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STABILITY OF A HIGH-β TOKAMAK TO UNIFORM VERTICAL DISPLACEMENTS

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Abstract

A simple model of a toroidal, high-β quasi-uniform current model of a tokamak with elliptic cross-section, is investigated for stability against uniform vertical displacements. On the basis of asymptotic theory it is shown that for high-β / toroidal effects to be important β must be of order the equilibrium limit. For a given β and ellipticity the mode can always be stabilised by a conducting wall placed sufficiently close to the plasma. Although the plasma is displaced rigidly the toroidicity leads to both $m = 1$ and $m = 2$ harmonics in the perturbed vacuum field.

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1. INTRODUCTION

In this paper we consider the stability of a toroidal elliptic cross-section model of a tokamak maintained in equilibrium by a vacuum field, against uniform vertical displacements ($n=0$, $m=1$). The interest in this mode has arisen through the necessity to design field configurations or feedback systems [1] to produce tokamak equilibria. Several authors have investigated analytically the "$n=0$", $m=1$ mode in a straight tokamak. The toroidal problem, however, has been studied numerically. The object of the present work is to investigate analytically the stability of a toroidal, high-β, quasi-uniform current model of tokamak, with elliptic cross-section, against this mode.

We begin by briefly reviewing the earlier work. Rutherford [2] has examined the straight uniform current elliptic cross-section plasma and finds it to be unstable to rigid shifts in the direction of the larger semi-axis, b say. More precisely, the model is unstable to the $m=1$, $k=0$ mode, for $b>a$. In their recent paper, Laval et al. [3] have investigated the same model in the presence of a conducting wall. They find this mode to be stabilised by a wall placed at or within a critical distance from the plasma. This distance is a function of the ellipticity. The above results are confirmed numerically by Lackner and MacMahon [4], who also find that for a sufficiently tight torus stability pertains for $b/a < 1.25$, even in the absence of a wall. This effect, which is obviously toroidal, has also been found by Okabayashi and Sheffield [5]. Recently, Wesson and Sykes [6] have studied numerically the equations of motion for $n=0$ perturbations of an approximately elliptical cross-section plasma with a quasi-uniform current and find the motion to be vertical but non-uniform. It is of interest, however, to consider a uniform displacement, since for this mode, the investigation of a toroidal high-β system can be carried through analytically using asymptotic expansions.

2. MODEL AND EQUILIBRIUM

We consider an axisymmetric plasma of elliptic cross-section containing both a poloidal and toroidal magnetic field. Introducing cylindrical coordinates, R, φ, Z based on the axis of symmetry, the centre of the ellipse is taken to be at the point $R = R_o$, $Z = 0$ (see Fig. 1). If we further introduce the cartesian coordinates x, y such that

$$x \equiv R - R_o \quad \text{and} \quad y \equiv Z ,$$

then the equation for the plasma boundary can be expressed as

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} , \tag{1}$$

where a, b are the semi-axes of the ellipse. External to the plasma it is assumed that there exists a vacuum magnetic field capable of maintaining the required pressure balance at the surface.

Within the plasma the magnetic field $\underline{B}$ can be written in the form

$$\underline{B} = \frac{F(\psi)}{R} \underline{e}_\varphi + \frac{\underline{e}_\varphi \times \nabla\psi}{R} , \tag{2}$$

where $\underline{e}_\varphi$ is the unit vector in the toroidal direction, ψ the poloidal flux, and $F(\psi)$ the current stream function. It is well known that ψ satisfies the equation

$$R \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial Z^2} + FF'(\psi) + R^2 p'(\psi) = 0 . \tag{3}$$

In the present paper we consider its solution for the choices

$$\left.\begin{aligned} F(\psi) &= \frac{\alpha\psi}{a} \\ \text{and} \qquad p(\psi) &= \frac{\eta}{2R_o^{\,2}a^2}(\psi_B^2 - \psi^2) , \end{aligned}\right\} \tag{4}$$

the latter ensuring that the pressure vanishes at the plasma boundary $\psi = \psi_B$. The dimensionless quantities η and α are free parameters. Using the forms defined in Eq. (4), Eq. (3) becomes

