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FINITE AMPLITUDE EFFECTS ON HYDROMAGNETIC WAVES

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FINITE AMPLITUDE EFFECTS ON HYDROMAGNETIC WAVES

by

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A B S T R A C T

Hydromagnetic waves travelling along an axial magnetic field in a cylindrical plasma bounded by a conductor have a frequency ω determined by the exciter. The non-linear terms appearing in the equation of motion and in Ohm's law, usually neglected in wave theory, will induce a small amplitude wave of frequency 2ω accompanying the basic wave. In addition the boundary conditions at the exciter will produce further double-frequency waves (df waves), but moving at speeds different from the basic wave. In this paper the amplitudes of these df waves are calculated for all the components of the magnetic field and for the plasma density. The existence of certain critical frequencies, at which resonance between the basic and df waves occurs, is established. In the limit as $\Omega \equiv \omega/\omega_{ci}$ tends to zero, the amplitudes of the magnetic df waves are shown to depend on Ω/ε, where ε^{-1} is the damping length of the wave, whereas the corresponding density wave amplitude is found to be independent of both ε and Ω.

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C O N T E N T S

1. INTRODUCTION

An understanding of non-linear processes is important in plasma physics since most instabilities are in fact detected through their non-linear effects, since proposals to heat plasmas with various kinds of waves inevitably involve (finite) large amplitude waves, and since non-linear wave interactions form the basis of many theories of hydromagnetic turbulence and collisionless shocks.

In this paper we consider a very elementary process in which a basic wave interacts with itself to produce a higher harmonic. Aside from the fact that this problem is mathematically tractable, it also has the virtue that it can be tested by simple experiments. All that is required is that the amplitude of the waves in existing experiments be increased[1,2]. Let B_o be the magnitude of the steady magnetic field and let the amplitude of the perturbations in $\underset{\sim}{B}$ be proportional to CB_o, then the present theory is correct to second order in the amplitude parameter C.

It is frequently stated (e.g. see Spitzer[3]) that transverse Alfvén waves are not subject to amplitude effects with incompressible motions and when the frequency ratio, $\Omega \equiv \omega/\omega_{ci}$, is negligible. However the actual effects of removing these restrictions on the density ρ and on Ω have not been discussed in the literature as far as this author has been able to determine. For example it is not clear, a priori, whether these are effects of $O(C^2)$ or $O(C^3)$.

The non-linear problem we shall study is that of determining the relative amplitude of the double frequency (df) wave we anticipate will be induced by the (squared) non-linear terms occurring in the equation of motion and in Ohm's law. This amplitude, α say, will normally be $O(C^2)$, but we should not be surprised to find that at certain frequencies a resonance between the basic wave and the df wave will lead to much larger amplitudes than this order would suggest. In fact one of the resonance frequencies proves to be the low frequency limit $\Omega = 0$, which at first sight seems to contradict the result quoted above from Spitzer's text. However the introduction of some resistivity, η, is found to modify this singular behaviour, so that if the limits $\Omega \to 0$ and $\eta \to 0$ are taken in this order, α is found to be zero for all components of the magnetic field. This result holds (for the transverse wave only) even when the density is permitted to fluctuate.

2. THE NON-LINEAR DIFFERENTIAL EQUATION FOR B_z

Adopting the usual notation and mks units, we take the equations for a pressureless, isotropic, fully-ionized plasma to be

$$\nabla \times \underset{\sim}{B} = \mu \underset{\sim}{j} , \quad \nabla \times \underset{\sim}{E} = - \partial \underset{\sim}{B}/\partial t \qquad \ldots (1)$$

$$\eta \underset{\sim}{j} = \underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B} - (B_o/\omega_{ci})(\partial \underset{\sim}{v}/\partial t + \underset{\sim}{v} \cdot \nabla \underset{\sim}{v}) \qquad \ldots (2)$$

$$\rho \partial \underset{\sim}{v}/\partial t + \rho \underset{\sim}{v} \cdot \nabla \underset{\sim}{v} = \underset{\sim}{j} \times \underset{\sim}{B} \qquad \ldots (3)$$

and

$$\partial \rho/\partial t + \nabla \cdot (\rho \underset{\sim}{v}) = 0 , \qquad \ldots (4)$$

where ω_{ci} is the ion-cyclotron frequency corresponding to B_o. In the steady state $\underset{\sim}{E}$, $\underset{\sim}{v}$ and $\underset{\sim}{j}$ are taken to be zero, $\rho = \rho_o$ and $B = B_o \underset{\sim}{n}$, where $\underset{\sim}{n}$ is unit vector along the axis of the cylinder.

Now let $\underset{\sim}{v}$, $\underset{\sim}{j}$, $\underset{\sim}{E}$, ρ and $\underset{\sim}{B}$ denote perturbation quantities, then (1) is unchanged in form, while (2), (3) and (4) become

$$\eta \underset{\sim}{j} = \underset{\sim}{E} + B_o \underset{\sim}{v} \times \underset{\sim}{n} - (B_o/\omega_{ci}) \dot{\underset{\sim}{v}} + (B_o/\mu \rho_o \omega_{ci}) \underset{\sim}{Q} , \qquad \ldots (5)$$

$$\rho_o \dot{\underset{\sim}{v}} = B_o \underset{\sim}{j} \times \underset{\sim}{n} + \underset{\sim}{T}/\mu \qquad \ldots (6)$$

$$\dot{\rho} = - \rho_o \nabla \cdot \underset{\sim}{v} - \nabla \cdot (\rho \underset{\sim}{v}) \qquad \ldots (7)$$

where the dots denote time derivatives and $\underset{\sim}{T}$ and $\underset{\sim}{Q}$ are non-linear terms defined by

$$\underset{\sim}{T} \equiv \mu \underset{\sim}{j} \times \underset{\sim}{B} - \mu \rho_o \underset{\sim}{v} \cdot \nabla \underset{\sim}{v} - \mu \rho \dot{\underset{\sim}{v}} , \qquad \ldots (8)$$

$$\underset{\sim}{Q} \equiv (\mu \rho_o \omega_{ci}/B_o) \underset{\sim}{v} \times \underset{\sim}{B} - \mu \rho_o \underset{\sim}{v} \cdot \nabla \underset{\sim}{v} . \qquad \ldots (9)$$

