CLM - P 377

CLM - P 377

UKAEA RESEARCH GROUP

Preprint

PARAMETRIC EXCITATION OF ALFVÉN AND ION ACOUSTIC WAVES

C N LASHMORE-DAVIES
R S B ONG

CULHAM LABORATORY
Abingdon Berkshire

1974

CLM-P 377

PARAMETRIC EXCITATION OF ALFVÉN AND ION ACOUSTIC WAVES

C.N. Lashmore-Davies

and

R.S.B. Ong*

(Submitted for publication in Physical Review Letters)

ABSTRACT

The threshold is derived for parametric excitation of Alfvén waves in a uniform plasma in which the background magnetic field is modulated sinusoidally in time. Including the plasma pressure we show that ion acoustic waves can also be excited directly. Both cases are absolutely unstable for any interaction length. Finally the subsequent decay of the excited Alfvén wave into an ion acoustic and another Alfvén wave is analyzed.

* Mathematical Institute, Oxford University and UKAEA Culham Laboratory

† This work is partially supported by a United Kingdom SRC visiting Fellowship and by Grant AF-AFOSR-72-2224.

UKAEA Research Group,
Culham Laboratory,
Abingdon,
Berks.

February 1974

The growth rate for the parametric excitation of Alfvèn waves was calculated by Vahala and Montgomery[1]. The Alfven waves were excited by oscillating the background magnetic field. This phenomenon was later observed experimentally by Lehane and Paoloni[2]. The purpose of this letter is three-fold; first we derive the Alfvèn wave instability threshold; secondly we show that by taking the plasma pressure into account not only Alfven waves may be excited directly but also ion acoustic waves; thirdly we obtain the threshold for the subsequent decay of the finite amplitude Alfven wave so excited into an ion acoustic wave and another Alfvèn wave.

We shall follow Vahala and Montgomery[1] and use simple magneto-hydrodynamic equations with the addition of a plasma pressure term and resistivity. These equations are the following

$$\rho \frac{\partial \underline{v}}{\partial t} + \rho(\underline{v}.\underline{\nabla})\underline{v} = - \underline{\nabla}p + \frac{1}{\mu_o} (\underline{\nabla} \times \underline{B}) \times \underline{B} \qquad (1)$$

$$\frac{\partial \underline{B}}{\partial t} = \underline{\nabla}\times(\underline{v} \times \underline{B}) - \frac{\eta}{\mu_o} \underline{\nabla} \times (\underline{\nabla} \times \underline{B}) \qquad (2)$$

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}.(\rho \underline{v}) = 0 \qquad (3)$$

and we take the equation of state to be $p = c_s^2 \rho$.

The model considered by Vahala and Montgomery[1] was that of a uniform column of plasma of radius a with a uniform magnetic field pointing in the z-direction. The source of energy for the parametric excitation was then a perturbation to the background magnetic field which was approximately uniform spatially but periodic in time. This perturbation to the background magnetic field then formed the new 'oscillating equilibrium' whose stability properties were then examined. Vahala and Montgomery constructed the following equilibrium:

$$B_o = \hat{i}_z B_o (1 + \epsilon \operatorname{Re} e^{-i\omega_o t})$$

$$\rho_o = \rho_o (1 + \epsilon \operatorname{Re} e^{-i\omega_o t}) , \quad v_o = 0$$

where ω_o is the modulation frequency and ϵ is a small parameter measuring the percentage modulation of the uniform magnetic field and mean plasma density. This zero order (or oscillating equilibrium) solution is not affected by the inclusion of plasma pressure. It is also unaffected by the resistivity term provided $\nu/\omega_o \lesssim \omega_{pe}^2\, a^2/c^2$ where ν is the electron-ion collision frequency and c and ω_{pe} have their usual meaning.