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{R_o + x}\frac{\partial\psi}{\partial x} + \frac{\partial^2\psi}{\partial y^2} + \frac{1}{a^2}\left[\alpha^2 - \eta\left(1 + \frac{x}{R_o}\right)^2\right]\psi = 0 . \tag{5}$$

Eq. (5) is now solved by an expansion in the inverse aspect ratio, $\epsilon = a/R_o$. Choosing η and α such that $\alpha^2 \sim \eta \sim 1$ and that $\eta - \alpha^2 \sim \epsilon$, and expanding ψ in the form

$$\psi = \psi_o(x,y) + \psi_1(x,y) + \ldots. , \tag{6}$$

then Eq. (5) can be solved order-by-order. If we set ψ_o = constant = ψ_B, then the leading-order equation is trivially satisfied, and to first-order,

$$\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial y^2} + \frac{1}{a^2}\left[\alpha^2 - \eta - 2\eta\frac{x}{R_o}\right]\psi_o = 0 . \tag{7}$$

It is straightforward to show that the solution of this equation for which

$\psi_1 = 0$ at the boundary, is

$$\psi_1 = -\psi_0\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\left\{\frac{\eta - \alpha^2}{2(1 + \frac{a^2}{b^2})} + \frac{\eta\epsilon x}{a(3 + \frac{a^2}{b^2})}\right\}, \tag{8}$$

a form similar to that obtained by Laval et al. [7] for a different model.

Defining the total β to be given by

$$\beta = \frac{2\int p dS}{A B_{\varphi o}^2}, \tag{9}$$

where the pressure is integrated over the minor cross-section, A is the cross-sectional area, and $B_{\varphi o}$ is the zero-order toroidal field, we obtain to leading-order,

$$\beta = \frac{1}{2} \frac{\eta - \alpha^2}{1 + \frac{a^2}{b^2}}. \tag{10}$$

Similarly, the total toroidal current I is given by

$$I = \frac{\pi \psi_o}{R_o} \frac{b}{a} (\eta - \alpha^2). \tag{11}$$

The safety factor for an equivalent cylinder of radius b is defined to be

$$q_o = \frac{2\pi\epsilon B_{\varphi o} b}{I}, \tag{12}$$

and can be expressed in the form

$$q_o = \frac{2\epsilon\eta^{\frac{1}{2}}}{\eta - \alpha^2}. \tag{13}$$

Now it is well-known that a second magnetic axis can enter the plasma when β approaches the equilibrium limit β_c [8]. In the model discussed here this occurs when the ratio $\eta/(\eta - \alpha^2)$ approaches the value

$$\left(\frac{\eta}{\eta - \alpha^2}\right)_c = \frac{1}{2\epsilon} \frac{3 + \frac{a^2}{b^2}}{1 + \frac{a^2}{b^2}}. \tag{14}$$

It follows from Eqs. (10), (13) and (14), that the limiting-β can be written as

$$\beta_c = \frac{\epsilon}{q_o^2} \frac{3 + \frac{a^2}{b^2}}{\left(1 + \frac{a^2}{b^2}\right)^2}, \tag{15}$$

which, allowing for differences in notation, is of the same form as that obtained by Laval et al. [9] . The equation for the flux-surfaces can now be written as

$$\psi_1 = - \frac{a R_o I}{2\pi b(1 + \frac{a^2}{b^2})} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right)\left(1 + \frac{\beta}{\beta_c}\frac{x}{a}\right). \tag{16}$$

Fig. 2 shows a typical plot of the flux-surfaces for the case $\beta/\beta_c = \frac{1}{4}$ and $b/a = 2.0$, the magnetic axis being displaced outwards by an amount 0.12a. The toroidal current density for the present model is quasi-uniform and shows the usual phenomenon of current reversal. This occurs when β approaches the value

$$\beta^+ = \frac{1 + \frac{a^2}{b^2}}{3 + \frac{a^2}{b^2}} \beta_c \ . \tag{17}$$

