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THE RELATIONSHIP BETWEEN THE MODULATIONAL INSTABILITY AND THE OSCILLATING TWO STREAM INSTABILITY

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Abstract

The stability of a finite amplitude monochromatic Langmuir wave is considered in one dimension. A dispersion relation is obtained which includes the decay, purely growing and modulational instabilities. It is shown that for an infinite wavelength Langmuir pump wave the modulational and oscillating two stream instabilities are the same. It is also pointed out that the threshold for the modulational instability is equal to the threshold for the inverse oscillating two stream instability, in which the Langmuir wave energy is converted into electromagnetic radiation.

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I INTRODUCTION

There are a number of situations in which long wavelength Langmuir turbulence can be generated in a hot collisionless plasma e.g. in the interaction of intense laser or electron beams with a plasma. This long wavelength turbulence then has the tendency to decay to longer and longer wavelengths by various non-linear processes resulting in an accumulation of the Langmuir waves in the $k = 0$ state. The problem then arises as to how this turbulent energy is dissipated since for such long wavelengths linear Landau damping is negligible and collisions are also ineffective. A possible resolution of this problem was proposed some time ago by Vedenov and Rudakov[1]. They noted the existence of an instability in which the initially uniform Langmuir turbulence would become modulated due to some initial density perturbation and in which shorter wavelength Langmuir waves are generated. This instability has become known as the modulational instability. The fully non-linear stages of this instability have been examined more fully by Zakharov[2] who has predicted a three dimensional focussing of the Langmuir waves with resultant generation of very intense high frequency electric fields and density depletion. The subject is now attracting a good deal of attention[3,4] and was recently investigated in a computer simulation[5].

In this paper we shall consider the simpler problem, originally discussed by Vedenov and Rudakov[1] and also by Zakharov[2], of the stability of a finite amplitude monochromatic Langmuir wave. The two principal aims of this paper are the following. We shall show explicitly the relationship between the modulational instability and the oscillating two stream instability. To the best of the authors' knowledge, this has not yet been demonstrated in the literature[6]. We shall also compare the modulational instability with the inverse oscillating two stream instability[7] (in which a finite amplitude Langmuir wave generates transverse electromagnetic radiation) and show that the threshold amplitude of the Langmuir wave is the same for the two instabilities.

II DERIVATION OF THE NON-LINEAR WAVE EQUATIONS

We shall use a two fluid description of the plasma and assume that the electrons are much hotter than the ions ($T_e \gg T_i$). A finite amplitude Langmuir wave of frequency and wavenumber ω_o, $\underset{\sim}{k}_o$ will be assumed present and we shall calculate the effect of this wave on other Langmuir waves and on ion acoustic waves. The fluid equations we use are as follows

$$\frac{d\underset{\sim}{v}_j}{dt} + \frac{\gamma_j \kappa T_j}{n_j m_j} \underset{\sim}{\nabla} n_j + \nu_j \underset{\sim}{v}_j = \frac{q_j}{m_j} \underset{\sim}{E} \tag{1}$$

$$\frac{\partial}{\partial t} n_j + \underset{\sim}{\nabla}.(n_j \underset{\sim}{v}_j) = 0 \tag{2}$$

$$\epsilon_o \underset{\sim}{\nabla}.\underset{\sim}{E} = \sum_j q_j n_j \tag{3}$$

where $j = i$ or e. We now perform a perturbation analysis on these equations in which we obtain the coupling between the Langmuir and the ion acoustic branches due to the presence of the finite amplitude Langmuir wave. First consider the equation for Langmuir waves. We shall consider the one dimensional problem in which all waves (including the finite amplitide Langmuir wave) vary as $\exp i(kz - \omega t)$. Eliminating all the Langmuir variables in favour of the wave electric field we obtain the following equation

$$\epsilon_\ell(\omega,k)E_\ell = \frac{e}{\omega \epsilon_o k} \frac{\partial}{\partial z} (n_e v_e) + \frac{n_o e}{\omega^2 \epsilon_o} v_e \frac{\partial v_e}{\partial z} - \frac{e \gamma_e \kappa T_e}{\omega^2 \epsilon_o n_o m_e} n_e \frac{\partial n_e}{\partial Z} + \frac{e\, \nu_e}{\omega \epsilon_o k} n_e \tag{4}$$