We shall use cylindrical co-ordinates (r,θ,z) and write $\underset{\sim}{\zeta} \equiv \nabla \times \underset{\sim}{v}$, $v_A^2 \equiv B_o^2/\mu\rho_o$ for the vorticity and Alfvén speed v_A. Then on applying the operators $\frac{\partial}{\partial t}\underset{\sim}{n} \cdot \nabla \times$ to (5), $\underset{\sim}{n}$. and ∇. to (6), and eliminating $\nabla \cdot \underset{\sim}{v}$ and $\underset{\sim}{n} \cdot \underset{\sim}{v} (= v_z)$ from the three resulting equations we obtain

$$\mathcal{L}_1 B_z - (B_o/v_A^2 \omega_{ci}) \ddot{\zeta}_z = \mathcal{R}(T_r - \dot{Q}_\theta/\omega_{ci})/B_o , \qquad \ldots (10)$$

where $\mathcal{L}_1$ and $\mathcal{R}$ are the operators

$$\mathcal{L}_1 \equiv (1 + \frac{\eta}{\mu v_A^2} \frac{\partial}{\partial t}) \nabla^2 - \frac{1}{v_A^2} \frac{\partial^2}{\partial t^2} , \quad \mathcal{R} \equiv \frac{1}{r} \frac{\partial}{\partial r} r .$$

Similarly, elimination of $\nabla \cdot \underset{\sim}{v}$, v_z and j_z from $\underset{\sim}{n} \cdot \nabla \times (6)$, $\underset{\sim}{n} \cdot (6)$ and $\underset{\sim}{n} \cdot \nabla \times (\nabla \times (3))$ gives

$$B_o \mathcal{L}_2 \zeta_z + (v_A^2/\omega_{ci}) \nabla^2 B_z'' = (v_A^2/B_o \omega_{ci}) \mathcal{R} \{ T_r' - Q_r' - \frac{\partial}{\partial r}(T_z - Q_z) \}' - \mathcal{R} \dot{T}_\theta / B_o , \qquad \ldots (11)$$

where

$$\mathcal{L}_2 \equiv \frac{\partial^2}{\partial z^2} + \frac{\eta}{\mu v_A^2} \frac{\partial}{\partial t} \nabla^2 - \frac{1}{v_A^2} \frac{\partial^2}{\partial t^2} .$$

and the dashes denote derivatives with respect to z.

Hence on eliminating ζ_z from (10) and (11) we get the non-linear differential equation

$$\mathcal{L}B_z = F_1 \equiv \frac{1}{B_0}\,\mathcal{R}\left\{\mathcal{L}_2(T_r - \dot{Q}_\theta/\omega_{ci}) + \frac{1}{\omega_{ci}^2}\frac{\partial^2}{\partial t^2}\left[\left\{T'_r - Q'_r - \frac{\partial}{\partial r}(T_z - Q_z)\right\}' - \frac{\omega_{ci}}{v_A^2}\dot{T}_\theta\right]\right\} , \quad \ldots (12)$$

where $\mathcal{L}$ is the linear operator

$$\mathcal{L} \equiv \mathcal{L}_1\mathcal{L}_2 + \frac{1}{\omega_{ci}^2}\frac{\partial^4}{\partial t^2 \partial z^2}\nabla^2 \quad \ldots (13)$$

3. THE SOLUTION TO THE LINEAR WAVE EQUATION

When $F_1 = 0$, (12) reduces to the linear equation $\mathcal{L}B_z = 0$, the wave solution of which has been given in an earlier paper[4]. The axial components of $\underset{\sim}{j}$, $\underset{\sim}{\zeta}$ and $\underset{\sim}{v}$ are found to satisfy the same linear equation. For the moment we shall ignore the resistive damping term, i.e. the term involving η appearing in the operators $\mathcal{L}_1$ and $\mathcal{L}_2$. Let ω be the frequency and k the real wave number, then it is convenient to introduce the following notation:

$$\psi \equiv kz - \omega t\,,\; \Omega \equiv \omega/\omega_{ci}\,,\; k_A \equiv \omega/v_A\,,\; \kappa^2 \equiv k^2 + k_c^2\,,$$

where k_c is a radial wave number to be determined by the boundary condition. Another parameter that arises in the linear theory is

$$f = f(\omega,k,k_c) \equiv (\kappa^2 - k_A^2)/\,k\kappa^2\Omega = k\Omega/(\kappa^2 - k_A^2)\;. \quad \ldots (14)$$

As our main interest here is in torsional and radial waves, we shall restrict attention to waves of zero azimuthal wave number ($m = 0$). This restriction is easily removed, but at the cost of a considerable increase in algebra.

One form of the solution to the (undamped) wave problem just described is

$$\left.\begin{aligned} \underset{\sim}{B} &= -B_0C\left\{kfJ_1\cos\psi\,,\; J_1\sin\psi\,,\; -k_cfJ_0\sin\psi\right\}, \\ \mu\underset{\sim}{j} &= B_0C\left\{kJ_1\cos\psi\,,\; \kappa^2 fJ_1\sin\psi\,,\; -k_cJ_0\sin\psi\right\}, \\ \omega\underset{\sim}{v} &= v_A^2C\left\{\kappa^2 f\cos\psi\,,\; k\sin\psi\,,\; 0\right\}J_1 \;, \end{aligned}\right\} \quad \ldots (15)$$

where C is a non-dimensional constant, and the argument of the Bessel functions J_0 and J_1, omitted for brevity, is k_cr. The equality in (14) is the dispersion relation, which can be written

$$I = I(\omega,k,k_c) \equiv (\kappa^2 - k_A^2)(k^2 - k_A^2) - \kappa^2k^2\Omega^2 = 0\;, \quad \ldots (16)$$

and it has the two roots k_1 and k_2 given by

$$k_\sigma^2(\omega,k_c) = (k_A^2 - \tfrac{1}{2}G_\sigma)/(1 - \Omega^2) - \tfrac{1}{2}k_c^2\;, \quad \sigma = 1,2\;, \quad \ldots (17)$$

where $$G_\sigma \equiv \pm\left\{k_c^4(1-\Omega^2)^2 + 4k_A^4\Omega^2\right\}^{\frac{1}{2}}, \quad \begin{matrix}(+\text{ is } \sigma = 1,\text{ fast wave})\\(-\text{ is } \sigma = 2,\text{ slow wave})\end{matrix}\;.$$

