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NEW PERTURBATION THEORY OF RANDOM ACCELERATION AND FORMALLY SIMILAR PROBLEMS

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Abstract

We show how an equation, due to Orszag and Kraichnan, satisfied by the average of the solution of a linear random equation may be obtained very straightforwardly from an improved perturbation theory. This equation is the first member of a sequence of conditions, the imposition of which optimises the convergence properties of the expansion.

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Introduction

This paper is about equations which can be written symbolically

$$(L_o + \tilde{L})G = I \ , \tag{1}$$

where L_o and $\tilde{L}$ are linear operators, L_o being non-random and $\tilde{L}$ being random and zero-mean. $\tilde{L}$ is assumed "smaller" than L_o. The sense in which this is so varies with the nature of the problem described by equation (1).

For example, the propagator of a charged particle in a random electric field satisfies

$$\left(\frac{\partial}{\partial t} + v.\frac{\partial}{\partial r} + \frac{e}{m}\,\underline{E}.\frac{\partial}{\partial v}\right) G(\underline{r}, \underline{v}, t\,;\ \underline{r}', \underline{v}', t') = \delta(\underline{r} - \underline{r}')\delta(\underline{v} - \underline{v}')\delta(t - t'), \tag{2}$$

which is of the form (1) if $\underline{E}(\underline{r}, t)$ is a zero-mean random function. Equation (2) arises in connection with plasma turbulence. Other problems which are formally similar include wave propagation in random media (KELLER 1969: FRISCH 1968), turbulent diffusion (ROBERTS 1961, BOURRET 1962a) and the turbulent dynamo (MOFFATT, 1972).

The problem is to determine the statistical moments of G, given those of $\tilde{L}$. For our purpose it is necessary only to study the first moment $\langle G \rangle$, since higher moments may also be obtained from linear random equations like equation (1).

The most natural way to proceed is to replace $\tilde{L}$ by $\epsilon\tilde{L}$, where ϵ is an ordering parameter which is set equal to one at the end of the analysis, and seek an expansion of G in powers of ϵ about $G_o (\equiv L_o^{-1})$ (the Neumann series), followed by averaging. This method fails on account of secular terms. To circumvent this problem the following equation is often used

$$[L_o - \langle \tilde{L}\, G_o\, \tilde{L} \rangle] \langle G \rangle = I \ . \tag{3}$$

Originally derived by BOURRET (1962 b) by the summation of a subset of diagrams representing terms of the averaged Neumann series, equation (3) may also be obtained by formal operator methods (KELLER 1964, FRISCH 1968). A very similar equation may be obtained by two time-scale perturbation theory (PAPANICOLAOU and KELLER 1971).

For the random acceleration problem (equation (2)) equation (3) is

$$\left(\frac{\partial}{\partial t} + v.\frac{\partial}{\partial r} \right) \langle G(\underline{r}, \underline{v}, t ; \underline{r}', \underline{v}', t') \rangle - \frac{e^2}{m^2} \frac{\partial}{\partial v^\alpha} \int_{t'}^{t} ds \int d\underline{y}\, d\underline{u}\; G_o(\underline{r},\underline{v},t; \underline{y},\underline{u},s) \langle E^\alpha(\underline{r},t) E^\beta(\underline{y},s) \rangle$$

$$\times \frac{\partial}{\partial u^\beta} \langle G(y, u, s; r', v', t') \rangle = \delta(\underline{r} - \underline{r}')\delta(v - \underline{v}')\delta(t - t') \qquad (4)$$

where:

$$G_o(\underline{r}, \underline{v}, t;\ \underline{y}, \underline{u}, s) \equiv \delta\big(\underline{r} - \underline{y} - \underline{v}(t - s)\big)\delta(\underline{v} - \underline{u}) \qquad (5)$$

When the time-scale of the fluctuating force is much smaller than the time-scale of the equation, equation (4) can be reduced to a Markovian form:

$$\left(\frac{\partial}{\partial t} + v.\frac{\partial}{\partial \underline{r}} - \frac{e^2}{m^2} \frac{\partial}{\partial v^\alpha} \int_0^\infty Q^{\alpha\beta}(\underline{v}\, s, s) \frac{\partial}{\partial v^\beta} \right)$$

$$\times \langle G(\underline{r}, \underline{v}, t; \underline{r}', \underline{v}', t') \rangle = \delta(\underline{r}-\underline{r}')\delta(\underline{v} - \underline{v}')\delta(t - t') \qquad (6)$$

where

$$Q^{\alpha\beta}(\underline{r}, t) \equiv \langle E^\alpha(\underline{r}_o + \underline{r},\ t_o + t) E^\beta(\underline{r}_o, t_o) \rangle \qquad (7)$$

(and we have assumed $\underline{E}(\underline{r}, t)$ is stationary).