where the variables have been split up into a uniform, constant part and a wave part. The wave damping has been treated as a perturbation to the linear undamped wave solution along with the non-linear interaction between these waves. The quantity $\epsilon_\ell(\omega,k)$ is the dielectric function in the plasma where

$$\epsilon_\ell(\omega,k) = 0$$

gives the Langmuir wave dispersion relation

$$\omega_\ell^2 = \omega_{pe}^2 + \gamma_e k_\ell^2 v_{Te}^2 \tag{5}$$

We now introduce the slowly varying amplitude functions given by

$$E_\ell^{\pm}(z,t) = \mathcal{E}_\ell^{\pm}(z,t) \quad e^{i(k_\ell z \mp \omega_\ell t)} \tag{6}$$

and expand the dielectric function about the ω,k values given by equation (5). Making the identification $\delta\omega \rightarrow i\partial/\partial t$ and $\delta k \rightarrow -i\partial/\partial z$ we obtain the equation

$$e^{i(k_\ell z - \omega_\ell t)} \left(\frac{\partial}{\partial t} + \gamma_e \frac{k_\ell v_{Te}^2}{\omega_\ell} \frac{\partial}{\partial z} + \gamma_\ell\right) \mathcal{E}_\ell^{+}(z,t) = -\frac{i\,e}{2\omega_\ell^2 \epsilon_o k_\ell} \frac{\partial}{\partial z}(n_e v_e) - \frac{i\,e\,n_o}{2\omega_\ell^3 \epsilon_o} v_e \frac{\partial v_e}{\partial z}$$

$$+ \frac{i\,\gamma_e \kappa T_e}{2\omega_\ell^3 \epsilon_o n_o m_e} n_e \frac{\partial n_e}{\partial z} \tag{7}$$

where the notation $\mathcal{E}_{\ell}^{+}$ denotes the Langmuir wave propagating in the direction k_{ℓ} and $\mathcal{E}_{\ell}^{-}$ represents the wave travelling in the reverse direction. We can now simplify the right hand side of equation (7) by picking out those terms which are resonant or nearly resonant. We do this by means of the wave number and frequency matching conditions

$$k_o = k_1 + k_s \tag{8}$$

$$\omega_o \approx \omega_1 + \omega_s \tag{9}$$

where (ω_1, k_1) refer to a Langmuir wave and (ω_s, k_s) to an ion acoustic wave. Equation (8) is taken to be satisfied exactly whereas equation (9) is only required to be satisfied approximately, thus allowing for a frequency mis-match. We take the finite amplitude Langmuir wave to be

$$E_o(z,t) = \text{Re}\left\{\mathcal{E}_o^{+}\, e^{i(k_o z - \omega_o t)} + \mathcal{E}_o^{-}\, e^{i(k_o z + \omega_o t)}\right\} \tag{10}$$

and similar expressions for its associated density and velocity fields. The acoustic waves which we shall consider will be

$$n_{es}(z,t) = \text{Re}\left\{N_s^{+}(z,t)\, e^{i(k_s z - \omega_s t)} + N_s^{-}(z,t)\, e^{i(k_s z + \omega_s t)}\right\} \tag{11}$$

and again the associated fields for both ions and electrons. Notice that we are using the convention of distinguishing between left and right travelling waves by reversing the sign of ω (rather than k). In the above expressions for the wave fields this sign has been included explicity so that ω_0, ω_1 and ω_s are all positive definite quantities. At this point the signs of k_o, k_1 and k_s have still to be specified. We shall do that at a later point in the analysis. With the aid of the matching conditions (equations (8) and (9)) and equations (10) and (11) we can obtain the final form of the equation for $\mathcal{E}_1^{+}(z,t)$ where we have retained only the dominant contribution to the coupling coefficients

$$\left(\frac{\partial}{\partial t} + \gamma_e k_1 \frac{v_{Te}^2}{\omega_1}\frac{\partial}{\partial z} + \gamma_1\right)\mathcal{E}_1^{+}(z,t) = -ic_{\ell s}\,\mathcal{E}_o^{+}\left\{(N_s^{+})^{*} e^{-i(\delta-\omega_s)t} + (N_s^{-})^{*} e^{-i(\delta+\omega_s)t}\right\} \tag{12}$$

where $c_{\ell s} \equiv \frac{1}{4}\frac{e^2}{\epsilon_o m_e \omega_o}$ and $\delta \equiv \omega_0 - \omega_1$.