We shall use the notation f_σ to denote the value of $f(\omega,k_\sigma,k_c)$, and when the subscript σ is omitted from k,κ,f and other functions of k to be given later, the quantities in question can be the values either for the slow wave or for the fast wave. From the linear terms in (7) and the last of (15)

$$\omega \nabla \cdot \underset{\sim}{v} = C v_A^2 f k_c \kappa^2 J_0 \cos\psi , \qquad \ldots (18)$$

and

$$\rho = C \rho_0 f k_c (\kappa^2/k_A^2) J_0 \sin\psi . \qquad \ldots (19)$$

Resistivity has two effects on the solution given in (15). The first, and most important, is to replace C by $Ce^{-\varepsilon z}$, where C is now the amplitude factor at $z = 0$, and the absorption coefficient ε is related to the resistivity parameter $\delta \equiv (\eta\omega/\mu v_A^2)$ by

$$\varepsilon = \varepsilon(\omega,k,k_c) = \frac{\delta\kappa^2}{2k(1-\Omega^2)} (1 - 2\Omega^2 k_A^2/G) . \qquad \ldots (20)$$

The second effect is to replace the $\cos\psi$ in B_r and μj_r by $\cos\psi - (\varepsilon/k)\sin\psi$, and the $\sin\psi$ in v_θ by $\sin\psi + (\varepsilon/k)\cos\psi$; to first order in ε this is simply a change in phase.

We shall confine our attention to the case of a cylindrical plasma bounded by an infinitely conducting wall at $r = r_0$, for which the appropriate boundary condition is $B_r = 0$, i.e. from (15)

$$J_1(k_c r_0) = 0 . \qquad \ldots (21)$$

The roots of (21) in ascending order will be denoted by $k_{c1}, k_{c2}, \ldots$, and (15) can be immediately generalized by summing over these radial wave numbers. In the sequel by k_c we shall mean one of these roots, usually the first one, k_{c1}. This completes our survey of the linear theory.

4. AN APPROXIMATE SOLUTION FOR THE NON-LINEAR EQUATION

A linear approximation to (12) can be found by introducing the linear solution (15) into the right-hand side of (12), which is thus assumed to be small. After some algebra we find that

$$\mathcal{L}B_z = F_1 = B_0 k_c (f k_c C e^{-\varepsilon z})^2 \left\{ \alpha_1 \mathcal{R}(J_1 J_0) + 2\kappa^2 \mathcal{R}(J_1^2/k_c r) \right\} \cos 2\psi , \qquad \ldots (22)$$

where

$$\alpha_1 \equiv 6\kappa^2 \left\{ 2k^2(\kappa^2 - k_A^2)/(k_A^2 k_c^2) - 1 \right\} ,$$

and we have taken into account the attenuating effect of resistivity on C. In arriving at (22) it has been assumed that the resistivity is small enough to permit the ommission of terms of the order $(C^2\varepsilon/k)$, which requires that ε/k be $O(C^{1/2})$ or smaller. With this ordering and $C \ll 1$, it follows that

$$1 \gg C \gg C\,\varepsilon/k \gg C^2 \gg C^2\,\varepsilon/k$$

It will be noted that the time independent terms arising in $\underset{\sim}{T}$ and $\underset{\sim}{Q}$ have disappeared, and that the non-linear term F_1 has become a forcing term with a frequency twice that of the basic wave.

To solve (22) we shall adopt the orthogonal set of functions $J_0(k_{cs}r)$, $s = 1, 2, \ldots$ generated by the boundary condition, and expand functions of r in Fourier-Bessel series, e.g.

$$\mathcal{R}(J_1 J_0) = \sum_{s=1}^{\infty} k_{cs}\Upsilon_s\, J_0(k_{cs}r) \,. \qquad \ldots (23)$$

where (after an integration by parts)

$$\Upsilon_s = \int_0^{k_c r_0} x\, J_0(x) J_1(x) J_1(x k_{cs}/k_c)\,dx \Bigg/ \int_0^{k_c r_0} x\, J_0^2(x k_{cs}/k_c)\,dx \,,$$

and k_c is one of $k_{c1}, k_{c2}, \ldots$ The non-dimensional numbers Υ_s depend on both k_c and k_{cs}. It is practical to choose k_c to be the first root, $k_{c1} = 3.38\ldots/r_0$, because the higher radial modes are more rapidly dissipated by resistivity, however we shall continue to write k_c so as to leave the choice open. Table 1 gives values of k_{cs} and Υ_s for $k_c = k_{c1}$, while Table 2 lists the expansion coefficients for the various functions of r that arise in the theory.

TABLE 1

VALUES OF Υ_s and k_{cs}

s	1	2	3	4	5	6
Υ_s	0.176_1	0.438_8	0.009_1	-0.001_5	0.0004	-0.0002
$k_{cs}r_0$	3.832	7.016	10.174	13.324	16.471	19.616

The coefficients given in Table 2 enable us to write (22) in the form:

$$\mathcal{L}B_z = B_0 f\, k_c\, C^2\, e^{-2\varepsilon z} \cos 2\psi \sum_{s=1}^{\infty} \ell_s\, J_0(k_{cs}r) \,. \qquad \ldots (24)$$

where

$$\ell_s \equiv 2f\kappa^2 k_{cs}\Upsilon_s \left\{6k^2(\kappa^2 - k_A^2)/k_A^2 - k_c^2 - \tfrac{1}{2}\, k_{cs}^2\right\}. \qquad \ldots (25)$$

To solve (24) first replace $e^{-\varepsilon z}\cos 2\psi$ by $\exp\left\{2i[(k + i\varepsilon)z - \omega t]\right\}$, then a particular integral of the resulting equation is easily found to be

$$B_Z = B_O f\ k_c\ C^2\ e^{2i\{(k+i\varepsilon)z-\omega t\}} \sum_{s=1}^{\infty} a_s\ J_O(k_{cs}r)\ , \qquad \ldots (26)$$

where

$$a_s \equiv \ell_s(I_s - iE_s)/P_s^2\ , \qquad P_s \equiv (I_s^2 + E_s^2)^{\frac{1}{2}}\ , \qquad \ldots (26a)$$

$$I_s \equiv I(2\omega, 2k, k_{cs}) = 4(k^2 - k_A^2)(4k^2 - 4k_A^2 + k_{cs}^2) - 16\ k^2(4k^2 + k_{cs}^2)\Omega^2\ , \qquad \ldots (27)$$

and

$$E_s \equiv 2\delta(4k^2 + k_{cs}^2)(8k_A^2 - 8k^2 - k_{cs}^2) - 8\varepsilon k\left\{8k_A^2 - 8k^2 - k_{cs}^2 + 4\Omega^2(8k^2 + k_{cs}^2)\right\}. \qquad \ldots (28)$$

