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ABSTRACT

Starting from a cylindrical Tokamak equilibrium with a peaked parabolic current profile, and considering helical perturbations, a class of non-linear neighbouring equilibria is shown to exist corresponding to perturbations of the poloidal magnetic field only, which are localised in the current channel. The main contribution to the pertubed flux is independent from the helical coordinate and, when the value q_o of the safety factor on the axis decreases, it is associated with a negative change of the poloidal flux. A situation is considered in which q_o is time dependent and decreases as a consequence of the radial shrinking of the current channel; moreover the conducting shell (if it exists) surrounding the plasma, is slit in order that the poloidal flux can adjust to the new neighbouring equilibrium. In such a situation the sudden change which occurs in the poloidal flux when q_o crosses the bifurcation points of the nonlinear equilibria gives rise to voltage spikes (independent from the detailed shape of the peaked current profile), whose sign and magnitude are consistent with observations in Tokamaks when $\dot{q}_o(t) < 0$. It is shown that, in the case of a slightly peaked current density, the voltage spikes correspond to a relaxation of the current towards a flat profile. However, if the current density is strongly peaked, the new nonlinear equilibrium corresponds, rather than to a relaxation, to a collapse of the current distribution accompanied by a strong decrease of the poloidal field, so that the initial equilibrium is disrupted.

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1. INTRODUCTION

As is known, a typical effect observed in the present Tokamak experiments [1, 2, 3] is the so-called disruptive instability which manifests itself with negative spikes on the voltage driving the induced axial current, accompanied by sudden inward shifts of the major radius and relaxation [see Ref. 2] of the current density to a flatter profile.

While the small amplitude oscillations which precede the disruptive events can be explained in terms of neighbouring nonlinear kink equilibria or of nonlinear limits on tearing mode growth [4, 5, 6] in the frame of an equilibrium model with a homogeneous [7] or a rounded current profile [8], the disruptive process still escapes a theoretical understanding [9].

In the present paper we construct a class of nonlinear neighbouring equilibria whose existence is related to the fact that the initial unperturbed equilibrium has a current profile which is not homogeneous but peaked. In our model the perturbations of the equilibrium are localized inside the current channel, while the singular surface, where $\vec{k}.\vec{B}_o = 0$ $\{\vec{k} \equiv (0, \frac{m}{r}, k)\}$ is the wavenumber of the perturbation and $\vec{B}_o$ is the unperturbed magnetic field) is outside this channel. In section 2a the nonlinear equation describing the neighbouring equilibria is formulated in cylindrical geometry; in order to proceed analytically a plane approximation in the form of a sheet pinch is introduced in the subsection (b) and the space dependence of the coefficients is neglected. After these approximations the solution of the nonlinear problem and the bifurcation analysis can be performed exactly, in terms of the Weierstrass function, as shown in subsection 2c. The relevance of the solution so obtained for the description of the disruptive process is discussed in section 3. Here the change in time of the poloidal flux, associated with the radial shrinking of the current and the decrease of the safety factor is calculated together with the magnitude and the sign of the voltage induced in the z direction. One obtains negative voltage spikes when the safety factor crosses the bifurcation points, whose location depends on the form of the current profile. At the same time, for slightly peaked currents, there is a sudden flattening of the profile, while, if the equilibrium current is strongly peaked one has, on a longer time scale, a collapse of the initial current distribution.

Finally in section 4 the disruptive process is considered briefly from the more general point of view of the thermodynamics of the collisionless plasma and the relationship is noted between the neighbouring equilibria discussed here and those which can be derived with a thermodynamic method.

2. NONLINEAR NEIGHBOURING EQUILIBRIA

a. Formulation in cylindrical geometry

For the calculation of the helical equilibria neighbouring to a Tokamak cylindrical configuration, we start from the equation for the magnetic helical flux χ considered earlier by various authors([9] and [4]):

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \chi}{\partial r} + \frac{m^2}{r^2}\frac{\partial^2 \chi}{\partial \tau^2} = 2\,kB_z + mj(\chi) \ . \tag{2.1}$$