The equation for the Langmuir wave propagating in the opposite direction is obtained in a similar manner and is

$$\left(\frac{\partial}{\partial t} - \gamma_e k_1 \frac{v_{Te}^2}{\omega_1}\frac{\partial}{\partial z} + \gamma_1\right) \mathcal{E}_1^-(z,t)$$

$$= ic_{\ell s}\,\mathcal{E}_o^- \left\{ (N_s^+)^* e^{i(\delta+\omega_s)t} + (N_s^-)^* e^{i(\delta-\omega_s)t} \right\}. \tag{13}$$

The damping factor γ_1 appearing in equations (12) and (13) can be interpreted as the sum of the collisional and linear Landau damping. The finite amplitude Langmuir wave given by equation (10) couples together four waves - two Langmuir and two ion acoustic. The reason for this is that $\omega_s \ll \omega_1$ so that the off resonant acoustic wave is only off resonant by a small mis-match. If we consider smaller and smaller values of k_o then δ becomes smaller (or even negative) and both acoustic waves are off resonant.

The non-linear wave equations for the two ion acoustic waves can be obtained in a similar way. Eliminating all the acoustic wave variables in favour of the density perturbation of the electrons we obtain

$$\epsilon_s(\omega,k)n_{es} = i\,k\,\frac{\gamma_e v_{Te}^2}{\omega^2}\frac{\omega_{pi}^2}{\omega_{pe}^2}\frac{n_e}{n_o}\frac{\partial n_e}{\partial z} - \frac{i\,k n_o}{\omega^2}\frac{\omega_{pi}^2}{\omega_{pe}^2} v_e \frac{\partial v_e}{\partial z} - \frac{i}{\omega}\frac{\omega_{pi}^2}{\omega_{pe}^2}\frac{\partial}{\partial z}(n_e v_e)$$

$$- \frac{i}{\omega}\frac{\partial}{\partial z}(n_i v_i) - \frac{i\,k\,n_o}{\omega^2} v_i \frac{\partial v_i}{\partial z} - \frac{i\nu_i n_i}{\omega} \tag{14}$$

We proceed from here exactly as for the Langmuir wave case to obtain the desired equations for the two ion acoustic waves

$$\left(\frac{\partial}{\partial t} + \frac{k_s c_s^2}{\omega_s} + \gamma_s\right) N_s^+(z,t) = -\,ic_{\ell\ell}\left\{ \mathcal{E}_o^+(\mathcal{E}_1^+)^* e^{-i(\delta-\omega_s)t} + \mathcal{E}_o^-(\mathcal{E}_1^-)^* e^{i(\delta+\omega_s)t} \right\} \tag{15}$$

$$\left(\frac{\partial}{\partial t} - \frac{k_s c_s^2}{\omega_s} + \gamma_s\right) N_s^-(z,t) = ic_{\ell\ell}\left\{ \mathcal{E}_o^+(\mathcal{E}_1^+)^* e^{-i(\delta+\omega_s)t} + \mathcal{E}_o^-(\mathcal{E}_1^-)^* e^{i(\delta-\omega_s)t} \right\} \tag{16}$$

where the frequency $\omega_s \equiv |k_s| c_s$ and $c_{\ell\ell} \equiv k_s^2 n_o e^2/4\omega_s m_i m_e \omega_o \omega_1$. The condition for the validity of equations (12), (13), (15) and (16) is $\epsilon_o|\mathcal{E}_o|^2 \ll n_o k T_e$. These four equations describe the stability of the finite amplitude Langmuir wave $\underset{\sim}{E}_o$. In what follows we shall assume that $\mathcal{E}_o^{\pm}$ remains constant thus linearizing the equations. A relaxation of this condition would take account of the full non-linear behaviour of the coupled waves. However, we shall only be concerned to find the conditions for instability and the initial growth rates.

