CLM-P 381

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Preprint

THE ENERGETIC PARTICLE DISTRIBUTION IN A TOROIDAL PLASMA WITH NEUTRAL INJECTION HEATING

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1973

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(Submitted for publication in Physics of Fluids)

ABSTRACT

The energetic ion distribution resulting from the injection of high energy neutrals into a toroidal plasma has been derived. An appropriate kinetic equation which contains the angular scattering, friction and diffusion of the energetic ions by the background particles, charge exchange on the background neutrals, and acceleration of the ions by the electric field has been solved analytically by use of the W.K.B.J method. Collisions of the energetic ions with the faster moving electrons results in some of the ions increasing their energy above the injection energy. The width of this 'high energy tail' is shown to depend upon the electron temperature and the electric field. An estimate of the effect upon the width of this tail of collisions between the energetic ions themselves is also given. Finally illustrated examples of the various significant physical processes are presented.

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I INTRODUCTION

Several initial experiments[1,2,3] on neutral injection heating of Tokamak plasmas have just been completed and small increases in the background ion temperature were measured. The main diagnostic used so far in these experiments has been energy analysis of the charge exchanged energetic ions. From this measurement one can then obtain the shape of the injected energetic ion distribution. In this paper a detailed theoretical analysis of the energetic ion distribution is given so that a comparison with the experimental results may be made.

In previous papers the author[4,5] has derived expressions for the energetic ion distribution, however effects such as charge exchange and the acceleration of the ions by the electric field were ignored. Numerical solutions have been given by Clarke and Fowler[6] however these were primarily for the parameters of Ormak and no functional scalings were given. In this paper analytic expressions will be derived for each region of velocity space and where this is not possible, numerical results will be presented.

The most important classical collision process determining the motion of the energetic ions is their loss of energy by dynamic friction and scattering on the background ions and electrons. However the shape of the energetic ion distribution is not determined entirely by the dynamic friction and angular scattering terms, since velocity diffusion through collisions with the faster-moving electrons can result in some of the energetic ions gaining energy. As will be shown in

Section III (a) this leads to the setting up of a 'high energy tail' on the energetic ion distribution. Collisions of the energetic ions with each other are also found to contribute significantly to the width of this tail.

In a Tokamak the electric field parallel to the field lines can accelerate the ions in the case of parallel injection or decelerate them for anti-parallel injection. Thus for parallel injection the rate of slowing down (loss of energy) is slower than that of anti-parallel injection. This effect makes a significant difference to the width of the high energy tail of the distribution for parallel and anti-parallel injection and changes the shape of the distribution below the injection energy.

Charge exchange of the energetic ions with the neutral particles is also included in the analysis. Charge exchange is found to have a significant effect upon the shape of the ion distribution when the charge exchange time $\tau_{cx} \sim \tau_s$ the Spitzer slowing down time of an energetic ion.

The structure of the paper is as follows, first in Section II the energetic Fokker-Planck equation is derived and then solved analytically by use of a W.K.B.J. technique. Then the solutions for energies above and below the injection energy are presented in Section III,and finally in the latter part of that section plots of the complete solution for the parameters of the Cleo Tokamak are given.

II THE ENERGETIC ION FOKKER-PLANCK EQUATION

Our starting point is the Fokker-Planck equation for small angle binary collisions and the form that is used is that given by Rosenbluth, Macdonald and Judd[7] (henceforth abreviated to RMJ) for diffusion in an axisymmetric velocity space. The axis of symmetry in this problem is the direction of the magnetic field lines. The RMJ equation is valid for plasmas in which $\omega_{pe} > \omega_{ce}$, for in this approximation the interaction between the particles is limited to a distance by Debye shielding for which the electron orbit can be taken to be straight. The magnetic corrections to this equation have been examined by using a formulation supplied by Baldwin and Watson[8] but for the problem considered below are found to be very small for the usual Tokamak parameters ($\omega_{pe} \sim \omega_{ce}$).