Here τ is the helical coordinate $\tau = m\theta + kz$; χ is the magnetic flux through a helical ribbon in the direction $\vec{k} \equiv (0, \frac{m}{r}, k)$ and is defined by the relation

$$r\,B_r = -\frac{\partial \chi}{\partial \tau} , \tag{2.2a}$$

$$m\,B_\theta + kr\,B_z = \frac{\partial \chi}{\partial r} \ . \tag{2.2b}$$

The longitudinal magnetic field B_z is considered unperturbed and constant while the poloidal field B_θ includes a possible perturbation. At equilibrium the axial current $j(\chi)$ (a factor 4π was included in j) depends on space through the flux χ only. In the following we will look for the helical equilibria which are neighbouring to an initial equilibrium specified by a given function $\chi_o(r)$, satisfying the unperturbed equation

$$\frac{1}{r}\frac{d}{dr} r \frac{d}{dr}\chi_o = 2\,kB_z + mj(\chi_o) , \tag{2.3}$$

where $j(\chi_o)$ is a known function of χ_o depending on the current profile of the unperturbed equilibrium. The flux χ can then be split in the form $\chi = \chi_o(r) + \chi_1(r, \tau)$, with χ_1 satisfying the equation

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\chi_1 + \frac{m^2}{r^2}\frac{\partial^2 \chi_1}{\partial \tau^2} = m\,[j(\chi_o+\chi_1) - j(\chi_o)] + mj_1(r,\tau) \ . \tag{2.4}$$

Here $j_1(r,\tau)$ is the fundamental perturbation of the equilibrium current, namely it describes the departure of the current from the <u>given</u> functional dependence $j = j(\chi_o)$ which specifies the unperturbed equilibrium. It will be assumed that $|\chi_1| \ll |\chi_o|$. Then we expand $j(\chi_o + \chi_1)$ in MacLaurin's series and, up to third order in χ_1 , the equation above takes the form:

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} \chi_1 + \frac{m^2}{r^2}\frac{\partial^2\chi_1}{\partial\tau^2} = m\left(j'\chi_1 + \frac{j''}{2!}\chi_1^2 + \frac{j'''}{3!}\chi_1^3 \right) + mj_1 , \quad (2.5)$$

where the primes indicate differentiation of $j(\chi_o)$ with respect to χ_o. Now we assume that the effect of the fundamental perturbation $j_1(r,\tau)$ on the flux χ_1 is of secondary importance with respect to the effects resulting from the term $j(\chi_o + \chi_1) - j(\chi_o)$. This term describes that part of the current perturbation which preserves the form of the dependence between j and χ_o as is defined in the unperturbed equilibrium. Correspondingly we introduce a further splitting in χ_1, putting $\chi_1 = \chi_{10}(r) + \chi_{11}(r,\tau)$, where χ_{10} does not contain, by definition, any effect related to j_1, so that it satisfies the equation

$$\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} \chi_{10} = m[j(\chi_o + \chi_{10}) - j(\chi_o)] = m\left(j'\chi_{10} + \frac{j''}{2}\chi_{10}^2 + \frac{j'''}{6}\chi_{10}^3 \right) . \quad (2.6)$$

Then, remembering Eq. (2.5), the equation for χ_{11}, which comprises the contribution of the fundamental perturbation $j_1(r,\tau)$, takes the form

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} \chi_{11} + \frac{m^2}{r^2}\frac{\partial^2\chi_{11}}{\partial\tau^2} = m\left[j' + \frac{j''}{2}(2\chi_{10} + \chi_{11}) + \frac{j'''}{6}(3\chi_{10}^2 + 3\chi_{11}\chi_{10} + \chi_{11}^2) \right] \chi_{11} + mj_1 . \quad (2.7)$$

In the limit of an homogeneous current profile the termsinvolving the χ_o - derivatives of j vanish. Eq. (2.6) is then satisfied by $\chi_{10} = 0$ (this would be the only acceptable solution in view of the boundary conditions for χ_{10} considered below) and only Eq. (2.7) for χ_{11} remains. A solution of this equation for an homogeneous current profile (when only the term $m\, j_1$ exists in the r.h.s.) was found in Ref. 4.