Having described the essential properties of the plasma in our equilibrium, we now consider the behaviour at the interface. This is most conveniently discussed in terms of elliptic coordinates. Assuming $b > a$, we take

$$R - R_o = c \sinh u \cos v \quad \text{and} \quad Z = c \cosh u \sin v , \tag{18}$$

where c is a constant. Denoting the plasma surface by $u = u_o$, then c and u_o satisfy

$$c \cosh u_o = b \quad \text{and} \quad c \sinh u_o = a . \tag{19}$$

For the case $b < a$ we must take the definitions

$$R - R_o = c \cosh u \cos v \quad \text{and} \quad Z = c \sinh u \sin v , \tag{20}$$

where

$$c \cosh u_o = a \quad \text{and} \quad c \sinh u_o = b . \tag{21}$$

Apart from this minor change the analysis is the same as that for the case $b > a$. Thus the results obtained cover both cases. Deriving B_R and B_Z from Eq. (16) it is simple to show that the poloidal magnetic field B_p at the surface is given by

$$B_p(v) = \frac{I}{\pi b(1 + \frac{a^2}{b^2})} \left(1 + \frac{\beta}{\beta_c} \cos v\right) \left(\cos^2 v + (\frac{a}{b})^2 \sin^2 v\right)^{\frac{1}{2}} . \tag{22}$$

The present model having no skin-current, the magnetic field at $\psi = \psi_B$

must be continuous, and since $p = 0$ at the interface, this implies that the pressure-balance condition is automatically satisfied. In principle we could obtain the vacuum field required to maintain the plasma in equilibrium by solving for the vacuum flux-function and matching solutions at the interface using the continuity of the poloidal field (see for example [10]). For the present problem, however, this is unnecessary as the subsequent stability calculation depends only on the poloidal-vacuum field at the surface and not on a complete knowledge of the vacuum field.

For completeness we show the relationship between q_o and the true safety-factor at the surface, $q(u_o)$. The latter quantity is defined to be

$$q(u_o) = \left.\frac{\Delta\varphi}{\Delta v}\right|_{\delta v = 2\pi} , \qquad (23)$$

which to leading-order becomes

$$q = \frac{\epsilon B_{\varphi o}}{2\pi} \frac{b}{a} \int_o^{2\pi} \frac{\sqrt{\cos^2 v + \frac{a^2}{b^2} \sin^2 v}\, dv}{\hat{B}_p(v)} . \qquad (24)$$

Using the above form for $B_p(v)$ this leads to

$$q = \frac{q_o b}{2a} \left(1 + \frac{a^2}{b^2} \right) \frac{1}{\sqrt{1 - \left(\frac{\beta}{\beta_c}\right)^2}} . \qquad (25)$$

3. THE ENERGY PRINCIPLE FOR VERTICAL DISPLACEMENTS

We now apply the energy principle [11] to our equilibrium with the express purpose of studying uniform vertical displacements. In general, the potential energy δW resulting from a small displacement $\underline{\xi}$ comprises three terms:

$$\delta W = \delta W_p + \delta W_v + \delta W_s . \qquad (26)$$

Since we shall be concerned with incompressible perturbation ($\nabla . \underline{\xi} = 0$) the plasma energy δW_p takes the form

$$\delta W_p = \frac{1}{2}\int d\tau\left((\underline{\delta B})^2 + \underline{j}.\underline{\xi}\times\underline{\delta B}\right), \tag{27}$$

where $\underline{\delta B} = \nabla\times(\underline{\xi}\times\underline{B})$. The vacuum energy δW_v is given by

$$\delta W_v = \frac{1}{2}\int d\tau\ \hat{B}'^2 , \tag{28}$$

where $\underline{\hat{B}}'$ is the perturbed vacuum field. As the present model has no skin current and thus the magnetic field is continuous, the surface energy δW_s is zero.