To the real part of (26) we must add the complementary function, i.e. the solution to $\mathcal{L}B_Z = 0$ at a frequency of 2ω. This is

$$B_Z = B_O C^2 \sum_{s=1}^{\infty} \sum_{\sigma=1}^{2} e^{-\varepsilon_{\sigma s} z}\ k_{cs} f_{\sigma s} A_{\sigma s} \cos(\psi_{\sigma s} - \xi_{\sigma s})\ J_O(k_{cs}r)\ , \qquad \ldots (29)$$

where $A_{\sigma s}$, $\xi_{\sigma s}$ are constants, $\psi_{\sigma s} = k_{\sigma s}z - 2\omega t$, $k_{\sigma s} \equiv k_\sigma(2\omega\ , k_{cs})$, $f_{\sigma s} \equiv f(2\omega\ , k_{\sigma s}\ , k_{cs})$ and $\varepsilon_{\sigma s} \equiv \varepsilon(2\omega\ , k_{\sigma s}\ , k_{cs})$. Here we are simply doubling ω and replacing k_c by k_{cs} in (17), (14) and (20) . Thus equation (29) contains the fast and slow waves, propagating in the positive z-direction, that would occur naturally in the plasma if the exciter had a frequency of 2ω.

Our complete solution, viz. the sum of the linear solution, i.e. B_Z in (15), the real part of (26) and (29), can be written

$$B_Z = B_O C \left\{ f\ k_c e^{-\varepsilon z} \sin\psi\ J_O(k_c r) + C \sum_{s=1}^{\infty} \left[f k_c e^{-2\varepsilon z} \frac{\ell_s}{P_s} \cos(2\psi - \varphi_s) + \sum_{\sigma=1}^{2} f_{\sigma s} k_{cs} e^{-\varepsilon_{\sigma s} z} A_{\sigma s} \cos(\psi_{\sigma s} - \xi_{\sigma s}) \right] J_O(k_{cs} r) \right\}, \qquad \ldots (30)$$

where

$$\varphi_s \equiv \tan^{-1}(E_s/I_s)\ . \qquad \ldots (31)$$

To evaluate $A_{\sigma s}$ and $\xi_{\sigma s}$ we need boundary conditions, at $z = 0$ say, this is discussed below in section 5. Table 1 shows that it would be sufficiently accurate to terminate the series in (30) at $s = 2$.

TABLE 2

FOURIER-BESSEL COEFFICIENTS

F(r)	Coefficient	F(r)	Coefficient
$\mathcal{R}(J_1\ J_O)$	$k_{cs}\Upsilon_s$	$\mathcal{R}\ \frac{\partial}{\partial r}\mathcal{R}(J_1\ J_O)$	$-k_{cs}^3\Upsilon_s$
$\mathcal{R}(J_1^2/k_{cs}r)$	$2k_{cs}\Upsilon_s(1 - k_{cs}^2/4k_c^2)$	$\mathcal{R}\ \frac{\partial}{\partial r}\mathcal{R}(J_1^2/k_c r)$	$-2k_{cs}^3\Upsilon_s(1 - k_{cs}^2/4k_c^2)$
J_1^2	$2k_c\Upsilon_s(1 - k_{cs}^2/2k_c^2)/k_{cs}$	$\mathcal{R}(J_O'\ J_O'')$	$-k_{cs}\Upsilon_s(1 - k_{cs}^2/2k_c^2)$

5. TRANSVERSE COMPONENTS

An analysis similar to that given in section 2 yields the non-linear differential equation

$$\mathcal{L}(\mu j_z) = F_2 \equiv B_o^{-1}\left\{\mathcal{L}_1 \mathcal{R}[(\dot{T}_r - \dot{Q}_r)/\omega_{ci} - T_\theta]' - \frac{1}{\omega_{ci}}\nabla^2 \mathcal{R}\frac{\partial}{\partial t}[T_r + (\dot{T}_\theta - \dot{Q}_\theta)/\omega_{ci}]'\right\} \quad \ldots (32)$$

And on using (15) to find an approximate value for the right-hand side of (32) one finds

$$F_2 = 2\,B_o k_c^2\, C^2\, e^{-2\varepsilon z}\, f\, k_c \cos 2\psi$$

$$\times\left\{(\kappa^2/k_A^2)\left[12k^2(k_A^2 - k^2) + (3k^2 - 2k_A^2)\mathcal{R}\frac{\partial}{\partial r}\right]\mathcal{R}(J_o J_1) + k_c^2(-4k^2 + \mathcal{R}\frac{\partial}{\partial r})\mathcal{R}(J_1 J_o'')\right\} ,$$

which, with the help of Table 2 can be expressed

$$F_2 = -\,B_o k_c\, C^2\, e^{-2\varepsilon z} \cos 2\psi \sum_{s=1}^{\infty} \lambda_s\, J_o(k_{cs} r) , \quad \ldots (33)$$

with

$$\lambda_s \equiv 2 f k_{cs} \Upsilon_s \left\{(4k^2 + k_{cs}^2)[3\kappa^2(k^2 - k_A^2)/k_A^2 + k_c^2 - \tfrac{1}{2}k_{cs}^2] + \kappa^2 k_{cs}^2\right\} . \quad \ldots (34)$$

And now if we proceed to solve (32) by the method used for (24) we will find the solution

$$\mu j_z = -\,B_o C\left\{k_c e^{-\varepsilon z}\sin\psi\, J_o(k_c r)\right.$$

$$\left. + C\sum_{s=1}^{\infty}\left[k_c e^{-2\varepsilon z}\frac{\lambda_s}{P_s}\cos(2\psi - \varphi_s) + \sum_{\sigma=1}^{2} k_{cs} e^{-\varepsilon_{\sigma s} z} A_{\sigma s}\cos(\psi_{\sigma s} - \xi_{\sigma s})\right] J_o(k_{cs} r)\right\} \quad \ldots (35)$$