The present paper originates from the feeling that the part χ_{10} of χ_1 could be more important than χ_{11}, even in the case of a slightly inhomogeneous current profile. So we concentrate our attention only on the calculation of χ_{10}, which is determined by the nonlinear equation (2.6) and by the boundary conditions. In order to specify the latter we consider an initial equilibrium with a current profile $j(\chi_o) = \tilde{j}(r)$ which extends to $r = a$, while for $r > a$ we take $j = 0$. The plasma has an infinite conductivity and also exists in the region $r > a$ outside the current channel. However we will consider only perturbations which are localised in the current channel so that $\chi_{10} = 0$ for $r > a$. Let us introduce a function $\xi(r)$ describing the radial displacement of the constant χ surfaces, defined by the equality

$$\chi_o(r_o) = \chi_o(r_o + \xi) + \chi_{10}(r_o + \xi) . \quad (2.8)$$

At first order in ξ one has $\chi_{10} = -(d\chi_0 | dr)\xi$ and from the continuity of $\xi(r)$ across the surface of the current channel one obtains the condition

$$\chi_{10}(a) = 0 \ . \tag{2.9}$$

A second limiting condition is obtained by noting that χ_{10} is related to the poloidal field perturbation $B_{1\theta}$ by the relation

$$\frac{d\chi_{10}}{dr} = m \ B_{1\theta} \tag{2.10}$$

which is a consequence of Eq. (2.2b), remembering that B_z is not perturbed. It follows from Eq. (2.10) that

$$\left(\frac{d\chi_{10}}{dr} \right)_{r=0} = 0 \ . \tag{2.11}$$

We assume further that the singular surface where $\vec{k}.\vec{B}_o = 0$, falls outside the current channel. It follows that the coefficients of Eq. (2.6) are regular in the whole region $0 \leqslant r \leqslant a$ of our interest (see Eqs. 2.12). The solution χ_{10} is then completely determined by Eq. (2.6) and by the boundary conditions (2.9) and (2.11). The derivative of χ_{10} can be discontinuous at $r = a$, implying a discontinuity of $B_{1\theta}$ at the surface of the current channel. Moreover χ_{10} can be different from zero at $r = 0$, a property which will have important physical implications.

b. The plane approximation

In order to proceed analytically we use a simplified mathematical model in plane geometry. For the equilibrium current, in the z direction, we assume the following parabolic form:

$$j = j_o(1 - \gamma x^2) \qquad \text{for} \quad x < a \ ,$$

$$j = 0 \qquad \text{for} \quad x > a \ .$$

The parameter γ satisfies the limitation $\gamma a^2 \leqslant 1$ and will be taken as positive. So the case $\gamma a^2 \ll 1$ represents a slightly peaked current profile while for $\gamma a^2 \to 1$ the profile becomes strongly peaked. The field created by the current in the y direction simulates the poloidal field $B_{o\theta}$ at equilibrium. The unperturbed flux $\chi_o(x)$ (x now replaces the radial coordinate r) is expressed by the relation

$$\chi_o(x) = \frac{m \ j_o \ x^2}{2} \left(1 - \frac{\gamma x^2}{6} \right) + k \ B_z \ x^2$$

apart from an arbitrary constant. Solving the equation above with respect to x^2 (choosing the solution such that $x \to 0$ for $\chi_o \to 0$) the current profile can be expressed as a function of χ_o as follows

$$j(\chi_o) = j_o(1 - \gamma\, x^2(\chi_o)) = j_o + \frac{6}{m}\left(- \frac{mj_o}{2} - kB_z + \Delta^{\frac{1}{2}} \right)$$

with

$$\Delta \equiv \left(\frac{mj_o}{2} + kB_z \right)^2 - \frac{m\gamma j_o \chi_o}{3} = B^2_{o\theta}\, \frac{1}{x^2}(m - q(x))^2 ,$$

$$q(x) = q_o \left(1 - \frac{\gamma\, x^2}{3} \right)^{-1} .$$

Here q_o is the safety factor at $x = 0$ and is expressed in plane as well as in cylindrical geometry by $q_o = 2\, B_z / j_o\, R$ (henceforth $k = -1/R$). The χ_o derivatives of $j(\chi_o)$ are then expressed by the relation

$$\begin{aligned} j' &= -\, 2\gamma \left(m - q_o - \frac{m\gamma x^2}{3} \right)^{-1} , \\ j'' &= -\, \frac{4}{3j_o}\, \gamma^2 m \left(m - q_o - \frac{m\,\gamma x^2}{3} \right)^{-3} , \\ j''' &= -\, \frac{8}{3j_o^{\,2}}\, \gamma^3 m^2 \left(m - q_o - \frac{m\,\gamma x^2}{3} \right)^{-5} . \end{aligned} \qquad (2.12)$$

These expressions should be introduced into the equation for χ_{10}, which in plane geometry, instead of Eq. (2.6), takes the form

$$\frac{1}{m}\, \frac{d^2\chi_{10}}{dx^2} = j'\chi_{10} + \frac{j''}{2}\, \chi_{10} + \frac{j'''}{6}\, \chi_{10}^3 . \qquad (2.13)$$