Another approximation that will be used in the following is that the variation in magnetic field strength along the particle trajectory will be ignored. It has been shown by Connor and Cordey,[5] that for a Tokamak the geometry only has a significant effect upon trapped energetic ions for which the scattering operator was altered, and even in this case the results were similar to those obtained in the uniform field approximation. Thus in the present work the magnetic field is taken to be uniform and the RMJ equation for the energetic ion distribution can then be written in the following form:

$$\frac{\partial f}{\partial t} + \frac{Z_h e E^*}{m_h}\left[\frac{(1-\zeta^2)}{v}\frac{\partial f}{\partial \zeta} + \zeta\frac{\partial f}{\partial v}\right] = \Gamma\left\{(2v^2)^{-1}\frac{\partial}{\partial v^2}\left(v^2\frac{\partial^2 g}{\partial v^2} f\right) - v^{-2}\frac{\partial}{\partial v}\left[f\left(v^2\frac{\partial h}{\partial v} + \frac{\partial g}{\partial v}\right)\right] + (2v^3)^{-1}\frac{\partial g}{\partial v}\frac{\partial}{\partial \zeta}(1-\zeta^2)\frac{\partial f}{\partial \zeta}\right\} - \frac{f}{\tau_{cx}} + S\,\delta(v - v_o)\,K(\zeta) \qquad (1)$$

with $\Gamma = 4\pi e^4 Z_h^2 \log \Lambda/m_h^2$, $E^* = E(1-1/Z_{eff})$, $\zeta = \frac{v_{11}}{v}$, v_o the injected velocity and $Z_{eff} = \Sigma\, n_i Z_i^2 / n_e$.

The Rosenbluth potentials h and g are isotropic since both the background ion and electron distributions are assumed to be Maxwellian. The parameter E* the effective electric field acting on the energetic ions consists of two terms, the first part is the acceleration of the ions by the electric field E and the second part is the drag on the ions due to the electron motion in the electric field. This latter part was obtained by first writing h in the form:

$$h = h_o(v) + h_1(v)\,\zeta E$$

and a similar expression for g and then using the electron Fokker-Planck equation to determine $h_1(v)$ in the same manner as Spitzer and Härm [(9)]. The penultimate term on the r.h.s. of Eqn (1) is the charge exchange term. In this term the charge exchange time τ_{cx} is taken to be independent of energy which is a good approximation in the region considered here (energies in the range 1-30 keV). Finally the last term on the r.h.s. of Eqn (1) is the source of injected energetic ions; since the ions are injected almost monoenergetically the energy dependence is taken to be a δ function. However there will usually be quite a significant angular spread and this is represented by the function $K(\zeta)$.

The functions h and g which are given in RMJ may be simplified by use of the inequality $v_i < v << v_e$ where $v_{i,e}$ are the background ion and electron thermal velocities and v the energetic ion velocity. In the part of h and g which consists of collisions with background ions, only terms of the order of $\exp(-v^2/v_i^2)$ are neglected, this enables the theory to be extended down to energetic ion velocities $\sim 1.5\, v_i$. Using the above approximation h and g become:

$$v^2 \frac{\partial h}{\partial v} + \frac{\partial g}{\partial v} = -\frac{4}{3\pi^{1/2}} \frac{m_h}{m_e} \frac{n_e v^3}{v_e^{\,3}} - m_h \sum_j \frac{n_j Z_j^2}{m_j}$$

$$\frac{\partial g}{\partial v} = \Sigma\, n_j\, Z_j^2 = n_e\, Z_{eff}$$

$$v^2 \frac{\partial^2 g}{\partial v^2} = \frac{4}{3\pi^{1/2}} \left[\frac{n_e v^2}{v_e} + \frac{2}{v} \sum_j n_j Z_j^{\,2}\, v_j^{\,2} \right]$$