Taking the trial-function ξ_z = constant, which in the conventional notation of MHD stability theory is an $n = 0$, $m = 1$ mode, it is straightforward to show [12] that

$$\delta W_p = \frac{1}{2}\xi_z^2\int d\tau\left\{\left(\frac{\partial B_z}{\partial Z}\right)^2 + \frac{\partial B_z}{\partial R}\frac{\partial B_R}{\partial Z}\right\}. \tag{29}$$

Deriving $\frac{\partial B_z}{\partial Z}$, $\frac{\partial B_z}{\partial R}$ and $\frac{\partial B_R}{\partial Z}$ from Eq. (16), and evaluating this integral over the elliptic cross-section, we find δW_p to second-order to be given by

$$\delta W_p^{\epsilon\epsilon} = -\frac{\xi_z^2 I^2 R_o ab}{(b^2+a^2)^2}\left\{1 + \frac{1}{2}\left(\frac{\beta}{\beta_c}\right)^2\right\}. \tag{30}$$

We now consider the minimisation of the vacuum term. The vacuum region is assumed to be bounded by a perfectly conducting wall placed on the $u = u_1$ coordinate surface. In most experiments the discharge lifetime is long compared to the penetration time through the conducting shell. Thus the conducting shell should not influence the equilibrium. On the other hand it can be considered as perfectly conducting on the instability time scale and can therefore affect the stability. If δW_v is minimised subject to the constraint $\nabla.\underline{\hat{B}}' = 0$ then the minimising perturbations satisfy $\nabla\times\hat{B}' = 0$. Consequently $\underline{\hat{B}}' = \nabla V$ with V satisfying

$$\nabla^2 V = 0 . \tag{31}$$

The boundary condition for V at the interface is

$$(\underline{n}.\nabla V)_{u_o} = \underline{\hat{B}}.\nabla\xi - \xi\,\underline{n}.(\underline{n}.\nabla)\,\underline{\hat{B}} , \tag{32}$$

where $\underline{n}$ is the unit normal to the surface and is given by

$$n = \frac{\cos v \; \underline{e}_R + \frac{a}{b} \sin v \; \underline{e}_Z}{\left(\cos^2 v + \left(\frac{a}{b} \right)^2 \sin^2 v \right)^{\frac{1}{2}}} , \tag{33}$$

and ξ is the component of displacement normal to the surface. For the surface, the operator ∇ can be expressed as

$$\nabla = \frac{\underline{e}_\varphi}{R} \frac{\partial}{\partial \varphi} + \frac{1}{b\left(\cos^2 v + \left(\frac{a}{b} \right)^2 \sin^2 v \right)^{\frac{1}{2}}} \left(\underline{n} \frac{\partial}{\partial u} + \underline{\tau} \frac{\partial}{\partial v} \right) , \tag{34}$$

where $\underline{\tau}$ is the unit vector in the poloidal direction. From $\nabla . \underline{B} = 0$ it is straightforward to show that

$$\cos v \; \frac{\partial B_R}{\partial u} + \frac{a}{b} \sin v \; \frac{\partial B_z}{\partial u} = - \left(\cos^2 v + \left(\frac{a}{b} \right)^2 \sin^2 v \right)^{\frac{1}{2}} \frac{\partial B_p}{\partial v} . \tag{35}$$

Using Eqns. (33), (34) and (35), it follows that for an axisymmetric perturbation, the condition (32) can be put into the form

$$\left(\frac{\partial V}{\partial u} \right)_{u_o} = \frac{\partial}{\partial v} \left(\xi \; B_p(v) \right) . \tag{36}$$

We now have to relate ξ_z and ξ. Thus

$$\xi = \xi_z \; \underline{n} . \underline{e}_z = \frac{a}{b} \frac{\sin v}{\left(\cos^2 v + \left(\frac{a}{b} \right)^2 \sin^2 v \right)^{\frac{1}{2}}} \xi_z . \tag{37}$$

Using Eqs. (22), (36) and (37), we obtain

$$\left(\frac{\partial V}{\partial u} \right)_{u_o} = \frac{\xi_z \, I a}{\pi (a^2 + b^2)} \left[\cos v + \frac{\beta}{\beta_c} \cos 2v \right] . \tag{38}$$

To the order required, Laplace's equation takes the form

$$\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2} = 0 , \tag{39}$$

and the solution relevant to the boundary condition is

$$V = \left(C_1 e^{u} + C_2 e^{-u} \right) \cos v + \left(C_3 e^{2u} + C_4 e^{-2u} \right) \cos 2v . \tag{40}$$

Using Eq. (38) together with $\frac{\partial V}{\partial u} = 0$ at the conducting wall the coefficients