Let us now calculate the threshold for the excitation of Alfvén waves by the modulation of the equilibrium state. The ordering chosen by Vahala and Montgomery allows one to consider Alfven waves which propagate only in the direction of $\underline{B}_o$ and whose wavelength is long compared with the plasma radius. For simplicity we consider waves polarized such that $\underline{B}_1 = (B_x, 0, 0)$. Now taking the x-components of equations (1) and (2) we obtain

$$\rho_o \frac{\partial v_x}{\partial t} - \frac{B_o}{\mu_o}\frac{\partial B_x}{\partial z} = -\epsilon\rho_o\, \mathrm{Re}(e^{-i\omega_o t})\frac{\partial v_x}{\partial t} + \frac{\epsilon B_o}{\mu_o}\mathrm{Re}(e^{-i\omega_o t})\frac{\partial B_x}{\partial z} \tag{4}$$

$$\frac{\partial B_x}{\partial t} - \frac{\eta}{\mu_o}\frac{\partial^2 B_x}{\partial z^2} - B_o\frac{\partial v_x}{\partial z} = \epsilon B_o\, \mathrm{Re}(e^{-i\omega_o t})\frac{\partial v_x}{\partial z} \tag{5}$$

where we have included only those terms which couple the modulation to the Alfven waves.

We now put equations (4) and (5) into the coupled mode[3,4] form using the selection rules $\omega_o \approx \omega_1 + \omega_2$ and $k_o = k_1 + k_2$ where $k_o = 0$ and (ω_1, k_1), (ω_2, k_2) are the frequency and wave number of the two Alfven waves we are investigating.

The Alfvén wave solution B_1 varies approximately as Re exp i $(k_1 z - \omega_1 t)$ where $\omega_1 = |k_1| C_A$ and C_A is the Alfven speed. The wave equation obtained from equations (4) and (5) describing the coupling of this wave to the second Alfven wave B_2 and the modulation is

$$\left(\frac{\partial}{\partial t} + \frac{k_1^2\,\eta}{2\mu_o}\right) b_1 = i\,\frac{\epsilon}{4}\,\frac{k_2^2 C_A^2}{\omega_2}\, b_2^*\, e^{-i\phi t} \tag{6}$$

where $b_1(t)$ is a slowly varying amplitude (compared with ω_o) given by $B_1(z,t) = b_1(t)\, e^{i(k_1 z - \omega_1 t)}$ and $\phi \equiv \omega_o - \omega_1 - \omega_2$. Similarly, the equation for the second Alfvén wave, obtained from equations (4) and (5) is

$$\left(\frac{\partial}{\partial t} + \frac{k_2^2 \eta}{2\mu_o} \right) b_2 = i \frac{\epsilon}{4} \frac{k_1^2 C_A^2}{\omega_1} b_1^* e^{-i\phi t} \tag{7}$$

where $b_2(t)$ is the slowly varying amplitude of wave B_2 defined as for wave B_1. (Note that $\omega_2 \equiv |k_2| C_A$ and $k_2 = -k_1$, from the matching conditions). Equations (6) and (7) yield the dispersion relation

$$(\omega - \phi + i\gamma_A)(\omega + i\gamma_A) + \left(\frac{\epsilon}{4}\right)^2 \omega_1 \omega_2 = 0 \tag{8}$$

where $\gamma_A \equiv k_1^2 \eta / 2\mu_o$. This equation has an unstable solution when the amplitude of the modulation exceeds a threshold value given by

$$\left(\frac{\epsilon}{4}\right)^2 \left(\frac{\omega_o}{2}\right)^2 = \gamma_A^2 + \left(\frac{\phi}{2}\right)^2 \tag{9}$$

where we have used the fact that $\omega_1 = \omega_2 \approx \omega_o/2$. The minimum threshold occurs for perfect matching ($\phi = 0$) and is

$$\epsilon_{min.} = \frac{\omega_o \eta}{\mu_o C_A^2} \tag{10}$$

Well above this threshold the growth rate of the two excited Alfven waves is

$$\gamma = \epsilon\, \omega_o / 8 \tag{11}$$

which is the result obtained by Vahala and Montgomery[1].

Now consider the second possibility i.e. the excitation of ion acoustic waves by the modulation of the equilibrium state. In this case, it is the modulation of the average density which couples to the acoustic waves. Again we consider waves which propagate along the uniform magnetic field. The wavelength of the excited acoustic waves depends on the value of $\beta\ (= 2\mu_o n_o \kappa T_e / B_o^2)$. The ion acoustic wave number k_s must satisfy $k_s a \sim \delta\, \beta^{-\frac{1}{2}}$ where δ is $a\omega_o / C_A$ and has been assumed to be $O(\epsilon^{\frac{1}{2}})$.

