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FURTHER REMARKS ON THE FINITE LARMOR RADIUS STABILIZATION THEORY FOR MIRROR MACHINES

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A B S T R A C T

Several existing theoretical treatments of the low frequency drift instability for low β and low density plasmas in mirror machines are compared. We point out the importance of boundary conditions, and show that finite Larmor radius stabilization is important for the m = 1 mode. The equation for the electrostatic potential for high densities is solved for some ratios $\frac{a}{r_o}$ of the Larmor radius to the plasma radius. It is found that in the Phoenix experiment, under the present operating conditions, a ratio of $\frac{a}{r_o} \geqslant 0.5$, corresponding to the field of $\leqslant 5$ kG is sufficient for stability.

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INTRODUCTION

Several theoretical treatments of the low frequency drift instability for low β and low density ($\varepsilon = 1$) plasmas in mirror machines have been published during the past few years. Many of them appear to be in disagreement without it being obvious where the difference between them lies. One such treatment has recently been published by us [1]. We have now succeeded in understanding its relationship with at least two other standard works on the subject; those by Rosenbluth, Krall and Rostoker [2] (R.K.R.) and by Mikhailovskii [3]. Apart from showing the crucial differences, the comparison elucidates the role of the finite Larmor radius stabilization for the $m = 1$ mode. It follows from our treatment that in the Phoenix experiment, under present operating conditions, the plasma should be stable for fields around 5 kG.

Our equation (14) in reference 1 for the perturbed electrostatic potential in cylindrical co-ordinates was obtained from the Poisson equation:

$$- \nabla^2 \Psi = 4\pi ec \, (n' - n'_e) \qquad \ldots (1)$$

and can be written in the form:

$$\left[1 + \frac{4\pi Mc^2}{B^2}\left(n + \frac{m\omega_c}{(\omega + m\Omega)} \frac{a^2}{2} \frac{1}{r} \frac{dn}{dr}\right)\right] \nabla^2 \Psi + \frac{4\pi Mc^2}{B^2} \frac{dn}{dr} \frac{\partial \Psi}{\partial r}$$

$$- \frac{4\pi Mc^2}{B^2} \frac{m^2 \Omega \omega_c}{\omega(\omega + m\Omega)} \frac{1}{r} \frac{dn}{dr} \Psi = 0 \, , \qquad \ldots (2)$$

where the symbols are defined as follows:

Ψ = electrostatic potential, assumed to be $\Psi = \psi(r) \exp (i\omega t + i m \varphi)$

M = ion mass

B = magnetic field

ω_c = ion cyclotron frequency

Ω = ion precession frequency due to the magnetic field gradient

a = Larmor radius

n = ion density

n' = perturbation in the ion density

n'_e = perturbation in the electron density.

This equation was obtained assuming that $\frac{1}{Br} \frac{dn}{dr}$ is independent of r. To simplify the comparison with the other works we shall have to assume that B is independent of r, the equation (2) will then apply to a parabolic density distribution:

$$n = N\left(1 - \frac{r^2}{r_o^2}\right) .$$

COMPARISON WITH MIKHAILOVSKII THEORY

Mikhailovskii [3] treated the case of an infinite plasma in Cartesian co-ordinates with the density varying in the y direction and the magnetic field pointing in the z direction. The electric field was in the x direction only and was of the form:

$$\vec{E} = \vec{E}_x \exp(-i\omega t + ikx) .$$

The resulting dispersion relation for ω was:

$$1 + \frac{4\pi Mc^2}{B^2}\left(n - \frac{a^2\omega_c k}{2\left(\omega + \frac{gk}{\omega_c}\right)}\frac{dn}{dy}\right) + \frac{4\pi Mc^2}{B^2}\frac{g}{\omega\left(\omega + \frac{gk}{\omega_c}\right)}\frac{dn}{dy} = 0$$

where g is some equivalent gravitational force and all other symbols are defined as before.