For the zeroth azimuthal mode $\mathcal{R}B_r = -\,B_z'$ and $\mathcal{R}B_\theta = \mu j_z$. Hence from (30) and (35) we deduce that

$$B_r = B_o C\left\{-kfe^{-\varepsilon z}(\cos\psi - \frac{\varepsilon}{k}\sin\psi)\, J_1(k_c r)\right.$$

$$\left. + C\sum_{s=1}^{\infty}\left[\frac{2fk_c k \ell_s}{k_{cs} P_s}e^{-2\varepsilon z}\sin(2\psi - \varphi_s) + \sum_{\sigma=1}^{2} f_{\sigma s} k_{\sigma s} e^{-\varepsilon_{\sigma s} z} A_{\sigma s}\sin(\psi_{\sigma s} - \xi_{\sigma s})\right] J_1(k_{cs} r)\right\} \quad \ldots (36)$$

$$B_\theta = -\,B_o C\left\{e^{-\varepsilon z}\sin\psi\, J_1(k_c r)\right.$$

$$\left. + C\sum_{s=1}^{\infty}\left[k_c e^{-2\varepsilon z}\frac{\lambda_s}{k_{cs} P_s}\cos(2\psi - \varphi_s) + \sum_{\sigma=1}^{2} e^{-\varepsilon_{\sigma s} z} A_{\sigma s}\cos(\psi_{\sigma s} - \xi_{\sigma s})\right] J_1(k_{cs} r)\right\} \quad \ldots (37)$$

on ignoring a small phase shift $2\varepsilon/k$ in the B_r df wave. If the transverse components of the current are required, they can be deduced from $\mu j_r = -\,B_\theta'$ and $\mu j_\theta = B_r' - \partial B_z/\partial r$.

To evaluate the constants in these equations for the components of $\underset{\sim}{B}$ we must employ boundary conditions. For example suppose the exciter is at $z = 0$, and produces waves symmetrically disposed about this plane, then B_z will be zero at $z = 0$ and for all time. By (30) this requires that $\xi_{\sigma s} = \varphi_s$ and

$$\frac{\Gamma k_s \ell_s}{k_{cs} P_s} + \sum_{\sigma=1}^{2} f_{\sigma s} A_{\sigma s} = 0 .$$

Further if the exciter causes a radial current to flow that contains no double-frequency terms, B'_θ will be zero at $z = 0$ at all t. Hence from (37)

$$\frac{2kk_c\lambda_s}{k_{cs} P_s} + \sum_{\sigma=1}^{2} k_{\sigma s} A_{\sigma s} = 0 .$$

This pair of equations gives

$$A_{\sigma s} = \frac{k_c}{k_{cs} P_s} \frac{2k\lambda_s f_{\alpha s} - \ell_s \Gamma k_{\alpha s}}{k_{\alpha s} f_{\sigma s} - k_{\sigma s} f_{\alpha s}} , \quad \begin{pmatrix} \sigma = 1,2 \\ \alpha = 2,1 \end{pmatrix} \qquad \ldots (38)$$

These values are appropriate for the usual method of exciting torsional waves by means of a radial discharge between a circular electrode on the tube wall and a point electrode on the axis[2]. If the plasma is compressed at $z = 0$ by a current loop, then $B_r = 0$ at $z = 0$ for all t; then from (36) and the symmetry condition on B_z we find that in place of (38)

$$A_{\sigma s} = \frac{k_c \, f \, \ell_s}{f_{\sigma s} k_{cs} P_s} \frac{2k - k_{\alpha s}}{k_{\alpha s} - k_{\sigma s}} , \quad \begin{pmatrix} \sigma = 1,2 \\ \alpha = 2,1 \end{pmatrix} .$$

Well away from the exciter the differential damping rates for the df waves in B will suppress some waves compared with others, but it is not easy to draw general conclusions from the analytic forms of ε and $\varepsilon_{\sigma s}$. At low values of Ω $\varepsilon_{2s} < \varepsilon_{1s} < 2\varepsilon$, provided $s = 1$, so that the slow, natural df wave will predominate at large z.

6. DENSITY FLUCTUATIONS

On eliminating $\nabla \cdot \underset{\sim}{v}$ from $\nabla \cdot (6)$ and $\frac{\partial}{\partial t}(7)$ we find

$$\ddot{\rho} = (B_0/\mu)\nabla^2 B_z + \chi , \qquad \ldots (39)$$

with

$$\chi \equiv - \left\{ \frac{\partial}{\partial t} \nabla \cdot (\rho \underset{\sim}{v}) + \frac{1}{\mu} (\mathcal{R}\, T_r + T'_z) \right\} .$$

An expression for χ can be calculated from (8), (15) and (19), with the result:

$$\chi \equiv \frac{1}{\mu} B_0^2 C^2 e^{-2\varepsilon z} \, f \, k_c \, (\alpha_2 \cos 2\psi - \alpha_3) , \qquad \ldots (40)$$

with

$$\alpha_2 \equiv \frac{k_c}{k\Omega} \left\{ k^2 J_1^2 + \tfrac{1}{2} k_c \mathcal{R}(J_1 J_0'') \right\} + \frac{\kappa^4 f}{k_A^2} \mathcal{R}(J_0 J_1) ,$$

and

$$\alpha_3 \equiv \frac{k_c^2}{2k\Omega} \left\{ 1 - \frac{2k^2(\kappa^2 - k_A^2)}{k_A^2} \right\} \mathcal{R}(J_1 J_0'') .$$

The secular term α_3 in (40) appears at first to be an embarrassment, but it should be associated with the steady state rather than with the perturbed state. It will then result in a modification of order C^2 to the assumed steady state, $\underset{\sim}{v}_0 = 0, \rho = \rho_0$ and $\underset{\sim}{B} = B_0\underset{\sim}{n}$, and when this new steady state is perturbed, terms of order C^3 will result from the secular term. We can therefore safely drop it from (40).