Equations (12), (13), (15) and (16) have time dependent coefficients and these equations can be simplified by introducing the new amplitudes:

$$\alpha_\ell^+ = \mathcal{E}_1^+ \quad ; \quad \alpha_\ell^- = \mathcal{E}_1^- \, e^{-i2\delta t}$$

$$\alpha_s^+ = (N_s^+)^* \, e^{-i(\delta - \omega_s)t} \quad ; \quad \alpha_s^- = (N_s^-)^* \, e^{-i(\delta + \omega_s)t}$$

In order to specify the problem completely we must choose the signs of the wave numbers and we take

$$k_o \geqslant 0 \ , \quad k_s > 0 \quad \text{and} \quad k_1 < 0 \ .$$

The final form of the equations is then

$$\left(\frac{\partial}{\partial t} - v_1 \frac{\partial}{\partial z} + \gamma_1\right) \alpha_\ell^+ = -ic_{\ell s}\, \mathcal{E}_o^+ (\alpha_s^+ + \alpha_s^-) \tag{17}$$

$$\left(\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial z} + \gamma_\ell + i2\delta\right) \alpha_\ell^- = ic_{\ell s}\, \mathcal{E}_o^- (\alpha_s^+ + \alpha_s^-) \tag{18}$$

$$\left(\frac{\partial}{\partial t} + c_s \frac{\partial}{\partial z} + \gamma_s + i(\delta - \omega_s)\right) \alpha_s^+ = ic_{\ell\ell} \left\{ (\mathcal{E}_o^+)^* \alpha_\ell^+ + (\mathcal{E}_o^-)^* \alpha_\ell^- \right\} \tag{19}$$

$$\left(\frac{\partial}{\partial t} - c_s \frac{\partial}{\partial z} + \gamma_s + i(\delta + \omega_s)\right) \alpha_s^- = -ic_{\ell\ell} \left\{ (\mathcal{E}_o^+)^* \alpha_\ell^+ + (\mathcal{E}_o^-)^* \alpha_\ell^- \right\} \tag{20}$$

where $v_1 \equiv \gamma_e |k_1| v_{Te}^2/\omega_1$.

III THE DISPERSION RELATION

Equations (17) - (20) can now be solved assuming a variation $\exp i\,(qz - \omega t)$ resulting in the following dispersion relation

$$(\Omega + qv_1 + \delta + i\gamma_1)(\Omega - qv_1 - \delta + i\gamma_1)(\Omega - qc_s + \omega_s + i\gamma_s)(\Omega + qc_s - \omega_s + i\gamma_s)$$
$$- 2c_{\ell s}c_{\ell\ell}(qc_s - \omega_s)\{|\mathcal{E}_o^-|^2(\Omega + qv_1 + \delta + i\gamma_1) - |\mathcal{E}_o^+|^2(\Omega - qv_1 - \delta + i\gamma_1)\} = 0 \tag{21}$$

where $\Omega \equiv \omega - \delta$. We will now examine this general dispersion relation for a number of special cases

(i) $q = 0$, $k_o \neq 0$.

(a) $|\mathcal{E}_o^-| = |\mathcal{E}_o^+| = \mathcal{E}_o$.

In this case the dispersion relation is

$$(\Omega + \delta + i\gamma_1)(\Omega - \delta + i\gamma_1)(\Omega^2 - \omega_s^2 + 2i\gamma_s\Omega) + 4c_{\ell s}c_{\ell\ell}|\mathcal{E}_o|^2\delta\omega_s = 0 \qquad (22)$$

which is well known[8] to give rise both to the decay instability ($\delta > 0$) and a purely growing instability ($\delta < 0$). The expression for the growth rate of the decay instability given by equation (22) agrees with the result first given by Oraevskii and Sagdeev[9]. In addition the threshold fields for the decay and purely growing instabilities are equal to the corresponding thresholds for a transverse pump[10].

(b) $\mathcal{E}_o^- = 0$, $\mathcal{E}_o^+ = \mathcal{E}_o$.

In this case the dispersion relation reduces to

$$(\Omega + \delta + i\gamma_1)(\Omega^2 - \omega_s^2 + 2i\gamma_s\Omega) - 2c_{\ell s}c_{\ell\ell}|\mathcal{E}_o|^2\omega_s = 0 . \qquad (23)$$

This equation is of the same form as the one derived for the inverse oscillating two stream instability[7] and has unstable roots both when $\delta > 0$ and $\delta < 0$. However, in contrast to the oscillating two stream instability the $\delta < 0$ solution is no longer purely growing. This is due to the fact that the travelling wave pump drives the excited Langmuir wave non-symmetrically such that it is frequency shifted below the pump frequency and the instability is converted to the decay type. This result is similar to that of Nishikawa et al.[11].

(ii) $q = 0$, $k_o = 0$.