One can attempt to apply the dispersion relation to cylindrical geometry by transforming:

$$x \to r\varphi, \quad y \to -r, \quad \text{and} \quad k \to \frac{m}{r} .$$

It follows then that $\frac{gk}{\omega_c} = m\Omega$. The transformed relation has the form:

$$\left[1 + \frac{4\pi Mc^2}{B^2}\left(n + \frac{m\omega_c}{(\omega + m\Omega)}\frac{a^2}{2}\frac{1}{r}\frac{dn}{dr}\right)\right]k^2 - \frac{4\pi Mc^2}{B^2}\frac{m^2\Omega\omega_c}{\omega(\omega + m\Omega)}\frac{1}{r}\frac{dn}{dr} = 0 \quad \ldots (3)$$

This is basically the same as the dispersion relation used by Damm et al [4] .

There is now no difficulty in seeing that the dispersion relation (3) follows from the equation (2) for a potential of the form $\Psi = C \exp(im\varphi)$, corresponding to the assumed field $\vec{E} = \vec{E}_x \exp(ikx)$. The method of using the dispersion relation (3) is however not at all obvious, since both n and k^2

depend on r. One tends to use their values at the boundary $r = r_0$. Such a procedure leads to a wrong result as can be seen by comparison with the result of equation (2) when solved properly.

We restrict the comparison to low densities, $(\frac{4\pi Mc^2}{B^2} n \ll 1)$, where finite Larmor radius effects can be neglected. The dispersion relation (3) then takes the form:

$$\alpha^2 = \frac{4\pi Mc^2}{B^2} \frac{m^2 \Omega \omega_c}{\omega(\omega + m\Omega)} \frac{1}{r} \frac{dn}{dr} = \frac{1}{r_0^2} \; . \qquad \ldots (4)$$

It was shown in reference 1, however, that the correct solution for cylindrical geometry should read

$$\alpha^2 = \frac{\arg^2[J_{m-1}(\alpha r_0) = 0]}{r_0^2} \; . \qquad \ldots (5)$$

This correction makes the plasma considerably more stable. The additional parameter in equation (5) appears naturally when the boundary conditions at $r = r_0$ are taken into account.

COMPARISON WITH R.K.R.

R.K.R.[2] used a density distribution of the form: $n = N \exp - (\frac{r}{r_0})^2$. This density distribution cannot be used directly in the equation (2) which is valid only for a parabolic density distribution. To see what changes are required in (2) one has to re-examine the calculation of n' .

The perturbed ion density n' used in equation (1) was obtained from the expression for the perturbed density of guiding centres n'_g , with the help of the relations:

$$n' = n'_g + \frac{a^2}{4} \nabla^2 n'_g \ldots \quad (6) \quad \text{and} \quad n_g = n - \frac{a^2}{4} \nabla^2 n \; , \qquad \ldots (7)$$

where n_g is the unperturbed density of guiding centres.

In calculating the finite Larmor radius correction, $\delta n'_g = n' - n'_g$, to the perturbed density n'_g, only the largest term was kept in the expression for n'_g.

$$n'_g \approx \frac{m}{(\omega + m\Omega)} \frac{1}{Br} \frac{dn_g}{dr} \Psi \, . \qquad \ldots (8)$$

With

$$n = N\left(1 - \frac{r^2}{r_o^2}\right)$$

one finds from (7) :

$$\frac{dn_g}{dr} = -\frac{2r}{r_o^2} N = \frac{dn}{dr}$$

The correction $\delta n'_g$ now easily follows from (8) :

$$\delta n'_g = \frac{a^2}{4} \frac{m}{(\omega + m\Omega)} \frac{1}{Br} \frac{dn}{dr} \nabla^2 \Psi \, .$$

This correction is incorporated in equation (2).