Using Table 2 to express a_2 as a series and calculating $\nabla^2 B_z$ from (30), we can express the right-hand side of (39) as a series of Bessel functions. Then integrating this we get

$$\rho = \rho_0 C\, k_A^{-2} \Bigg\{ f\, k_c e^{-\varepsilon z}[\kappa^2 \sin\psi + 2\varepsilon k \cos\psi]\, J_0(k_c r)$$

$$+ \tfrac{1}{4} C \sum_{s=1}^{\infty} \Bigg[e^{-2\varepsilon z} f\, k_c(4k^2 + k_{cs}^2) \left\{ \frac{\ell_s}{P_s} \cos(2\psi - \varphi_s) - q_s \cos 2\psi \right\}$$

$$+ \sum_{\sigma=1}^{2} f_{\sigma s} k_{cs} \kappa_{\sigma s}^2 e^{-\varepsilon_{\sigma s} z} A_{\sigma s} \cos(\psi_{\sigma s} - \xi_{\sigma s}) \Bigg]\, J_0(k_{cs} r) \Bigg\}, \qquad \ldots (41)$$

where

$$q_s \equiv k_{cs}\gamma_s \left\{ \frac{k_c^2 - \tfrac{1}{2}k_{cs}^2}{2k\Omega\, k_{cs}^2} + \frac{\kappa^4 f}{k_A^2(4k^2 + k_{cs}^2)} \right\}.$$

7. RESONANCE FREQUENCIES

The possibility that the amplitude of the df wave could be especially large for certain wave numbers will now be considered. From (30) and (35) such resonances will occur at minimum values of P_s, defined in (26a). As E_s^2 is $O(\varepsilon^2)$, and therefore must be small if the wave is to be propagated any distance, minimum values of P_s will lie near the zeros of I_s. Thus from (16) and (27) we should seek values of k that make both I and I_s zero. Combining these quadratics in k^2 we find

$$k^2 = k_A^2\, U/(U + S) = k_A^2(U - N)/U, \qquad \ldots (42)$$

where

$$U \equiv 4k_c^2 - k_{cs}^2 + \Omega^2(k_{cs}^2 - 16k_c^2) + 12k_A^2\Omega^2,$$

$$S \equiv 4\Omega^2 \left\{ 12k_A^2 + k_{cs}^2 - k_c^2 + \Omega^2(4k_c^2 - k_{cs}^2) \right\},$$

and

$$N \equiv \Omega^2 k_A^{-2} \left\{ 5k_A^2(k_{cs}^2 - 4k_c^2) + 3(4k_A^2 - k_c^2 k_{cs}^2) \right\}.$$

The equation $U^2 = (U + S)(U - N)$ is the condition that $I = 0$ and $I_s = 0$ have one root in common. It is a quintic in ω^2, and for the special case $k_c = k_{c1}$ it can be written

$$x\left\{16x^4 - 40x^3 + (13 - 4Z^{-1})x^2 + 5(1 - Z)x - Z\right\} = 0, \qquad \ldots (43)$$

where $x \equiv (\omega/k_c v_A)^2$ and $Z \equiv (\omega_{ci}/k_c v_A)^2$. To make further progress we need to assign a value to Z and then to look for the real positive roots of (43). The negative sign for the term of zero degree (-Z) ensures that (43) has at least one positive real root.

One conclusion of a general nature can be deduced from the first of (42). It is that as S is positive for $s = 1$ and 2 (see Table 1, which shows that higher values of s can be ignored), then $k < k_A$. By (17) this inequality means that these resonances can occur only if both the basic wave and the df wave are fast waves (see Fig.1), except of course near the root $x = 0$, i.e. $\Omega = 0$ of (43). In this case (42) reduces to $k^2 = k_A^2$; this slow wave resonance is investigated further in the next section.

Let R denote the ratio of the amplitude of the df wave to the amplitude of the basic wave, then at a resonance R will be large. However the theory given in sections 4 and 5 depends on R being small compared with unity, and so it will fail at a resonance unless the resistivity is sufficient to make $R < 1$.

8. THE RESONANCE NEAR $\Omega = 0$

(a) The Slow Wave

To study this resonance in more detail we shall assume that Ω^2 and ε are comparable small numbers, and that higher order terms like $\varepsilon\Omega^2$ can be neglected. From (14) we find that $k^2 \approx k_A^2$ and $f \approx k\Omega/k_c^2$, then it follows from (26a) and (27) that to this order $I_s \approx w_1\Omega^2$, and $E_s \approx w_2\varepsilon$, where

$$w_1 \equiv 4k_A^2 \left\{ k_{cs}^2(k_A^2 - 3k_c^2)/k_c^2 - 16k_A^2 \right\}, \qquad \ldots (44)$$

and

$$w_2 \equiv 4k_A k_{cs}^2(2k_c^2 - k_{cs}^2 - 2k_A^2)/(k_A^2 + k_c^2) ,$$

so that

$$P_s = (w_1^2\Omega^4 + w_2^2\varepsilon^2)^{1/2} . \qquad \ldots (45)$$

Also from (25), (34) and (31)

$$\begin{Bmatrix} \ell_s \\ \\ \lambda_s \end{Bmatrix} \approx C\, e^{-\varepsilon z} \left(\frac{k_A\Omega}{k_c}\right)\left(\frac{k_{cs}}{k_c}\right) \gamma_s \begin{Bmatrix} (k_c^2 + k_A^2)(10k_c^2 - k_{cs}^2) \\ \\ (2k_A^2 + k_{cs}^2)(4k_c^2 - k_{cs}^2) \end{Bmatrix} \qquad \ldots (46)$$

and

$$\varphi_s \approx \tan^{-1}\left(\frac{w_2\varepsilon}{w_1\Omega^2}\right) \qquad \ldots (47)$$

First notice that if $\varepsilon = 0$, i.e. if resistivity is neglected, then both (ℓ_s/P_s) and (λ_s/P_s) behave like $1/\Omega$ near $\Omega = 0$, so that the amplitudes of the df waves in (36) and (37) tend to infinity as Ω tends to zero. However equation (44) shows that this surprising singular behaviour is eliminated by the presence of a small amount of resistivity, for then these amplitudes tend to zero with Ω. The theory given in section 5 is slightly inconsistent in that some small phase shift of order ε/k has been neglected in the df wave, whereas φ_s has been retained. We have done this because (47) shows φ_s to be quite large if $|w_1\Omega^2| < |w_2\varepsilon|$ and equal to 90^0 in the limit $\Omega = 0$.

In experimental work with torsional waves one frequently finds that $k_c \gg k_A$, in which case

$$\frac{\lambda_s}{P_s} \approx - C \,\frac{\Upsilon_s}{12}\left(\frac{k_{cs}}{k_A\Omega}\right)\left(4 - \frac{k_{cs}^2}{k_c^2}\right) , \qquad \dots (48)$$

assuming $|w_1\Omega^2| \gg |w_2\varepsilon|$.

From (44) resonance occurs at $|w_1|\Omega^2 = |w_2|\varepsilon$, when $P_s = \sqrt{2}\, w_1\Omega^2$, and an equation like (48) holds if $k_c \gg k_A$, except that Υ_s is replaced by $\Upsilon_s/\sqrt{2}$. The phase lag φ_s is then 45^0.