If the singular surface $q(x) = m$ is far away from the surface of the current channel, namely if $m - q_o \gg \gamma m x^2/3$, the x dependence in the coefficients of Eq. (2.13) can be neglected. We shall see later that neighbouring nonlinear equilibria of practical interest exist for values $m - q_o$ which are $24/\pi^2$ times larger than $\gamma m a^2/3$. Although the coefficients (2.12) cannot be considered as strictly x independent, a first insight into the form of the solution of Eq. (2.6) can be gained assuming the homogeneity of its coefficients. In fact the rigorous treatment of Eq.(2.6) (in cylindrical geometry) by means of the computer shows that the inhomogeneity does not strongly affect the general structure of the solution, although it may somewhat affect the numerical results.

After this approximation Eq. (2.13) can be solved exactly in terms of the elliptic Weierstrass function [11]. Indeed from Eq. (2.13) follows the first integral

$$\frac{1}{m}\left(\frac{d\chi_{10}}{dx}\right)^2 = j'\chi_{10}^2 + \frac{j''}{3}\chi_{10}^3 + \frac{j'''}{12}\chi_{10}^4 + C . \qquad (2.14)$$

Passing to the dimensionless variables

$$y \equiv \frac{j''}{j'}\chi_{10} = \frac{2m}{3j_o}\gamma(m - q_o)^{-2}\chi_{10} , \quad u \equiv x(m|j'|)^{\frac{1}{2}} = x\left(\frac{2m\gamma}{m - q_o}\right)^{\frac{1}{2}} , \qquad (2.15)$$

Eq. (2.14) takes the form

$$\left(\frac{dy}{du}\right)^2 = -\frac{1}{4}y^4 - \frac{1}{3}y^3 - y^2 + \eta , \qquad (2.16)$$

where

$$\eta = \frac{Cj''^2}{m|j'|^3} = \frac{2}{9}C\frac{\gamma m}{(m - q_o)^3} > 0 . \qquad (2.17)$$

Clearly, as a consequence of our homogeneous approximation for the coefficients, one can take for q_o any value of $q(x)$ in the interval $0 \leqslant x < a$. Let y_o be one of the two zeros adjacent to the origin $y = 0$ of the algebraic equation

$$F(y_o) \equiv -\frac{1}{4}y_o^4 - \frac{1}{3}y_o^3 - y_o^2 + \eta = 0 . \qquad (2.18)$$

Then the solution of Eq. (2.16) satisfying the condition (2.11), namely $dy/du = 0$ at $u = x = 0$, is given by the expression [see Ref. 12, p. 453]:

$$y(u) - y_o = -\frac{1}{4}(y_o^3 + y_o^2 + 2y_o)\left\{\wp(u, g_2, g_3) + \frac{1}{8}y_o^2 + \frac{y_o}{12} + \frac{1}{12}\right\}^{-1} , \qquad (2.19)$$

where $\wp(u, g_2, g_3)$ is the Weierstrass function and the invariants g_2 and g_3 are given by the equalities [12]:

$$g_2 = 12^{-1}(1 - 3\eta), \quad g_3 = 6^{-3}(1 + \frac{15}{2}\eta) . \qquad (2.20)$$

The root y_o of Eq. (2.18) depends on the parameter η and then on the integration constant C. This constant is determined by the remaining limiting condition $y(u_a) = \chi_{10}(a) = 0$ (Eq. 2.9), which implies, from Eq. (2.19)

$$1 = \frac{1}{4}(y_o^2 + y_o + 2)\left\{\wp(u_a, g_2, g_3) + \frac{1}{8}y_o^2 + \frac{y_o}{12} + \frac{1}{12}\right\}^{-1} , \qquad (2.21)$$

or

$$\wp(u_a, g_1, g_2) - \frac{5}{12} = \frac{y_o^2}{8} + \frac{y_o}{6} . \qquad (2.22)$$

After solving Eq. (2.18), y_o is known as function of η, and the equation above determines η as function of the parameter

$$u_a \equiv a\left(\frac{2m\gamma}{m - q_o}\right)^{\frac{1}{2}} . \qquad (2.23)$$