Repeating the above procedure for the density distribution used by R.K.R., one finds:

$$\frac{dn_g}{dr} = \left[1 + 2\frac{a^2}{r_o^2}\left(1 - \frac{r^2}{2r_o^2}\right)\right] \frac{dn}{dr} \, ,$$

and

$$\delta n'_g = \frac{a^2}{4} \frac{m}{(\omega + m\Omega)} \frac{1}{Br} \frac{dn}{dr} \left(\nabla^2 \Psi - \frac{4r}{r_o^2} \frac{\partial \Psi}{\partial r} + \frac{4}{r_o^2} \Psi\right) .$$

It is seen that this contains correction terms to $\frac{\partial \Psi}{\partial r}$ and Ψ as well as the correction in $\nabla^2 \Psi$ equal to the one for the parabolic distribution used in (2). At this point one can also add the term

$$-\frac{\Omega}{\omega_c} \frac{m}{(\omega + m\Omega)} \frac{1}{Br} \frac{dn}{dr} \Psi$$

which was neglected in our original calculation. It is believed to be of no importance as it is of order $\frac{\Omega}{\omega_c}$, but it is introduced here to facilitate the comparison.

For the exponential density distribution of R.K.R., the corrected equation now becomes:

$$\left[1 + \frac{4\pi Mc^2}{B^2}\left(n + \frac{m\omega_c}{(\omega + m\Omega)}\ \frac{a^2}{2}\ \frac{1}{r}\frac{dn}{dr}\right)\right]\nabla^2\Psi$$

$$-\ \frac{2}{r_o^2}\ \frac{4\pi Mc^2}{B^2}\left(n + \frac{m\,\omega_c}{(\omega + m\Omega)}\ \frac{a^2}{2}\ \frac{1}{r}\frac{dn}{dr}\right) r\ \frac{\partial\Psi}{\partial r}$$

$$+\ \frac{4\pi Mc^2}{B^2}\ \frac{m\omega_c\left[\left(\frac{a^2}{r_o^2} - \frac{\Omega}{\omega_c}\right)\omega - m\Omega\right]}{\omega(\omega + m\Omega)}\ \frac{1}{r}\frac{dn}{dr}\ \Psi = 0\ .$$

By neglecting 1 compared to $\frac{4\pi Mc^2}{B^2}\,n$ and substituting $\frac{dn}{dr} = -\frac{2r}{r_o^2}\,n$, one finally obtains the R.K.R. equation:

$$\nabla^2\Psi - \frac{2}{r_o^2}\left(r\,\frac{\partial\Psi}{\partial r} - \nu\,\Psi\right) = 0\ , \qquad \ldots (9)$$

with

$$\nu = m\ \frac{\dfrac{m\Omega\,\omega_c - \left(\frac{a^2}{r_o^2} - \Omega\right)\omega}{\omega(\omega + m\Omega)}}{1 - \dfrac{a^2}{r_o^2}\ \dfrac{m}{(\omega + m\Omega)}} \qquad \ldots (10)$$

While the equation (9) is identical to the one of R.K.R., there is a slight discrepancy in ν, the correction terms $\frac{a^2}{r_o^2}$ and Ω being a factor of 2 smaller than in the R.K.R. paper. This difference may be attributed to the different velocity distribution assumed in the two calculations, and does not affect the general conclusions of this discussion.

R.K.R. used the solution $\Psi = r^m e^{im\varphi}$ for $0 \leqslant r \leqslant \infty$ and obtained the condition $\nu = m$. By setting $\nu = m$ in equation (10), one obtains the dispersion relation:

$$\omega^2 + \left[(m-1)\,\Omega - (m-1)\,\frac{a^2}{r_o^2}\right]\omega - m\Omega\omega_c = 0\ ,$$

which shows that for $m = 1$, there is no Larmor radius stabilization and

$$\omega = \pm\, i\left(\frac{m\omega_c}{\Omega}\right)^{1/2} = \pm\, i\,(kg)^{1/2}\ .$$

The same result follows from our equation (2) if one uses $\Psi = r^m e^{im\varphi}$ as a solution, which satisfies (2) for all densities. This solution, however, does not satisfy the boundary conditions determined by the density distribution

$n = N\,(1 - \frac{r^2}{r_o^2})$, which requires $\Psi = r^{-m}\, e^{im\varphi}$ beyond the plasma radius r_o. In general if realistic boundary conditions are taken into account, a finite Larmor radius correction exists even for $m = 1$.