C_j can be determined. By Gauss' theorem δW_V can be expressed as

$$\delta W_V = -\pi R_0 \int_{u=u_0} V \frac{\partial V}{\partial u} dv . \tag{41}$$

Substituting the solution for V it is straightforward to show that to $O(\epsilon^2)$, δW_V is given by

$$\delta W_V^{\epsilon\epsilon} = \frac{\xi_z^2 I^2 R_0 a^2}{(a^2+b^2)^2} \left\{ \frac{1+e^{2(u_0-u_1)}}{1-e^{2(u_0-u_1)}} + \frac{1}{2}\left(\frac{\beta}{\beta_c}\right)^2 \frac{1+e^{4(u_0-u_1)}}{1-e^{4(u_0-u_1)}} \right\} . \tag{42}$$

Combining the results for $\delta W_p^{\epsilon\epsilon}$ and $\delta W_V^{\epsilon\epsilon}$ leads to

$$\delta W^{\epsilon\epsilon} = \frac{\xi_z^2 I^2 R_0 a^2}{(a^2+b^2)^2} \left\{ \frac{1+e^{2(u_0-u_1)}}{1-e^{2(u_0-u_1)}} - \frac{b}{a} + \frac{1}{2}\left(\frac{\beta}{\beta_c}\right)^2 \left[\frac{1+e^{4(u_0-u_1)}}{1-e^{4(u_0-u_1)}} - \frac{b}{a} \right] \right\} . \tag{43}$$

4. DISCUSSION OF STABILITY

From Eq. (43) the necessary and sufficient condition for stability against uniform vertical displacements is

$$\frac{1+t}{1-t} - \frac{b}{a} + \frac{1}{2}\left(\frac{\beta}{\beta_c}\right)^2 \left[\frac{1+t^2}{1-t^2} - \frac{b}{a} \right] \geqslant 0 , \tag{44}$$

where $t = e^{2(u_0-u_1)}$. Stability depends on b/a, β, and t - the effective wall distance. For an ellipse with $b > a$, $u_0 = \tanh^{-1} a/b$, and placing a conducting wall on the coordinate surface $u = u_1$ corresponds to an elliptic cross-section wall with semi-axes $a_w = a \sinh u_1/\sinh u_0$ and $b_w = b \cosh u_1/\cosh u_0$. Thus although we could discuss stability in terms of the actual wall position, it is more convenient to use t. We note that $t = 0$ corresponds to a wall at infinity, whilst $t = 1$ corresponds to a wall on the plasma. In criterion (44) the terms involving β are a direct consequence of the toroidicity of the equilibrium. Thus for $\beta \ll \beta_c$ the toroidal effects vanish and the stability criterion for a straight ellipse is recovered [3], that is

$$t \geqslant \frac{\frac{b}{a} - 1}{\frac{b}{a} + 1} . \tag{45}$$

We now briefly **review** criterion (45). In the absence of conducting walls the criterion becomes $1 - b/a \geqslant 0$. Clearly a flat ellipse $(b/a < 1)$

is stable to an upward shift whereas a vertical ellipse ($b/a > 1$) is always unstable to a vertical displacement. We note that a circular plasma ($b/a = 1$) is marginally stable to the order of the present calculation, and in the absence of conducting walls it would be necessary to work to $O(\epsilon^4)$ to ascertain the correct result. The vertical ellipse, however, can always be stabilised by a conducting wall placed sufficiently close to the plasma. A circular plasma is stable with a wall placed at any finite distance.

We now return to the full stability condition (44). For toroidal effects to be significant β must be of order the equilibrium limit, and this is far from the case in most experiments. Since however an ultimate reactor would operate at as high a β as possible it is of interest to ascertain the theoretical position regarding a vertical shift at $\beta \sim \beta_c$.