The equations required to describe this process are the z-component of equation (1) and equation (3) (NB We have added a phenomenological collision term to the z-component of equation (1) in order to simulate the damping of the ion acoustic mode). We now proceed as for the previous case using v_z as the wave amplitude and calculating the perturbation to the two acoustic waves varying as $v_{z1} \sim \mathrm{Re}\,\exp i\,(k_{s1}z - \omega_{s1}\,t)$ and $v_{z2} \sim \mathrm{Re}\,\exp i\,(k_{s2}z - \omega_{s2}t)$ where $\omega_{s1,2} \equiv |k_{s1,s}|\,c_s$. The resulting coupled mode equations are

$$\left(\frac{\partial}{\partial t} + \frac{\nu}{2}\right) V_1 = - i \frac{\epsilon}{4} \frac{\omega_o\omega_{s2}}{\omega_{s1}} V_2^* e^{-i\phi_s t} \tag{12}$$

$$\left(\frac{\partial}{\partial t} + \frac{\nu}{2}\right) V_2 = - i \frac{\epsilon}{4} \frac{\omega_o\omega_{s1}}{\omega_{s2}} V_1^* e^{-i\phi_s t} \tag{13}$$

where $V_{1,2}$ are the slowly varying amplitudes of the acoustic waves $v_{z1,2}$ defined as for the previous case and $\phi_s \equiv \omega_o - \omega_{s1} - \omega_{s2}$ where we have again used the selection rules $\omega_o \approx \omega_{s1} + \omega_{s2}$ and $0 = k_{s1} + k_{s2}$ and ν is the phenomenological collision frequency. (It is expected that the main damping mechanism of the ion acoustic waves will be ion and electron Landau damping). The analysis of equations (12) and (13) proceeds as before giving as the threshold for instability

$$\left(\frac{\epsilon}{4}\right)^2 \omega_o^2 = \gamma_s^2 + \left(\frac{\phi_s}{2}\right)^2 \tag{14}$$

where $\gamma_s \equiv \nu/2$. The minimum threshold is again for perfect matching ($\phi_s = 0$)

$$\epsilon_{min.} = \frac{4\gamma_s}{\omega_o} \tag{15}$$

The growth rate of the two ion acoustic waves (one travelling in the + z direction the other in the - z direction) well above threshold is given by

$$\gamma = \frac{\epsilon}{4}\,\omega_o \tag{16}$$

This process excites ion acoustic waves of frequency $\omega_o/2$ (the same as the Alfvén waves) but usually with a much shorter wavelength than the Alfvén waves, unless $\beta \sim O(1)$.

In other parametric instabilities where ion acoustic waves are excited only a small fraction of the pump energy goes to the acoustic wave (due to the Manley-Rowe relations). However, in the direct excitation we have described the energy from the pump flows entirely to the acoustic waves. (The competition for the pump energy would be between different parametric instabilities rather than the decay products of one instability). In view of this, the instability described above could be an important plasma heating mechanism.

We now describe the subsequent decay of the excited Alfvén waves into another Alfvén wave and an ion acoustic wave. For a low-β plasma this process produces a long wavelength low frequency ion acoustic wave, compared with the ion acoustic wave excited directly. The decay of an Alfvén wave in the manner described was first analyzed several years ago by Galeev and Oraevskii[5]. However, they gave only the initial growth rate whereas we shall calculate the threshold both for the decay instability and a purely growing (or modified decay[6]) instability which is a magnetic analog of the oscillating two stream mode[7].

We again obtain coupled mode equations for the non-linear interaction of Alfvén and ion acoustic waves giving the dispersion relation

$$(\Omega + \delta + i\gamma_2)(\Omega - \delta + i\gamma_2)(\Omega^2 - \omega_s^2 + 2i\gamma_s\Omega) + K\frac{\delta}{\omega_2} = 0 \qquad (17)$$

where $K \equiv k_1|k_2|k_s^2\, C_A^2|b_1|^2/4\rho_o\mu_o$, $\gamma_2 \equiv k_2^2\,\eta/2\mu_o$, $\delta \equiv \omega_1 - \omega_2$ and $\Omega \equiv \omega - \delta$ and where we have taken the usual selection rules. In order to satisfy these relations for Alfvén and ion acoustic waves we have taken $k_1 > 0$, $k_s > 0$, $k_2 < 0$ and $\omega_1, \omega_2, \omega_s$ are all positive. The amplitude of the Alfvén wave (a standing wave) excited by the modulation of the equilibrium state is denoted by b_1. We have therefore taken our pump wave b_1 to be

$$b_1(z,t) = \mathrm{Re}\left\{b_1 e^{i(k_1 z - \omega_1 t)} + b_1 e^{i(k_1 z + \omega_1 t)}\right\}$$

