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ABSTRACT

It is argued that Dupree's diffusion theory of random acceleration in turbulent plasma is valid only in the same parameter range as is the "classical" theory.

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We wish to consider the validity of Dupree's theory of diffusion in strong turbulence [1-3]. To fix our ideas it is convenient to consider the simple case of independent particles accelerated by an electric field $\underset{\sim}{E}(\underset{\sim}{x})$ which is a stationary random function of position with rms value Eo and correlation length R. (The argument can be extended to cover more complicated circumstances.) In both the conventional Fokker-Planck theory (hereinafter called the "classical" theory) and Dupree's theory the velocity distribution function satisfies a diffusion equation:

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial}{\partial v^\alpha} \left\{ D^{\alpha\beta}(\underset{\sim}{v}) \frac{\partial f}{\partial v^\beta} \right\} \tag{1}$$

Here

$$D^{\alpha\beta}(\underset{\sim}{v}) = 2 \frac{e^2}{m^2} \int_0^\infty dt \; C^{\alpha\beta}(t) \tag{2}$$

and $C^{\alpha\beta}(t)$ is an approximation to the autocorrelation function of the electric field evaluated along the orbit.

The classical theory is valid when the deflection of a particle from its unperturbed orbit on traversing the correlation length is very small, i.e. when [4]

$$\mu \equiv \frac{eEoR}{\frac{1}{2}mv^2} \ll 1 . \tag{3}$$

In this case one may take $C^{\alpha\beta}(t)$ to be spatial autocorrelation function of $\underset{\sim}{E}(\underset{\sim}{x})$ evaluated along the unperturbed orbit:

$$C^{\alpha\beta}(t) = Q^{\alpha\beta}(\underset{\sim}{v}t) \tag{4}$$

$$= \int d\underset{\sim}{k} \; k^\alpha \, k^\beta \, h(\underset{\sim}{k}) \exp\,[i\,\underset{\sim}{k}.\underset{\sim}{v}\,t] \tag{5}$$

Here $Q^{\alpha\beta}(\underset{\sim}{x})$ is the autocorrelation function of the electric field and $h(\underset{\sim}{k})$ is the corresponding spectral function of the potential.

In Dupree's theory equations (1) and (2) remain unchanged but equation (5) is replaced by

$$C^{\alpha\beta}(t) = \int d\underset{\sim}{k}\ k^{\alpha}\ k^{\beta}\ h(\underset{\sim}{k})\ \exp\left\{ i\underset{\sim}{k}.\underset{\sim}{v}t - \frac{1}{6} D^{ij}k^{i}k^{j}t^{3} \right\} \qquad (6)$$

(Note that our definition of $D^{\alpha\beta}$ differs by a factor 2 from that of Dupree.)

The issue we are concerned with here is: for what values of μ are equations (1), (2) and (6) jointly valid?

The argument is most easily presented in terms of time scales. Let τ_c be the correlation time of the field seen by a particle. A random walk analysis gives

$$D \sim \frac{e^2}{m^2} E_o^2 \ \tau_c \qquad (7)$$

(Here $D \equiv \sum_{\alpha} D^{\alpha\alpha}$)

In the classical theory the only effect contributing to the fall-off in the field correlation is unperturbed motion through the random field. Thus in this case

$$\tau_c \sim \frac{R}{V} \ \ (\equiv \tau_0, \text{ say}) \qquad (8)$$

Combining equations (7) and (8) we obtain the classical result:

$$D \sim \frac{e^2}{m^2} E_o^2 \cdot \frac{R}{V} \qquad (9)$$

(The same result is, of course, obtained on performing the integrations in equations (2) and (5) with a numerical factor, close to unity, depending upon the precise shape of h (k).)

Let τ_R be a relaxation time obtained from equation (1):

$$\tau_R \sim \frac{V^2}{D(\underset{\sim}{v})} \tag{10}$$

If a diffusion equation is to be an appropriate description of the phenomena, the behaviour of the mean square velocity increment must satisfy

$$< \Delta v^2 > \ = \ Dt \tag{11}$$

where t is a time satisfying the following inequalities

$$\tau_c \ll t \ll \tau_R \tag{12}$$

Combining equations (7), (10) and (12), we infer that the following inequality must hold if the theory is valid

$$\left(\frac{\mu \ \tau_c}{\tau_o} \right)^2 \ll 1 \tag{13}$$

In the classical theory $\tau_c \approx \tau_o$ and inequality (13) simply reduces to inequality (3). If we are to escape from this small μ limitation, and develop a diffusion theory of strong turbulence, we must find some means of reducing the τ_c/τ_o ratio. It is clear from equation (6) that the additional decorrelating effect invoked by Dupree is the orbit modifications produced by the diffusion itself.

There are two objections one can take to this idea.