It is interesting to take the limit $\Omega \to 0$ in (41). If $k_c \gg k_A$, the result for the forced wave is

$$\rho = \rho_0 \,\frac{C^2 e^{-2\varepsilon z}}{24\, k_A^2} \cos 2\psi \sum_{s=1}^{\infty} \Upsilon_s \frac{k_{cs}^2}{k_c^2}\,(k_c^2 + 2k_{cs}^2)\;\; J_0(k_{cs}r),$$

so that double-frequency density fluctuations occur even when such non-linear effects are absent in the magnetic field components.

(b) The Fast Wave

For the fast wave (14) gives $k \approx (k_A^2 - k_c^2)^{\frac{1}{2}}$ and $f \approx - k_c^2/(kk_A^2\Omega)$, near $\Omega = 0$. Then (25) and (34) yield

$$\left\{\begin{matrix} \ell_s \\ \\ \lambda_s \end{matrix}\right\} \approx 2\,C\,\frac{k_c^2 k_{cs}\,\Upsilon_s}{k\,\Omega\,k_A^2} \left\{\begin{matrix} (k_c^2 + \frac{1}{2}k_{cs}^2)k_A^2 \\ \\ (8k_c^2 + k_{cs}^2)k_A^2 - 8k_c^4 + \frac{1}{2}k_{cs}^2 \end{matrix}\right\} ,$$

while from (31) and (26a), $\varphi_s \approx 0$, and $P_s \approx I_s = 4k_c^2(4k_c^2 - k_{cs}^2)$. Thus for this wave the resonance at $\Omega = 0$ is not suppressed by resistivity.

9. EXAMPLES

I. To illustrate the effects discussed in sections 7 and 8 we have chosen the values - typical of some of the Culham Laboratory experiments - $v_A = 5 \times 10^6$ cms/sec, $\omega_{ci} = 10^6$ rads/sec, $k_c = k_{c1} = 0.4$ cm^{-1}, $\delta = 0.05\, k_A$. Fig.1 shows the resulting dispersion curves for the fast and slow waves. The line OAB of constant phase velocity intersects the fast wave curve at points A and B such that ω and k at B are twice the values at A. Thus I and I_1 are both zero at these points and the resonance frequency is therefore about 3.2 ω_{ci}. This value is confirmed by (43) in which Z has the value 0.25. This fast wave resonance is also shown in Fig.2, the vertical ordinate of which is $A = (2k_1 f\ell_1/P_1)$. From (36) we see that B_oC^2 times A will give the amplitude of the driven df wave in the first radial mode in the component B_r. (Recall that C is the ratio of the basic B_θ wave amplitude to B_o). Fig.3 shows values of A when ω_{ci} is increased to 10^7 rads/sec., the other quantities being kept constant.

The resonance at cut-off, i.e. at $k_A = k_c$ revealed in both these figures, is due to the fact that as $k_A \to k_c, k \to 0$, $f \to \infty$ and $\ell_s \to \infty$. Resistivity does not suppress this resonance, so experiments near cut-off may have some chance of verifying this part of the above theory.

II. In Fig.4, for the same values of v_A, ω_{ci} and k_c given above, but with $\delta = 0$ and $0.25\ k_A$, we have plotted the ratio (λ_1/P_1) for the slow wave; (37) shows that B_oC^2 times this number gives the amplitude of the driven df wave in the first radial mode of B_θ. The resonance at $\Omega = 0$, and its damping by resistivity is clearly shown. The equation following (48) relating Ω and ε gives a resonance frequency in agreement with that shown in the figure. The figure also shows a resonance at $\Omega = 1$, which is suppressed by resistivity. This resonance arises because at $\Omega \to 1$, $k \to \infty$ for the slow wave, and so $\lambda_s \to \infty$.

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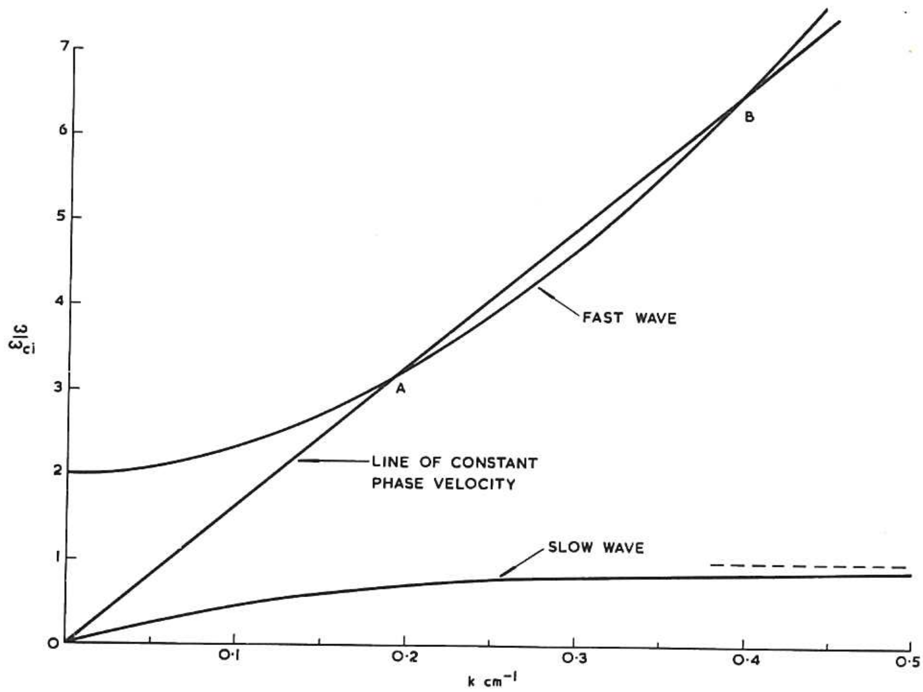
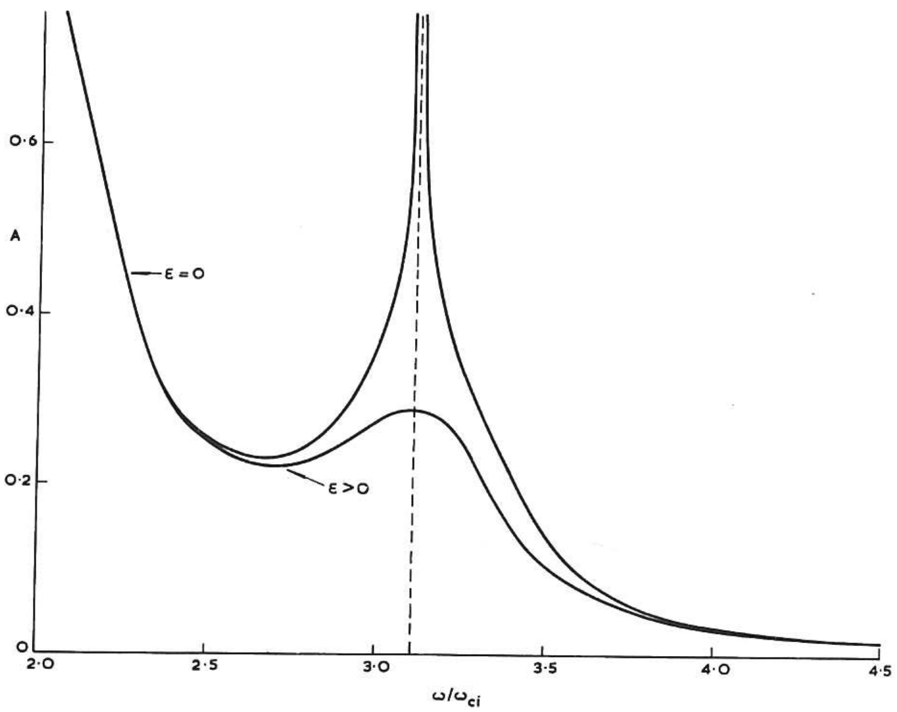
11. REFERENCES