The two remaining summations over j the ion species in the above equation may usually be replaced by the dominant specie (say the ith) of the plasma. Then after using quasi neutrality we find $\Sigma\, n_j Z_j^2/m_j \simeq n_e/m_i$ and $\Sigma\, n_j Z_j^{\,2}\, v_j^{\,2} = n_e v_i^{\,2}$ (it has been assumed that the dominant specie has $Z_i = 1$ which will be the case for experiments in hydrogen, deuterium etc.) Substituting the above expressions for h and g into Eqn (1) and writing the equation in non-dimensional for with $u = v/v_o$ we have:

$$\delta a \frac{\partial^2 f}{\partial u^2} + b \frac{\partial f}{\partial u} + df + \delta ar\, Lf - \frac{\varepsilon(1-\zeta^2)}{u} \frac{\partial f}{\partial \zeta} = -\tau_s S\delta(u-1)K(\zeta) \quad (2)$$

where $a(u) = 1 + 2\beta/u^3$, $b(u) = u_c^{\,3}/u^2 + u - \varepsilon\zeta - 4\delta\beta/u^3 + 4\delta/u$.

$$d = 3 - \tau_s/\tau_{cx} + 4\delta\beta/u^5 + 2\delta/u^2, \quad r(u) = m_i Z_{eff} u_c^{\,3}/(2m_h u^3\, \delta a)$$

$$u_c^{\,3} = v_c^{\,3}/v_o^{\,3}, \quad v_c = [.75\pi^{\frac{1}{2}} m_e/m_i]^{\frac{1}{3}} v_e, \quad \varepsilon = Z_h e E^* \tau_s/(m_h v_o)$$

$$L \equiv \frac{\partial}{\partial\zeta}(1-\zeta^2)\frac{\partial}{\partial\zeta}, \quad \delta = \tfrac{1}{2}\frac{m_e v_e^{\,2}}{m_h v_o^2} = \tfrac{1}{2}\frac{T_e}{\varepsilon_o}, \quad \beta = \frac{v_e v_i^{\,2}}{v_o^3}$$

and $$\tau_s = \frac{3v_e^3\, m_e\, m_h}{16\pi^{\frac{1}{2}}\, e^4 Z_h^2\, n_e\, \mathrm{Ln}\Lambda}$$

The second term on the l.h.s. of Eqn (2) may be eliminated by the transformation

$$f = y(u,\zeta)\ \exp\left(-\int_1^u b/(2\delta a)du\right)$$

and Eqn (2) then becomes:

$$\frac{\partial^2 y}{\partial u^2} - \left[\frac{b^2}{4\delta^2 a^2} - \frac{d}{\delta a} + \frac{1}{2\delta}\frac{\partial b/a}{\partial u}\right]y + \frac{\exp\left(\int_1^u b/(2\delta a)du\right)}{\delta\alpha}\left[L\{\ y\ \exp\left(-\int_1^u b/(2\delta a)du\right)\}\right.$$

$$\left. - \varepsilon(1-\zeta^2)\frac{\partial}{\partial\zeta}\{y\ \exp\left(-\int_1^u b/2\delta a\ du\right)\}\right] = -\ \tau_s S\ \exp\left(\int_1^u b/2\delta a\right)\delta(u-1)K(\zeta)/\delta a \quad (3)$$

To solve equation (3) extensive use will be made of the small parameter δ ($\equiv T_e/2\varepsilon_o$) and also for most cases of interest ε will also be small. The form of Eqn (3) suggests that a WKBJ solution is appropriate and for the remainder of this section this method of solution will be discussed. If terms of order ε are neglected in Eqn (3) and the Legendre operator replaced by a constant α^2 then the two WKBJ solutions of Eqn (3) are

$$y_{\pm} = \frac{1}{D^{\frac{1}{2}}}\ \exp\left(\pm\int_1^u D du\right)$$

where $D^2 = b^2/4\delta^2 a^2 - \ d/\delta a + 1/(2\,\delta)\frac{\partial b/a}{\partial u} + \alpha^2/\delta a$

These give the two distributions f_+ and f_-

$$f_{\pm} = \frac{A_{\pm}}{D^{1/2}}\quad \exp\left[\int_1^u(-b/2\delta a \pm D)du\right] \qquad (4)$$