As a consequence of the nonlinearity, the period of $\wp(u)$ and then of $y(u)$ depends, as known, on the amplitude C (or on the parameter η). It is just this circumstance which allows to determine the wavelength of $y(u)$ in such a way that the solution can be fitted in the interval $0 \leqslant x \leqslant a$, while satisfying the boundary conditions (2.9) and (2.11) for arbitrary values of the physical parameters q_o, γ and m.

c. Discussion of the solution

For $\eta \ll 1$ the function $\wp(u, g_2, g_3)$ has period 2π and can be expressed in terms of trigonometric functions as follows [13]

$$\wp(u, 12^{-1}, 6^{-3}) = -\frac{1}{12} + \frac{1}{4}\sin^{-2}\left(\frac{u}{2}\right) \qquad (2.24)$$

so that the solution (2.19) takes the form

$$y(u) = y_o\left(1 - \frac{\wp(u_a) + \frac{1}{8}y_o^2 + \frac{1}{12}y_o + \frac{1}{12}}{\wp(u) + \frac{1}{8}y_o^2 + \frac{1}{12}y_o + \frac{1}{12}}\right) = y_o\left(1 - \frac{\sin^{-2}\left(\frac{u_a}{2}\right) + \frac{1}{2}y_o^2 + \frac{1}{3}y_o}{\sin^{-2}\left(\frac{u}{2}\right) + \frac{1}{2}y_o^2 + \frac{1}{3}y_o}\right) .$$

(2.25)

The behaviour of $y(u)$ is sketched in fig. 1. Eq. (2.25) is a good approximation of the solution also for $\eta \neq 0$ because the dependence of $\wp(u, g_2, g_3)$ on η is very slow and can be neglected even when η and y_o are rather larger than unity. Eq. (2.22) then yields at once $y_o \equiv y(u = 0)$ as function of $u_a(q_o, m)$:

$$y_o(q_o, m) = \frac{1}{2}\left[-\frac{4}{3} \pm \left(-\frac{104}{9} + 32\,\wp(u_a, 12^{-1}, 6^{-3})\right)^{\frac{1}{2}}\right] . \qquad (2.26)$$

This expression is of immediate physical significance because $y_o(q_o, m)$ is related to the change ψ_p occurring in the poloidal flux when the neighbouring equilibrium is realized in the system. Indeed, remembering Eq. (2.10) one has, after integration across the current channel

$$\psi_p \equiv \int_0^a B_{1\theta}\, dx = -\frac{1}{m}\chi_{10}(0) = -\frac{3}{2}\frac{j_o}{\gamma}\left(\frac{m-q_o}{m}\right)^2 y_o(q_o, m) . \qquad (2.27)$$

Starting from the initial unperturbed equilibrium with $y_o = 0$, the neighbouring equilibrium arises when, for a variation of the physical parameters and in particular of the safety factor q_o (the change of q_o actually occurs in the Tokamak experiments where q_o is time dependent as a consequence of the shrinking of the current channel) the function $y_o(q_o, m)$ becomes different from zero. We then look closely at the properties of the function $y_o(q_o, m)$. Since one should start from the value $y_o = 0$, that branch of the square root should be chosen in Eq. (2.26) which admits a zero of $y_o(q_o, m)$, namely the branch corresponding to the positive sign. The function $y_o(q_o, m)$ has then a zero for $\wp(u_a, 12^{-1}, 6^{-3}) = \frac{5}{12}$ or $u_a = \frac{\pi}{2}$. Remembering Eq. (2.23) one has

$$q_o = m\left(1 - \frac{2\gamma a^2}{u_a^2}\right) \qquad (2.28)$$

and for $u_a = \frac{\pi}{2}$ one obtains a critical value of q_o

$$q_{oc} = m\left(1 - \frac{8\gamma a^2}{\pi^2}\right) \qquad (2.29)$$

which represents a bifurcation point for the equilibrium, This is a point where a new neighbouring equilibrium is generated by a variation of q_o from q_{oc}, so that $y_o(q_o, m)$, which is taken to be zero for $q > q_{oc}$, becomes different from zero in a continuous way (but with discontinuous derivative) from $y_o(q_{oc}, m) = 0$. The location of q_{oc} depends on γa^2 and one has from Eq. (2.29) $q_{oc} < m$; for $\gamma a^2 \ll 1$ q_{oc} is very near to $q_o = m$, but for $\gamma a^2 \to 1$(peaked current) one has $q_{oc} \to \sim 0.2m$.