Equation (6) was derived by STURROCK (1966) by Fokker-Planck methods. It is also an essential ingredient in the quasi-linear theory of micro-instabilities (DRUMMOND 1965, BERNSTEIN and ENGELMANN 1966).

The arguments leading to equation (3) are valid only if $\tilde{L}$ is much smaller than L_o. ORSZAG and KRAICHNAN (1967) suggested that a more generally valid equation may be obtained from equation (3) by replacing G_o by $\langle G \rangle$:-

$$(L_o - \langle \tilde{L} \langle G \rangle \tilde{L} \rangle) \langle G \rangle = I \ . \qquad (8)$$

An equation similar to equation (8) appears in DUPREE's (1966) turbulence theory. Orszag and Kraichnan give a discussion of the relationship between their work and that of Dupree.

Equation (8) was first obtained by ROBERTS (1961) in two different ways: as a consequence of Kraichnan's direct interaction approximation, and by summing additional classes of diagrams (see also FRISCH, 1968). Orszag and Kraichnan show that equation (8) is satisfied exactly by the mean Green function of a model equation which has some structural similarities to equation (2). All three methods are of considerable analytical complexity, yet they leave the question of the circumstances in which equation (8) is both valid and an improvement over equation (3) unanswered. In addition, these methods do not lend themselves to the generation of a sequence of successively better approximations, of which equation (8) is the first member.

The purpose of this paper is to show that equation (8) is a straightforward consequence of a very simple and reasonable re-ordering of equation (1), followed by a perturbation expansion. Equation (8) and higher order generalisations thereof occur very naturally as conditions guaranteeing the "goodness" of the expansion.

Perturbation Theory

The convergence properties of a perturbation series for G will be improved if the expansion is about, not G_o, but some (non-random) propagator Γ which is chosen to be closer (in a sense to be explained below) to $\langle G \rangle$ than is G_o.

So, introducing a non-random operator D via the definition

$$(L_o - D)\Gamma \equiv I , \tag{9}$$

we rewrite equation (1) as

$$\left\{(L_o - D) + \epsilon\tilde{L} + \epsilon^2 D^{(2)} + \epsilon^3 D^{(3)} + \epsilon^4 D^{(4)} + \ldots\right\} G = I , \tag{10}$$

where

$$D \equiv D^{(2)} + D^{(3)} + D^{(4)} + \ldots . \tag{11}$$

If G is expanded in powers of ϵ :-

$$G = \Gamma + \epsilon G^{(1)} + \epsilon^2 G^{(2)} + \epsilon^3 G^{(3)} + \epsilon^4 G^{(4)} + \ldots. , \tag{12}$$

and substituted into equation (10), we obtain (to order ϵ^4) :

$$G^{(1)} = - \Gamma \tilde{L} \Gamma \tag{13}$$

$$G^{(2)} = - \Gamma\left\{- \tilde{L} \Gamma \tilde{L} + D^{(2)}\right\}\Gamma \tag{14}$$

$$G^{(3)} = \Gamma\left\{\tilde{L}\Gamma [-\tilde{L}\Gamma\tilde{L} + D^{(2)}] + D^{(2)}\Gamma\tilde{L} - D^{(3)}\right\}\Gamma \tag{15}$$

$$G^{(4)} = - \Gamma\tilde{L}\Gamma\left\{\tilde{L}\Gamma[-\tilde{L}\Gamma\tilde{L}+D^{(2)}] + D^{(2)}\Gamma\tilde{L} - D^{(3)}\right\}\Gamma + \Gamma D^{(2)}\Gamma\left\{-\tilde{L}\Gamma\tilde{L} + D^{(2)}\right\}\Gamma + \Gamma D^{(3)}\Gamma\tilde{L}\Gamma - \Gamma D^{(4)}\Gamma \tag{16}$$