Although the plasma is displaced vertically ($m = 1$), high-β / toroidal effects lead to an additional $m = 2$ harmonic in the perturbed vacuum field. This is seen from the boundary condition (38), and from the $\frac{1+t^2}{1-t^2}$ wall stabilisation term in Eq. (44) - the wall stabilisation terms being equivalent to the $\frac{1+(r_p/r_w)^{2m}}{1-(r_p/r_w)^{2m}}$ - vacuum terms in the cylindrical pinch. As before, a flat ellipse is stable. In the absence of conducting walls the stability criterion is identical with that for the straight case, that is, $1 - b/a \geqslant 0$. Thus high-β / toroidal effects can only influence stability when a conducting wall is present. It is clear that with conducting walls present vertical shifts can always be stabilised for high-β. For if

$$t^2 \geqslant \frac{b-a}{b+a}\,, \tag{46}$$

condition (44) is certainly satisfied. Criterion (46) is however, over stringent, and it can be shown that the precise stability criterion is

$$t \geqslant \frac{\sqrt{1+\left(1+\frac{1}{2}\left(\frac{\beta}{\beta_c}\right)^2\right)^2\left(\frac{b^2}{a^2}-1\right)}-1}{\left(1+\frac{1}{2}\left(\frac{\beta}{\beta_c}\right)^2\right)\left(1+\frac{b}{a}\right)}\,. \tag{47}$$

Fig. 3 shows the marginal stability lines for this criterion for the cases $b = a$, $2a$ and $4a$. Stability requires that t be chosen to lie above the appropriate curve. For $b = a$ our model is stable for all $t \neq 0$.

For "flat" current profiles in the absence of conducting walls recent numerical investigations have indicated stability against uniform vertical shifts for values of b/a larger than unity. Thus, Lackner and MacMahon [4] find stability for $b/a < 1.25$, and Okabayashi and Sheffield [5] find stability for $b/a < 1.6$. Since both sets of workers consider plasmas with inverse aspect ratio $\epsilon \sim \frac{1}{2}$ this effect is presumably due to the strong toroidicity. Since the present work relies on an asymptotic expansion, it is not surprising that we do not recover their results.

5. PHYSICAL INTERPRETATION OF δW_F

We now discuss the physics of δW_F. Since we have shown toroidal effects to be unimportant for $\beta \ll \beta_c$, and this is the condition which pertains in present Tokamaks, it is reasonable to consider the simpler case of a straight system. In fact for this case we are able to consider plasmas of arbitrary cross-section and arbitrary current profile. Thus the discussion is applicable to any device for which toroidal effects can be neglected. As before it is assumed that the equilibrium is maintained by a prescibed vacuum field. The analysis is carried through in rectangular coordinates x, y, z. The x, y axes are set in the plane of the minor cross-section, and $\partial/\partial z \equiv 0$. The y-axis is defined to be the direction of the uniform displacement. Defining the stream-function ψ such that

$$B_x = -\frac{\partial\psi}{\partial y} \quad \text{and} \quad B_y = \frac{\partial\psi}{\partial x}, \tag{48}$$

it is well-known that the MHD equilibrium equation is

$$\nabla^2\psi = j_z(\psi),$$

where

$$j_z(\psi) = -p'(\psi) - B_z(\psi)B_z'(\psi). \tag{49}$$

For ξ_y = constant, δW_F can be written as

$$\delta W_F = \frac{1}{2}\xi_y^2\int d\tau\left\{\left(\frac{\partial B_x}{\partial x}\right)^2 + \frac{\partial B_y}{\partial x}\frac{\partial B_x}{\partial y}\right\}. \tag{50}$$

Using $\nabla.\underline{B} = 0$ this integral can be put in the symmetric form

$$\delta W_F = \frac{1}{4}\xi_y^2\int d\tau\left\{\left(\frac{\partial B_x}{\partial x}\right)^2 + \left(\frac{\partial B_y}{\partial y}\right)^2 + 2\frac{\partial B_y}{\partial x}\frac{\partial B_x}{\partial y}\right\}, \tag{51}$$

which, using $\underline{j} = \nabla \times \underline{B}$, can be expressed as

$$\delta W_F = \frac{1}{4}\,\xi_y^{\,2}\int d\tau\ \nabla.\left[\nabla\left(\frac{1}{2}B_p^{\,2}\right) - \underline{e}_x\,B_y\,j_z + \underline{e}_y\,B_x\,j_z\right], \tag{52}$$

where $B_p^{\,2} = B_x^{\,2} + B_y^{\,2}$. By Eq. (48) and using Gauss' theorem this becomes