The behaviour of f_+ and f_- may be found by expanding D as a series in δ, f_- decays rapidly with u, $f_- \sim \exp(-\frac{1}{\delta}\int_1^u(b/a)du)$ whereas f_+ is more slowly varying $f_+ \sim \exp\left[\int(\alpha^2-d)/bdu\right]$. The boundary condition at $u = \infty$ is $u^3 f \to o$, thus for velocities above the injected velocity $(u>1)$ it can be shown that the only possible solution is f_-. Below the velocity of injection ($u<1$) the solution which satisfies the boundary condition f finite at $u = o$ is f_+. By matching f and its derivative at the source the constants A_- and A_+ may be determined in terms of the source strength **S.** In the next section the solutions above and below the injection velocity are discussed in more detail and their properties examined.

III THE SOLUTION

(a) u>1, velocities above the injection velocity

The solution satisfying the boundary conditions at $u = \infty$ for the region $u>1$ is f_-. Expanding the function D as a series in δ, f_- in zero order is

$$f_- = A_- \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} \exp\left(-\int_1^u b/\delta a du\right) \tag{5}$$

This expression is independent of α so only the first two terms on the l.h.s. of Eqn (3) actually contribute to the solution in this region. Hence to zero order in δ the solution in this region $u>1$ is given by Eqn (5) with the parameter A_- a function of ζ. Since δ is very small this solution decays very rapidly, the width of this decay $\langle u \rangle$ can be estimated from Eqn (5) and is

$$\langle u \rangle = \delta a(1)/b(1) = \delta(1 + 2\beta)/(1 + u_c^3 - \varepsilon\zeta) \sim \frac{m_e v_e^2}{m_h v_o^2} \tag{6}$$

From this expression it can be seen that for injection parallel to the electric field ($\varepsilon>o$) the width of this 'high energy tail' is greater than that for antiparallel injection ($\varepsilon<o$). This difference is illustrated in Fig 1 where the complete distributions for parallel and antiparallel injection are compared. This effect has also been observed on the Cleo experiment[1].

The absolute experimental width was found to be a factor of four greater than the theoretical width. A possible explanation for this difference is that in this model, collisions between the energetic particles themselves have been neglected, however, this type of collision may be included fairly easily into the theory. The most important contribution from self collisions of the energetic particles is to the g potential in the Fokker-Planck equation. Now the form for g_h which has to be added to the existing g is:

$$g_h(v,\zeta) = \frac{2}{\pi} \int_0^\infty dv' v'^2 \int_{-1}^{1} d\zeta' \, f(v',\zeta') \left[v^2+v'^2-2vv'(\zeta\zeta'-\{(1-\zeta^2)(1-\zeta'^2)\}^{\frac{1}{2}}) \right]$$

$$\times \, E\left[\frac{4vv' \left\{(1-\zeta^2)(1-\zeta'^2)\right\}^{\frac{1}{2}}}{v^2+v'^2-2vv'\left\{\zeta\zeta'-[(1-\zeta^2)(1-\zeta'^2)]^{\frac{1}{2}}\right\}} \right] \qquad (7)$$

where E is the complete elliptic function.

If the above expression was substituted as it stands into Eqn (1), the equation would then be an integro-differential equation. However we shall now show how an estimate of g_h can be obtained. This is possible because most of the energetic ion distribution lies in the region $u<1$ and its form in this region does not depend strongly on g_h. Hence a good approximation to g_h can be obtained by substituting in Eqn (7) the expression for f_h in $u<1$ that is derived in the next part of this section. After the substitution is completed it is found that:

$$\left.\frac{\partial^2 g_h}{\partial v^2}\right|_{u=1} \simeq \frac{3n_h}{v_o}$$

where n_h is the energetic ion number density. Adding this term to the original expression for $\frac{\partial^2 g}{\partial v^2}$ it is found that the new width of the high energy tail is

$$\langle u \rangle = \frac{m_e v_e^2}{2 m_h v_o^2}\left(1+2\beta+4\,\frac{n_h v_e}{n_e v_o}\right) \Big/ \left(1 + u_c^3 - \varepsilon\zeta\right) \qquad (8)$$

This expression gives a result which is much closer to the measured width in the Cleo experiment.