As seen from Eq. (2.22) a real solution for y_o only exists for $\wp(u_a) > 13/36$ or $u_a < \sim 1{\cdot}6$ and is negative for $\frac{\pi}{2} < u_a$ or $q_o > q_{oc}$. When u_a and q_o decrease below the bifurcation point, y_o increases from zero and is positive, thus corresponding to a negative change ψ_p of the poloidal flux (see Eq. (2.27), recalling that j_o has the same sign as $B_{o\theta}$). Fig.2 shows the behaviour of $y_o(q_o)$ obtained by solving the basic equation (2.6) with the aid of the computer (taking into account the full inhomogeneous and non linear term $j(\chi_o - \chi_{01}) - j(\chi_o)$).

For the forthcoming discussion of the physical implications of the above solution it will be of interest to calculate the derivative dy_o/dq_o. Using Eq. (2.24), one finds for values of q_o in the neighborhood of q_{oc} :

$$\frac{dy_o}{dq_o} \approx - \frac{3}{m\gamma a^2} \quad . \tag{2.30}$$

It is noted that when the current profile is slightly peaked ($\gamma a^2 \ll 1$) the derivative above is very high, so that y_o becomes a sensitive function of q_o.

We observe that a positive solution y_o of $F(y_o) = 0$ (Eq. (2.18)), corresponding to a negative poloidal flux always exists for all values of η and is unique. This solution increases to ∞ with increasing η or decreasing q_o.

A positive solution of $F(y_o) = 0$ also exists and is unique and unbounded for $\eta \to \infty$ when higher order terms are taken into account in the MacLaurin expansion for $j(\chi_o + \chi_{01})$ because, as can be verified, they contribute to $F(y_o)$ with terms of degree higher than the fourth, all with the same negative sign. Then one should not expect that the higher order terms will change essentially the structure of the solution. This is confirmed by the rigorous numerical solution of the basic Eq.(2.6). In fact the solution retains its general form near the bifurcation point also when the term y_o^4 of $F(y_o)$ is neglected.

Finally we observe that in the case of an initial equilibrium with a skin current distribution ($\gamma < 0$) it can be verified that the neighboring nonlinear equilibria studied above do not exist.

3. THE NEGATIVE VOLTAGE SPIKES AND THE FLATTENING OR COLLAPSE OF THE CURRENT PROFILE

We now consider a situation in which, as a consequence of external factors, the current channel is shrinking in time, so that the equilibrium parameters are time dependent. In particular, if the total current remains constant, as in the Tokamak experiments, the safety factor q_o decreases with time.

We also assume that the conducting shell (if it exists) surrounding the plasma column is slit in order to admit poloidal flux, so that the poloidal magnetic field can adjust as required by the equilibrium configuration. The poloidal flux (2.27) which is zero for $q_o(t) > q_{oc}$ becomes suddenly time dependent at $q_o = q_{oc}$. An axial electric field is then induced in the plasma and is given at $x = a$ by the expression

$$E_z = \frac{d\psi_p}{dt} . \tag{3.1}$$

It is seen from Eqs. (2.27), (2.30) and (3.1) that at the bifurcation point, where $(q_{oc} - m)/m = (8/\pi^2)\gamma a^2$, a voltage spike occurs with magnitude

$$V = 2\pi RE_z = \frac{9.64}{\pi^3} R a^2 j_o \frac{1}{m} \frac{dq_o}{dt} . \tag{3.2}$$

Since $dq_o/dt < 0$, the voltage spike is opposite to the driving voltage of the axial current j_o. Experimentally one has $dq_o/dt \approx -10^3 \text{ sec}^{-1}$ and with a total current $I \equiv a^2 \pi j_o/4\pi = 50$ kA one finds easily a negative voltage spike of the order of 10^2 Volts. It should be noted that this result is independent from γ and is then insensitive to the detailed current profile.