STABLE REGION AT LOW FIELDS

We have shown in reference 1 that the proper solution to the equation (2) for low densities is $\Psi = J_m(\alpha r)e^{im\varphi}$, but no solution that satisfies the boundary conditions has yet been found for higher densities $\left(\frac{4\pi Mc^2}{B^2}\, N \gg 1\right)$. It is easy to see, however, that under certain circumstances the solution for higher densities is again a Bessel function and that finite Larmor radius effects exist and are important, even for $m = 1$.

To prove the statement one only has to assume that

$$A = \frac{a^2 m\,\omega_c}{r_o^2(\omega + m\Omega)} \gg 1. \qquad \ldots (11)$$

The equation (2) for a parabolic density distribution then becomes:

$$\nabla^2\Psi - \frac{2m\Omega}{a^2\omega}\,\Psi = 0\ ,$$

and the resulting solution is:

$$\Psi = J_m(\alpha r)e^{im\varphi}\ ,$$

with

$$\alpha^2 = -\frac{2m\,\Omega}{a^2\omega}\ .$$

The dispersion relation is

$$\omega = -\,2\left(\frac{r_o}{a}\right)^2 \frac{m\Omega}{\arg^2[J_{m-1}(\alpha r_o) = 0]}\ .$$

For $m = 1$ this becomes

$$\omega = -\,2\left(\frac{r_o}{a}\right)^2 \frac{\Omega}{5.76}\ . \qquad \ldots (12)$$

The solution for ω is real and the plasma is stable provided the assumption (11) can be satisfied and if the second order correction in Larmor radius is sufficient. By inserting the expression for ω back into the equation (2) one finds:

$$\nabla^2\Psi = -\,\frac{5.76}{r_o^2}\,\Psi\ ,$$

and stopping at the second term in the expansion $\Psi + \frac{a^2}{4} \nabla^2\Psi\ldots$ seems well justified for r_o of the order of 10 cm. One still has to check the assumption (11) which, using (12), becomes:

$$A = \frac{V^2}{r_o^2} \frac{1}{\omega_c\Omega} \frac{1}{\left(1 - \frac{2}{5.76} \frac{r_o^2}{a^2}\right)} \gg 1 .$$

For Phoenix and a hydrogen plasma of 20 keV thermal energy:

$$A \approx \frac{4}{\left(1 - \frac{22}{a^2}\right)} .$$

An 'A' of 10 requires a field of 5 kG which gives $\frac{a}{r_o} = 0.5$. Such a large ratio of the Larmor radius to the plasma radius is of course not satisfactory but there are several means available to improve on it. One obvious possibility is to decrease the precession frequency Ω. That would, however, result in a lower trapping efficiency in Phoenix. In principle one could also increase the stability of the plasma by placing a conducting wall very near to the plasma. Under such conditions the dispersion relation (5) becomes:

$$\alpha^2 = \frac{\arg^2[J_m(\alpha r_o) = 0]}{r_o^2} ,$$

which changes A into:

$$A = \frac{4}{\left(1 - \frac{8.6}{a^2}\right)} \text{ for } m = 1 .$$

This requires a field of about 8 kG and $\frac{a}{r_o} = \cdot 31$.

Apart from these possibilities there is the experimental evidence that the plasma is actually more stable in the high magnetic field region than the theory predicts, probably due to the finite length of the plasma as opposed to the infinite one assumed in the theory. This suggests that the required ratio of $\frac{a}{r_o}$ for stability is possibly smaller than 0.5. More experimental data is required for magnetic fields below 15 kG to prove that conclusively. As far as the theory is concerned stability can probably be obtained with $A > 1$ and not necessarily $A \gg 1$. The equation (2) is now being solved numerically and that should give an answer to the required A and provide the stability boundary.

CONCLUSION

We have shown that the three apparently different treatments of the low frequency drift instability [1,2,3] are basically the same. The crucial difference between them lies in the boundary conditions which are closely connected to the density distribution chosen. For a cylindrical plasma of a finite radius the finite Larmor radius effects can be important even for $m = 1$, contrary to the common belief.

Detailed comparison with the experimental behaviour must await a treatment which takes into account the motion parallel to the magnetic field and the finite size of the plasma in that direction.

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