In this case there is no difference between E_o^+ and E_o^- and so we put $\mathcal{E}_o^+ = \mathcal{E}_o^- = \mathcal{E}_o$ when the dispersion relation becomes

$$(\Omega + \delta + i\gamma_1)(\Omega - \delta + i\gamma_1)(\Omega^2 - \omega_s^2 + 2i\gamma_s\Omega) + 4c_{\ell s}c_{\ell\ell}|\mathcal{E}_o|^2\omega_s\delta = 0 . \qquad (24)$$

For this case $k_1 = -k_s$ and δ is necessarily negative. In this case, we no longer have the possibility of the decay instability - just the purely growing mode for which $\delta < 0$. Again using reference 8 the minimum threshold for this instability is when $\delta = -\gamma_\ell$ and is given by

$$\frac{\epsilon_o |\mathcal{E}_o|^2}{n_o \kappa T_e} = 4\gamma_e \frac{\nu_e}{\omega_{pe}} \tag{25}$$

Neglecting the damping terms in equation (24) we can easily obtain an expression for the growth rate of the instability when $|\delta| \ll \omega_s$. Using the fact that $\delta \approx -\gamma_e k_1^2 v_{Te}^2/2\omega_{pe}$ we obtain

$$\gamma = \left[\frac{1}{8}\frac{\epsilon_o |\mathcal{E}_o|^2}{n_o m_e}\right]^{\frac{1}{2}} k_s \tag{26}$$

which is exactly the result given by Vedenov and Rudakov[1] for the instability of a cold gas of Langmuir plasmons.

For an infinite wavelength pump this instability is indistinguishable from the oscillating two stream instability. Therefore, for this simple case of a monochromatic Langmuir pump wave the modulational instability can be interpreted as the excitation of a pair of finite wavelength Langmuir waves propagating in opposite directions and at the same frequency as the pump. In other words, the excited Langmuir waves form a standing wave perturbation at the pump frequency. Associated with the excited Langmuir waves is a growing density perturbation at zero frequency which results from a down shifting in frequency of the two ion acoustic waves.

(iii) $q \neq 0$, $k_o = 0$, $\mathcal{E}_o^+ = \mathcal{E}_o^-$.

The dispersion relation in this case reduces to

$$(\Omega + qv_1 + \delta + i\gamma_1)(\Omega - qv_1 - \delta + i\gamma_1)\{\Omega^2 - (q c_s - \omega_s)^2 + 2i\gamma_s \Omega\}$$
$$- 4c_{\ell s} c_{\ell\ell} |\mathcal{E}_o|^2 (qc_s - \omega_s)(qv_1 + \delta) = 0 . \tag{27}$$

Looking for solutions $\Omega = iy$ we can again obtain the threshold minimum for instability

$$K_m = -2\gamma_1\omega_s \left(\frac{q}{k_s} - 1\right) \tag{28}$$

where $q/k_s \ll 1$. The inclusion of $q > 0$ evidently lowers the the threshold for the instability.

IV CONCLUSION

We are now in a position to make an interesting comparison. It has been shown in the above analysis that the modulational instability is equivalent to the oscillating two stream instability when $k_o = 0$ and the pump is monochromatic. However, a finite amplitude Langmuir wave of the type considered here can give rise to another instability[7]. This is the inverse oscillating two stream instability in which the Langmuir wave pump excites a standing electromagnetic wave and a zero frequency density perturbation. The minimum threshold amplitude for this process is[7]

$$\epsilon_o |\mathcal{E}_o|^2 / n_o \kappa T_e = 4 \gamma_e \nu_e / \omega_{pe}$$

which is the same as the threshold for the modulational instability given by equation (25)! The growth rate for the inverse oscillating two stream instability is

$$\gamma = \left[\frac{1}{8} \frac{\epsilon_o |\mathcal{E}_o|^2}{n_o m_e} \right]^{\frac{1}{2}} \frac{c\, k_T}{\gamma_e^{\frac{1}{2}} \, v_{Te}}$$

where k_T is the wavenumber of the electromagnetic wave which is excited in a direction perpendicular to $\underset{\sim}{E}_o$. For the condition of minimum threshold the above growth rate is also equal to that for the modulational instability given by equation (26). It is therefore possible that the inverse oscillating two stream instability could be just as potent a mechanism for the dissipation of long wavelength Langmuir turbulence as the modulational instability. Whereas the latter effect transfers the Langmuir energy to shorter wavelength Langmuir waves, where it is eventually Landau damped, the inverse oscillating two stream instability would result in the Langmuir energy being lost by radiation.

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