When $q_o(t)$ decreases below the bifurcation point, the system assumes the configuration of the neighbouring equilibrium which is associated with the increasing positive $y_o(q_o, m)$ and a lower poloidal field and magnetic energy. The change in the poloidal field, averaged in space, can be calculated from Eq. (2.27). In the neighbourhood of the bifurcation point, ($q_o \approx q_{oc}$ and $y_o \ll 1$) this equation gives

$$\overline{B}_{1\theta} \equiv \frac{1}{a}\int_0^a B_{1\theta}\, dx \approx -\frac{3\cdot 32}{\pi^4} j_o \gamma a^3 y_o \approx - B_{o\theta} \gamma a^2 y_o . \tag{3.3}$$

The modification of the current profile associated with the neighbouring equilibrium is also easily calculated in terms of the solution y(u) :

$$j(\chi_o + \chi_{10}) - j(\chi_o) = -3\frac{m - q_o}{m} j_o y\left(1 + \frac{1}{2}y + \frac{1}{2}y^2\right) \approx -\frac{24}{\pi^2} j_o \gamma a^2 y\left(1 + \frac{1}{2}y + \frac{1}{2}y^2\right) . \tag{3.4}$$

A quantity which is of interest to us is the difference Δj between the values of the current on the axis $x = 0$ and near the edge $x = a$ of the current channel. One has

$$\Delta j = j_o \gamma a^2 - \frac{24}{\pi^2} j_o \gamma a^2 y_o\left(1 + \frac{1}{2} y_o + \frac{1}{2} y_o^2\right) . \tag{3.5}$$

We first consider the case when $\gamma a^2 \ll 1$, so that the initial equilibrium is slightly peaked at the centre. One sees from Eq. (3.5) that when y_o increases the current profile becomes more flat and the original peak at the centre is completely removed for $y_o \approx 0.35$ when $\Delta j \approx 0$. For higher values of y_o the value of the current at the centre is further decreased. However, on the one hand, the validity of the approximation based on the McLaurin expansion for $j(\chi_0 + \chi_{10})$ (which results in a series of powers of y) becomes doubtful, in this case. On the other hand, the external circuit, which is highly inductive, tends to keep constant the level of the total current and then to restore the original unperturbed equilibrium with $y = 0$. It is conceivable that if the modification of the current profile related to the neighbouring equilibrium is not too strong, namely if $\gamma a^2 \ll 1$, the original equilibrium can be re-established after the spike, however with a flatter current profile and a value of $q_o(t)$ lower than q_{oc}. Then, as the current channel continues to shrink, on one hand the current profile becomes peaked again; on the other hand $q_o(t)$ reaches the bifurcation point corresponding to the lower value of m after a time of the order of 10^{-3} sec. At this point y_o starts suddenly to increase again while $q_o(t)$ decreases and a negative spike occurs followed possibly by the restoration of the equilibrium. The existence of the shrinking then gives rise to a characteristic cyclic process; the negative voltage spikes, however, should increase proportionally to m^{-1}. The duration Δt of the relaxation process towards a flat current profile, related to the negative spike, can be calculated from the relation

$$\Delta y_o \approx \frac{dy_o}{dq_o} \frac{dq_o}{dt} \Delta t \approx - \frac{3}{m\gamma a^2} \frac{dq_o}{dt} \Delta t \qquad (3.6)$$

where $\Delta y_o \approx 0 \cdot 3$, $dq_o/dt \approx - 10^3 \text{ sec}^{-1}$ and $\gamma a^2 \approx 10^{-1}$. One finds that Δt is proportional to m and is of the order of 10 μsec.

The process sketched above takes a quite different character when the current profile is strongly peaked, namely when γa^2 approaches unity. Indeed the current's modification (3.4) is proportional to γa^2 and for the same value of Δy_o but after a time somewhat longer than 10μsec (depending on the value of γa^2) the peak is completely removed. But now, rather than a flattening, the process is a collapse of the current profile, as indicated in fig. 3. Moreover the perturbation of the poloidal field, which is also proportional to γa^2(see Eq.(3.3)) becomes very great. It is conceivable that in such a situation the external circuit is unable to restore the original equilibrium, which is then disrupted. One can also understand the fact that the collapse is occurring at the lower m, at the end of the shrinking phase, when the current profile is very peaked.

4. DISCUSSION

The model presented above, notwithstanding the approximations involved, seems to provide a satisfactory basis for the description of the disruptive process observed in the Tokamak experiments. In particular the existence of the relaxation of the current profile agrees with the measurement of Bowers et al. [2] and the correct time scale of the relaxation can be obtained when the value of $\dot{q}(t)$ known experimentally is used in Eq. (3.6). As shown in Fig. 2, when the current profile becomes more peaked, the voltage spikes tend to concentrate, for a given m, at the lower values of q_o. In practice the first spike corresponds to the value of m associated with the first branch at the left of the q_o value associated with the initial unperturbed equilibrium. In this connection it should be remembered that the voltage spikes are inversely proportional to m. The sudden decrease of the poloidal magnetic field in the current channel (when q_o crosses the bifurcation point) while this field remains constant at the plasma surface, implies a similar decrease in the internal inductance per unit length of the plasma column. This can explain the major radius inward shift.