The next step is to average the $G^{(n)}$. It is immediately seen that $\langle G^{(1)} \rangle$ vanishes. The average of equation (12) reduces to

$$\langle G \rangle = \Gamma + \epsilon^2 \langle G^{(2)} \rangle + \epsilon^3 \langle G^{(3)} \rangle + \epsilon^4 \langle G^{(4)} \rangle + O(\epsilon^5) , \qquad (17)$$

where

$$\langle G^{(2)} \rangle = - \Gamma \left\{ - \langle \tilde{L} \Gamma \tilde{L} \rangle + D^{(2)} \right\} \Gamma \qquad (18)$$

and

$$\langle G^{(3)} \rangle = - \Gamma \left\{ \langle \tilde{L} \Gamma \tilde{L} \Gamma \tilde{L} \rangle + D^{(3)} \right\} \qquad (19)$$

and

$$\begin{aligned} \langle G^{(4)} \rangle = {} & \Gamma \langle \tilde{L} \Gamma \tilde{L} \Gamma \tilde{L} \Gamma \tilde{L} \rangle \Gamma - \Gamma \langle \tilde{L} \Gamma \tilde{L} \rangle \Gamma D^{(2)} \Gamma \\ & - \Gamma \langle \tilde{L} \Gamma D^{(2)} \Gamma \tilde{L} \rangle \Gamma + \Gamma D^{(2)} \Gamma \left\{ - \langle \tilde{L} \Gamma \tilde{L} \rangle + D^{(2)} \right\} \Gamma \\ & - \Gamma D^{(4)} \Gamma . \end{aligned} \qquad (20)$$

Γ and the $D^{(n)}$s have not yet been specified and we propose to use this freedom to ensure that Γ is as close as possible to $\langle G \rangle$, thus hopefully optimising the convergence properties of the expansion. To this end we choose $D^{(n)}$ so as to cause the corresponding $\langle G^{(n)} \rangle$ to vanish. Clearly, from equations (18), (19) and (20), the appropriate choices are

$$D^{(2)} = \langle \tilde{L} \Gamma \tilde{L} \rangle \qquad (21)$$

$$D^{(3)} = - \langle \tilde{L} \Gamma \tilde{L} \Gamma \tilde{L} \rangle \qquad (22)$$

$$D^{(4)} = \langle \tilde{L} \Gamma \tilde{L} \Gamma \tilde{L} \Gamma \tilde{L} \rangle - \langle \tilde{L} \Gamma \tilde{L} \rangle \Gamma \langle \tilde{L} \Gamma \tilde{L} \rangle - \langle \tilde{L} \Gamma \langle \tilde{L} \Gamma \tilde{L} \rangle \Gamma \tilde{L} \rangle . \qquad (23)$$

With this choice of $D^{(2)}$, $D^{(3)}$ and $D^{(4)}$ equation (17) becomes

$$\langle G \rangle = \Gamma + O(\epsilon^5) \qquad (24)$$

and (using equation (11)) equation (9) can be written

$$\begin{aligned} \Big\{ L_o & - \langle \tilde{L} \Gamma \tilde{L} \rangle + \langle \tilde{L} \Gamma \tilde{L} \Gamma \tilde{L} \rangle \\ & - [\langle \tilde{L} \Gamma \tilde{L} \Gamma \tilde{L} \Gamma \tilde{L} \rangle - \langle \tilde{L} \Gamma \tilde{L} \rangle \Gamma \langle \tilde{L} \Gamma \tilde{L} \rangle - \langle \tilde{L} \Gamma \langle \tilde{L} \Gamma \tilde{L} \rangle \Gamma \tilde{L} \rangle] \\ & - O(\epsilon^5) \Big\} \Gamma = I . \end{aligned} \qquad (25)$$

Equations (24) and (25) are the principal results of this paper.

Comments

The present method of deriving equation (8) is simple and concise and it is quite clear how successively better approximations may be generated. We claim only that it is an improved perturbation theory. There is no reason to suppose that it will suffice for the description of situations in which $\tilde{L}$ is large. The region of convergence of the expansion can only be determined by a detailed analysis of the several equations represented formally by equation (1).

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