(b) $u<1$, velocities below the injection velocity

In this region the solution is of the form f_+ given by Eqn (4). If this solution is expanded in terms of δ it is found that to zero'th order all the terms of Eqn (3) contribute to the solution with the exception of the first term on the l.h.s. This equation is not separable in u and ζ in this form and a further expansion in ε the electric field term has to be made to give a separable equation. Thus putting

$$y = y^0 + \varepsilon y^1 + \varepsilon^2 y^2 + \ldots$$

in Eqn (3), the zero'th order equation is

$$\frac{\partial^2 y^0}{\partial u^2} - h y^0 + r\, L y^0 = -\,\tau_s\, S \exp\left(\int_1^u b/(2\delta a)\,du\right)\, \delta(u-1) K(\zeta)/\delta a \qquad (9)$$

where $h(u) = b^2/4\delta^2 a^2 - d/\delta a + \frac{\partial (b/a)}{\partial u}/2\delta$. Eqn (9) can now be solved by expanding in Legendre polynomials

$$y^0 = \sum_{n=o}^{\infty} c_n^{(0)}(u)\; P_n(\zeta)$$

and the $c_n(u)$ are given by the differential equation

$$\frac{d^2 c_n^{(0)}}{du^2} - (h + n(n+1)r)\, c_n^{(0)} = -\,\tau_s\, S \exp\left(\int_1^u b/(2\delta a)\,du\right)\delta(u-1) \int_{-1}^{1} K(\zeta) P_n(\zeta)\, d\zeta/\delta a \; .$$

The W.K.B.J. solution of this equation satisfying the boundary conditions at u = o is

$$c_n^{(0)} = \frac{C_n}{(h+n(n+1)\ r\ )^{\frac{1}{4}}} \exp\left(\int_1^u (h+n(n+1)\ r\ )^{\frac{1}{2}} du\right)$$

where the C_n are constants.

Expanding h and r in δ and completing the integration over u gives

$$c_n^{(0)} = \frac{C_n u^{\rho n(n+1)}\ t}{(u^3 + u_c^3)^{(\rho n(n+1)+d)/3}} \tag{10}$$

where $\rho = m_i Z_{eff}/(2m_h)$ and $t(u) = \exp\left[\int_1^u b/(2\delta a)du\right]$.

The first order correction in ε to the c_n is obtained by substituting $c_n^{(o)}$ in Eqn (3) and integrating, the solution is

$$c_n^{(1)} = \frac{C_n\ u^{\rho n(u+1)}\ t}{(u^3 + u_c^3)^{(\rho n(u+1)\ +\ d)/3}} \int_1^u \gamma_n du \tag{11}$$

where

$$\gamma_n = u^{1-\rho n}(u^3 + u_c^3)^{\rho n/3-1}\left\{\frac{C_{n-1}}{C_n}\ \frac{n(n-1)}{2n-1}\left[\frac{n}{2n-1}\ \frac{\rho u_c^{\ 3}}{(u^3+u_c^3)} - 1\right]\frac{(u^3+u_c^3)^{\rho n/3}}{u^{\rho n}}\right.$$

$$\left. + \frac{C_{n+1}}{C_n}\ \frac{(n+1)(n+2)}{2n+3}\left[\frac{n+2}{2n+3}\ \frac{\rho u_c^{\ 3}}{u^3+u_c^3} + 1\right]u^{\rho(3n+2)}(u^3+u_c^3)^{\rho n/3}\right\}$$

For most tokamak parameters it is only necessary to go as far as the first order correction $c_n^{(1)}$ and the results that are given later in this section are correct to this order. It now remains to determine the constants C_n and A_- in terms of the source strength S , by matching the solution given by Eqn (5) for u>1 to the solution given by Eqns (10) and (11) for $u<1$ at the source of injected ions $u=1$.