ω_2, k_2 are the frequency and wave number of the excited Alfvén wave and γ_2 its damping factor. Equation (17) is exactly the form of Nishikawa's model dispersion relation[7]. Using his result we can immediately write down the minimum thresholds for the purely growing and decay instabilities respectively

$$K_m = 2\omega_s^2 \omega_2 \gamma_2 \tag{18}$$

and

$$K_m = 4\, \omega_s \omega_2 \gamma_s \gamma_2 \left[1 - \frac{\gamma_s^2}{4\omega_s^2} \right] \tag{19}$$

where equation (19) is only valid for $\gamma_2 \ll \omega_s$. The threshold for the decay mode is lower than that for the purely growing or modified decay instability.

The threshold for the decay of the excited Alfven wave can be written in terms of $|b_1|/B_o$. If we compare the threshold value of this quantity with the minimum threshold for the initial excitation of the standing Alfvén wave we obtain $|b_1|/\epsilon\, B_o \sim (c_s\, \gamma_s / C_A \gamma_2)^{\frac{1}{2}}$. If $\beta \ll \gamma_2/\gamma_s$ then the excited Alfven wave will decay at a much lower value of $|b_1|/B_o$ than the initial amplitude of modulation. In this case the decay of the Alfven wave into another Alfven wave and an ion acoustic wave will be an important saturation mechanism for the original Alfven instability. Comparing the threshold values of ϵ required for the excitation of Alfven waves and the direct excitation of ion acoustic waves equations (10) and (15) show that the Alfven threshold is much lower. However, if ϵ is well above both thresholds then we can see from equations (11) and (16 that the ion acoustic growth rate is double that of the Alfven instability.

We have also considered the spatial dependence of these two instabilities. In both cases the two excited waves travel in opposite directions and both instabilities are absolute. Also, in both cases, the pair of excited waves have equal and opposite group velocities. Following the analysis of Kroll[8] it turns out that both cases are absolutely unstable for any value of the interaction length L i.e. there is no critical length! (NB This result is only true for a uniform plasma).

Summarizing the main results of this letter we have shown that by modulating the background magnetic field sinusoidally in time both Alfvén and ion acoustic waves can be excited. We have calculated the threshold values for the percentage modulation of the magnetic field for these two cases. For a low-β plasma the ion acoustic waves excited will be very short wavelength compared with the Alfvén waves. In addition, the ion acoustic instability may be an effective method for heating a plasma since the pump energy flows entirely to the acoustic waves. Provided $T_e \gg T_i$ in the experiment of Lehane and Paoloni[2] the acoustic waves should have been produced in their experiment. Both the Alfvén instability and the ion acoustic instability have been shown to be absolutely unstable. In both cases there was no critical length. Finally we have calculated the threshold value of the amplitude of the Alfvén wave excited for further decay into another Alfvén wave and an ion acoustic wave. The ion acoustic wave excited in this decay has a wavelength comparable to the Alfvén wave.

An extension of this work to two dimensions will be forthcoming. This will be more readily comparable with experiment and may also be of relevance to experiments on transit time magnetic pumping.

REFERENCES

[1] G. Vahala and D. Montgomery, Phys. Fluids 14, 1137 (1971).

[2] J.A. Lehane and F.J. Paoloni, Plasma Physics 14, 461 (1972).

[3] W.H. Louisell, Coupled Mode and Parametric Electronics (John Wiley, N.Y. 1960).

[4] C.N. Lashmore-Davies, submitted for publication to J. of Plasma Physics.

[5] A.A. Galeev and V.N. Oraevskii, Doklady Akademii Nauk SSR 147, 71 (1962) [Sov. Phys. Dokl. 7, 988 (1963)] .

[6] A.A. Galeev and R.Z. Sagdeev, Nuc. Fusion 13, 603 (1973).

[7] K. Nishikawa, J. Phys. Soc. Japan 24, 916 (1968).

[8] N.M. Kroll, J. Appl. Phys. 36, 34 (1965).