It follows from the present model that the disruptive process is the result of two opposite factors, namely the existence of the shrinking of the current channel and its consequent peaking due to external conditions and the intrinsic tendency of the current density to assume a flat profile. This tendency can be established on general thermodynamic grounds, also in the case of a plasma without individual collisions [14]. The thermodynamic arguments [14] indicate that the relaxation towards a flat profile should occur independently from any specified toroidal effect (this conclusion also holds for the relaxation of a skin current profile). So we do not expect that the treatment in toroidal geometry will alter essentially the present picture.

It is also worthwhile to note that the Eq. (2.13) for the nonlinear equilibria has just the same form as the equation describing the nonlinear neighbouring equilibria arising in the case of electrostatic reactive marginal instabilities [see Ref 15 Eq. (42)]. This case can be treated by means of the thermodynamic method. In fact the thermodynamic nature of the neighbouring magnetic Tokamak equilibria considered above is concealed by the time dependence of the equilibrium parameters arising from the shrinking, with the consequent vast inductive effects, which make the whole process so peculiar.

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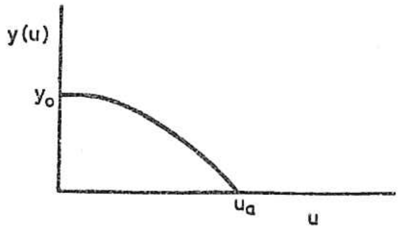

Fig.1 The function y(u).

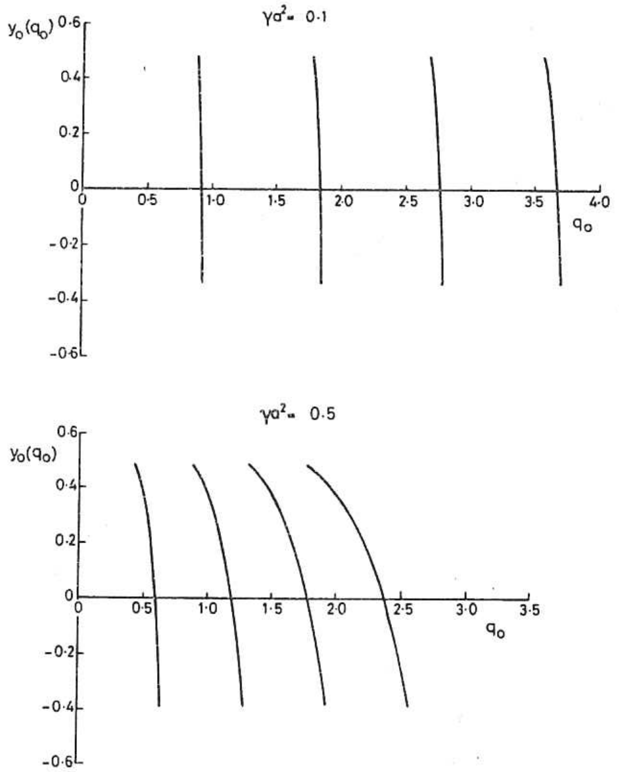

Fig.2 The branches of $y(q_o,m)$ (obtained by solving with a computer the equation (2.6)) for $1 \leq m \leq 4$ and $\gamma a^2 = 10^{-1}$ and 0.5. It is seen that when the current profile becomes more peaked, the branches of y_o and the related voltage spikes tend to concentrate at the lower values of q_o.

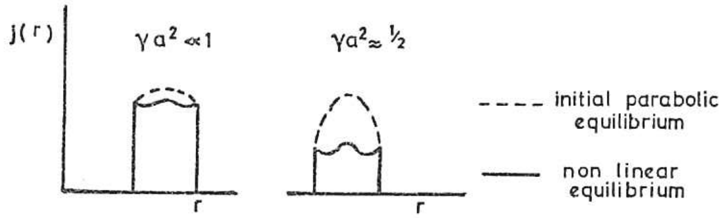

Fig.3 Modification of the current profile corresponding to the non-linear equilibrium. While in the case $\gamma a^2 \ll 1$, the modification is a flattening of the current profile, when γa^2 approaches unity the modification is a collapse.