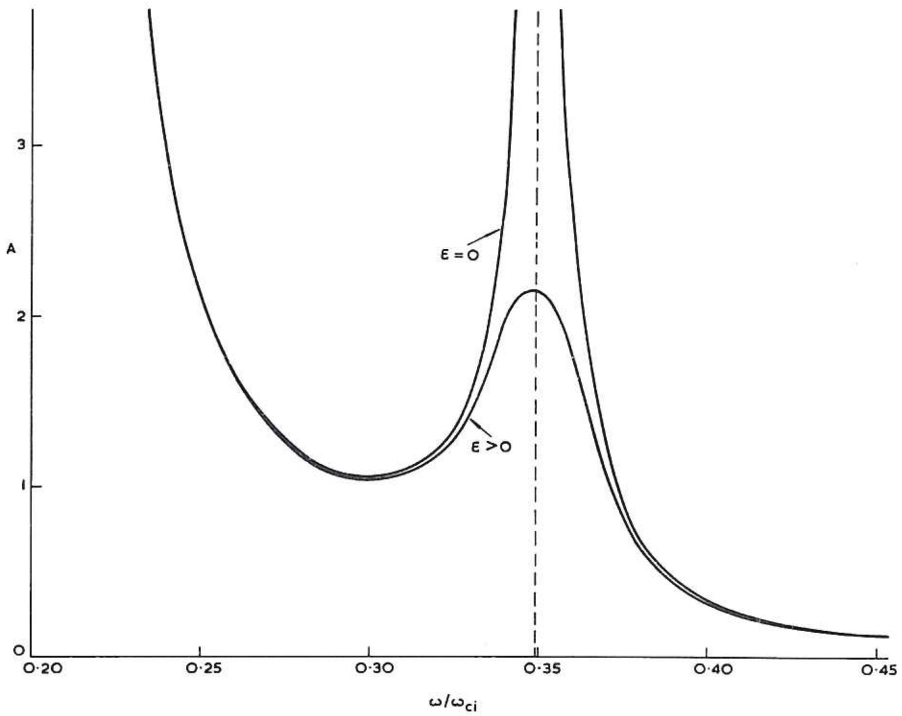
1. JEPHCOTT, D.F. and STOCKER, P.M., J. Fluid Mech., 13 (4) pp.587 - 596, 1962.
2. JEPHCOTT, D.F. Proc. Roy. Soc. (To be published)
3. SPITZER, L., Physics of Fully Ionized Gases, Interscience Pub. Inc., N.Y.
4. WOODS, L.C., J. Fluid Mech., 13, pt.4, pp.570 - 586, 1962.



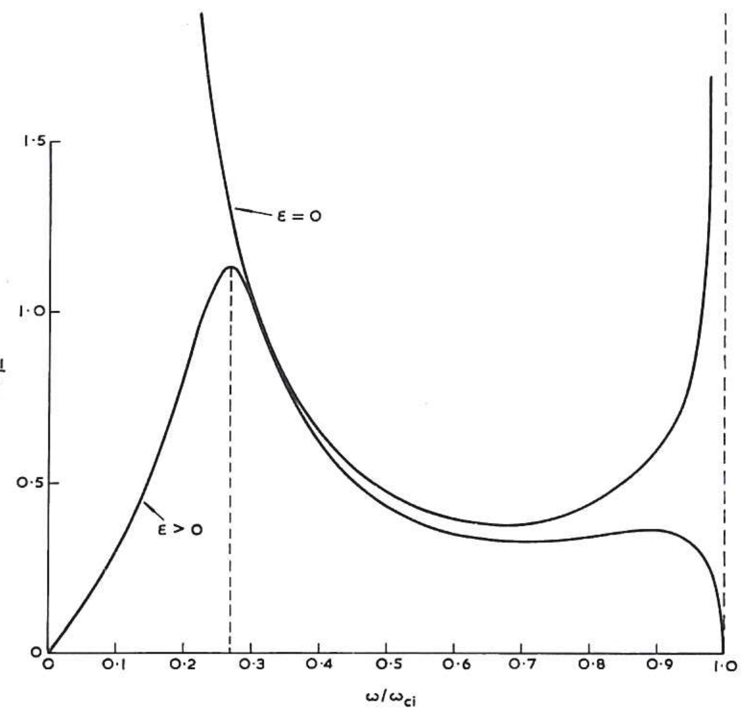
CLM-P41 Fig. 1
The dispersion curves : $\omega_{ci} = 10^6$



CLM-P41 Fig. 2
Amplitude factor, fast wave : $\omega_{ci} = 10^6$



CLM-P 41 Fig. 3
Amplitude factor, fast wave: $\omega_{ci} = 10^7$



CLM-P 41 Fig. 4 Amplitude factor, slow wave