Firstly, the mean propagator only assumes the form exp $\left[-k^\alpha k^\beta D^{\alpha\beta} t^3/6 \right]$

for times, t, satisfying inequality (12), whereas we are looking for an effect on the τ_c time scale. Thus, even if one accepts the idea of the relevance of orbit diffusion, the particular way in which this effect is introduced by Dupree is quite wrong.

Secondly, if we choose to ignore the preceding objection, it is in any case most implausible to suppose that diffusive effects, which (to avoid contradiction) we must suppose develop on a time scale longer than τ_c, can produce anything other than a minor reduction in τ_c itself. It follows that inequality (13) reduces to $\mu \ll 1$ for Dupree's theory also (as indeed it does for any theory which relies solely on orbit diffusion to reduce τ_c). An immediate corollary is that the classical and Dupree diffusion coefficients cannot differ much, except outside the domain of validity of the theories.

We now present a worked example as an illustration of the above argument.

Let the potential correlation function (the Fourier transform of $h(\underset{\sim}{k})$) be

$$h(\underset{\sim}{x}) = \frac{1}{3} (E_0 R)^3 \quad \exp \; [- x^2/2R] \tag{14}$$

(This corresponds to $\langle E^2 \rangle = E_0^2$, in three dimensions.)

Without loss of generality we may choose coordinates such that $\underset{\sim}{V} = (0,0,V)$. Considerations of symmetry show that the off-diagonal elements of $D^{\alpha\beta}$ vanish and that $D^{11} = D^{22} \equiv D_\perp$, say. The element D^{33} is zero in the classical theory and may be shown to be much smaller than $D_\perp$ in Dupree's theory for $\mu \lessdot 1$. (The details of this calculation will be presented elsewhere.) The integration in equation (6) may now be performed analytically. If the resulting $C^{\alpha\beta}(t)$ is substituted in equation (2) (neglecting D^{33}) one obtains

$$\hat{D}_{\perp}(\mu) = \frac{\mu^2}{6} \int_0^{\infty} dt \quad e^{-\frac{t^2}{2}} \left[1 + \frac{1}{3} \hat{D}_{\perp} t^3\right]^{-2} \tag{15}$$

$\hat{D}_{\perp}$ (μ) is a non-dimensional diffusion coefficient, related to the true diffusion coefficient as follows

$$D_{\perp}(v) \equiv \frac{v^3}{R} \hat{D}_{\perp}(\mu) \tag{16}$$

We have solved equation (15) numerically for a range of values of μ. The results are shown in table 1, together with the corresponding classical results (obtained by setting $\hat{D}_{\perp}$ to zero in the integrand in equation (15)).

It can be seen that the Dupree and the classical coefficients differ by about 10% at $\mu = 1$. Thus for $\mu < 1$ the difference between the classical and Dupree results is very slight and this implies (see equation (7)) that no significant reduction in τ_c has been achieved. Referring to inequality (13) we again conclude that the Dupree theory is valid only if $\mu \ll 1$.

We have performed a numerical simulation of particle acceleration in a two-dimensional random potential with a correlation function which is a 2-D version of (14). The results will be presented in detail elsewhere. They are in agreement with the analysis presented here.

These considerations strongly suggest that random acceleration in strongly turbulent fields cannot be described without resort to a non-Markovian kinetic equation (such as that of Orszag and Kraichnan[5]). It appears that such equations can be "Markovianised" and reduced to a diffusion equation only if the turbulence is so weak that the conventional Fokker-Planck theory is valid.

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1. T. H. Dupree, Phys. Fluids 9, 1773 (1966).
2. T. H. Dupree, Phys. Fluids 10, 1049 (1967).
3. J. Weinstock, Phys. Fluids 12, 1045 (1969).
4. U. Frisch, in Probabilistic Methods in Applied Mathematics Vol 1 edited by A. T. Bharucha - Reid (Academic Press, New York, 1968), p. 179.
5. S. A. Orszag and R. H. Kraichnan, Phys. Fluids 10, 1720 (1967).

TABLE 1

Classical and Dupree Diffusion Coefficients. μ is the electric field strength parameter. $\hat{D}_{\perp}$(Cl.) is the classical non-dimensional diffusion coefficient. $\hat{D}_{\perp}$(Du.) is the Dupree non-dimensional diffusion coefficient.

μ	$\hat{D}_{\perp}$ (Cl.)	$\hat{D}_{\perp}$ (Du.)
10^{-3}	$2.09 \cdot 10^{-7}$	$2.09 \cdot 10^{-7}$
10^{-2}	$2.09 \cdot 10^{-5}$	$2.09 \cdot 10^{-5}$
10^{-1}	$2.09 \cdot 10^{-3}$	$2.08 \cdot 10^{-3}$
1	$2.09 \cdot 10^{-1}$	$1.84 \cdot 10^{-1}$