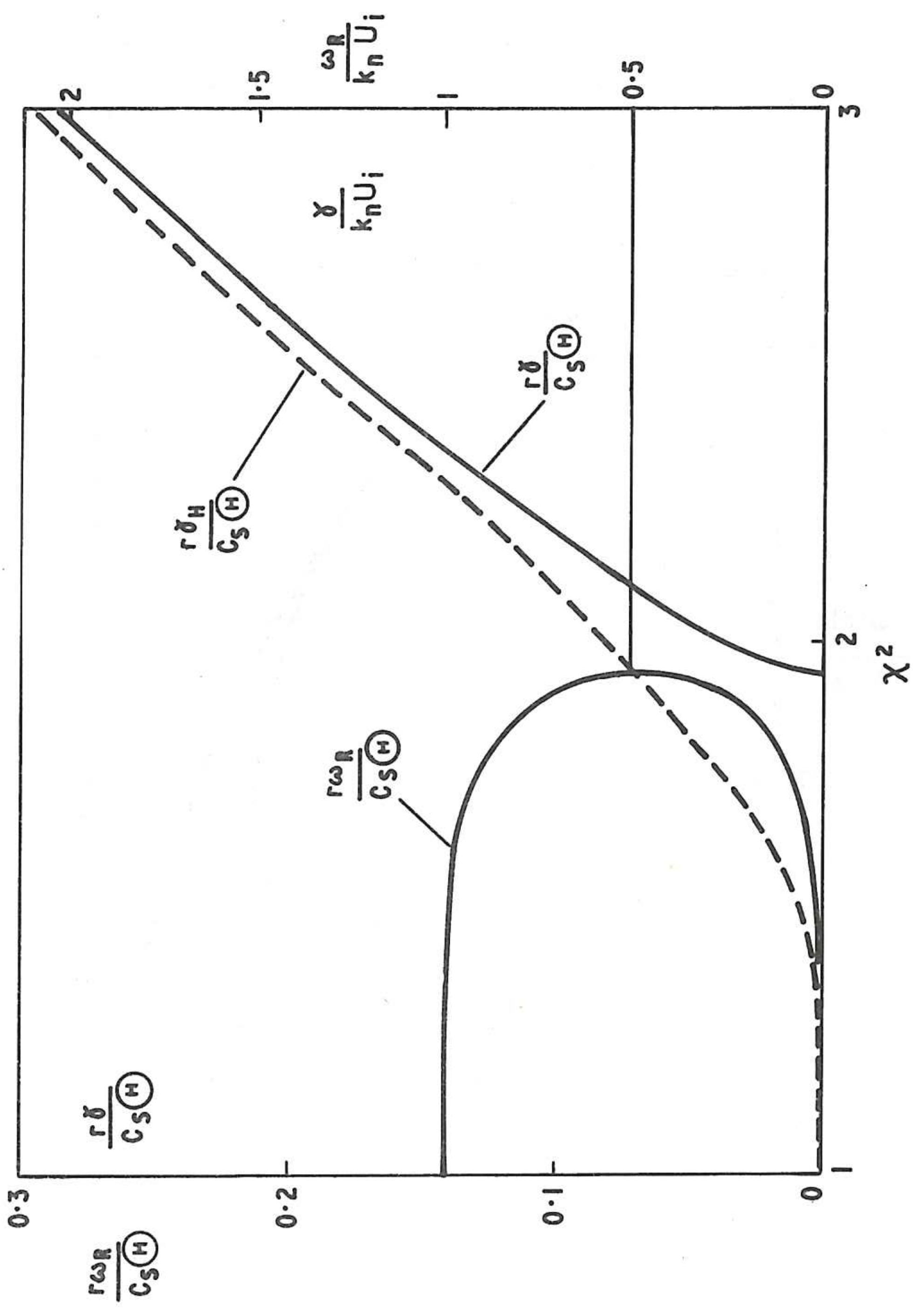

Fig.4. Radial Variation of the Mode Amplitude. $\chi^2 = 2$, $k_n \alpha^{\frac{1}{2}} = 0.04$.

The dashed curve shows the sine function approximation.