(c) The complete solution

The boundary conditions at the source region are

$$f\,(u,\zeta)\Big|_{u=1^-} = f\,(u,\zeta)\Big|_{u=1^+} \tag{12}$$

$$\delta a(1)\,\frac{\partial f}{\partial u}\Bigg|_{u=1^-} = \delta a(1)\,\frac{\partial f}{\partial u}\Bigg|_{u=1^+} + \tau_s\; S\, K(\zeta)\ . \tag{13}$$

Before substituting the two solutions given by Eqns (5) and (9) into the above, the arbitrary function $A_-(\zeta)$ in Eqn (5) is expanded in Legendre polynomials in the following form

$$\frac{A_-(\zeta)\ a^{\frac{1}{2}}}{b^{\frac{1}{2}}} = \Sigma\, A_n\, P_n(\zeta)\ .$$

After neglecting terms of order δ Eqns (12) and (13) give

$$A_n = \frac{S\tau_s}{2}\,\{(2n+1)K_n + \varepsilon\,[(n+1)K_{n+1} + nK_{n-1}]/(1+u_c^3)\}/(1+u_c^3)$$

$$C_n = A_n\,(1+u_c^3)^{(\rho n(n+1)+d)/3}$$

where $K_n = \int_{-1}^{1} K(\zeta)\ P_n(\zeta)d\zeta$.

With the above results we now have the complete solution.

Examples of this solution are given in Figs 1-3. These examples are for the parameters of the Cleo Tokamak experiment, however the general features of the energetic ion distribution will be similar for the parameters of other Tokamak experiments(2,3) with parallel injection. In the Cleo experiment the ions are injected almost parallel to the magnetic axis as a result the angular distribution of the source was taken to be of the form

$$K(\zeta) \begin{cases} = 1 - (\zeta - \zeta_0)^2/(\zeta_0 - \zeta_1)^2 , & \zeta_1 < \zeta < \zeta_2 \\ = 0 & \zeta < \zeta_1, \zeta > \zeta_2 \end{cases}$$

with $\zeta_o = (\zeta_1 + \zeta_2)/2$

and $\zeta_1 = 1$, $\zeta_2 = \cdot 8$. Two typical energetic ion distributions are shown in Fig 1. The continuous curve is for injection in the same direction as the electric field and the dotted one is for injection in the opposite direction to the magnetic field. The difference in the widths of the high energy tail for parallel and antiparallel injection is apparent from this Figure.

An idea of the effect of charge exchange can be seen in Fig 2 where a plasma with a high neutral particle density ($\tau_{cx} = 0.25\tau_s$) is compared with one with no neutrals ($\tau_{cx} = \infty$). By examining the analytic form of the solution of Eqn (9) it can be shown that charge exchange only becomes significant when $d<0$ that is for $\tau_{cx} < \tau_s/3$.

To show how the angular spreading depends on velocity, in Fig 3 a contour representation of f is given. The contours are highly anisotropic for velocities close to the injection velocity and become more isotropic at lower velocities ($u < u_c$). This anisotropy is strongly dependent upon Z_{eff}, for large Z_{eff} the distribution becomes more isotropic (this can be seen by inspection of equation (2)).

Using the above solution it has been possible to compare these results with the measured energetic ion distributions in the Cleo Tokamak experiment. This comparison will be discussed elsewhere.

CONCLUSION

The energetic ion distribution resulting from the injection of ions into a Tokamak has been derived. The dependence of the high energy tail upon the electric field and the electron temperature has been given (Eqn 5). A further correction due to the collisions of the energetic ions with themselves has been estimated (Eqn 8). The effect of charge exchange on the shape of the distribution is only found to be significant for plasmas in which $\tau_{cx} < \tau_s/3$. Impurities ($Z_{eff} \gg 1$) are found to make the distribution more isotropic.

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