$$\delta W_F = \frac{1}{4}\,\xi_y^{\,2}\int dS\left[\underline{n}.\nabla\left(\frac{1}{2}\,B_p^{\,2}\right) - j_z\,\underline{n}.\nabla\psi\right], \tag{53}$$

where $\underline{n}$ is the unit normal to the boundary. Since

$$j_z\ \underline{n}.\nabla\psi = -\ \underline{n}.\nabla\left(p + \frac{1}{2}\,B_z^{\,2}\right), \tag{54}$$

we have

$$\delta W_F = \frac{1}{4}\,\xi_y^{\,2}\int dS\ \underline{n}.\nabla\left(p + \frac{1}{2}\,B_p^{\,2} + \frac{1}{2}\,B_z^{\,2}\right). \tag{55}$$

The potential energy for the fluid can now be written in terms of the radius of curvature $\underline{\rho}$ of a magnetic line of force in the surface. Defining $\underline{\rho}$ to be the vector from a point on the line to its centre of curvature, then

$$\delta W_F = \frac{1}{2}\,\xi_y^{\,2}\int dS\ \frac{B^2}{\rho^2}\,\underline{n}.\underline{\rho}\ . \tag{56}$$

The sign of the integrand is determined by the direction of $\underline{\rho}$. Thus if at a given point on the surface $\underline{\rho}$ is inward ($\underline{n}.\underline{\rho} < 0$), that is the field line is concave towards the plasma, the integrand is destabilising at this point. Whereas if $\underline{\rho}$ is outward ($\underline{n}.\underline{\rho} > 0$), that is the field line is concave outward, the integrand is stabilising. For an ellipse $\underline{\rho}$ is everywhere towards the plasma and δW_F is clearly negative. In general, however, the overall sign of δW_F must be determined from the surface integral. We observe the similarity between this discussion and one given in the original energy principle paper [11]. The earlier work, however, is concerned with the stability of a plasma in which the magnetic field is zero. That is, plasma equilibrium is maintained by a skin-current. As a consequence of this model the destabilising part of the potential energy is due to δW_s.

For a model with arbitrary cross-section and arbitrary current distribution for which j_z vanishes at the boundary,

$$\delta W_F = \frac{1}{4}\,\xi_y^{\,2}\int dS\ \ B_p^{\,2}\ \frac{\underline{n}.\underline{\rho}}{\rho^2}\ , \tag{57}$$

where $\underline{\rho}$ is the radius of curvature vector for the poloidal field. In this case $\underline{n}.\frac{\underline{\rho}}{\rho} = \pm 1$. For an ellipse $\underline{\rho}$ is everywhere inward and $\underline{n}.\frac{\underline{\rho}}{\rho} = -1$.

We note that the magnitude of δW_F does not depend on the direction of the displacement. It follows, therefore, that the directional aspect of stability to a rigid shift must arise from the stabilising δW_V term. This is borne out by the results for the straight ellipse with quasi-uniform current.

6. CONCLUSIONS

We have derived a necessary and sufficient condition for stability to uniform vertical displacements of a high-β, toroidal, quasi-uniform current model of a tokamak with elliptic cross-section. It reduces to known results for the straight case. High-β / toroidal effects are only significant for β's comparable with the equilibrium limit. For $\beta \sim \beta_c$ and in the absence of conducting walls, however, the criterion is identical with that for the straight system. Thus within the limitations of asymptotic theory high-β / toroidal effects occur only in the presence of a conducting wall. For a given β and b/a the rigid shift can always be stabilised by a wall positioned at or within a critical distance, the value of which depends on β and the ellipticity. We note that the rigid shift of the plasma ($m = 1$) for $\beta \sim \beta_c$ leads to both $m = 1$ and $m = 2$ harmonics in the perturbed vacuum field.

Finally, we have considered the uniform displacement of a straight system with arbitrary cross-section and arbitrary current profile. It is demonstrated that the potential energy for the fluid can be expressed as an integral taken over the surface of the interface, the integrand involving the local curvature of the magnetic field lines in the surface. For an ellipse the integrand is everywhere negative and so the fluid contribution is destabilising, as exemplified by the quasi-uniform current model.

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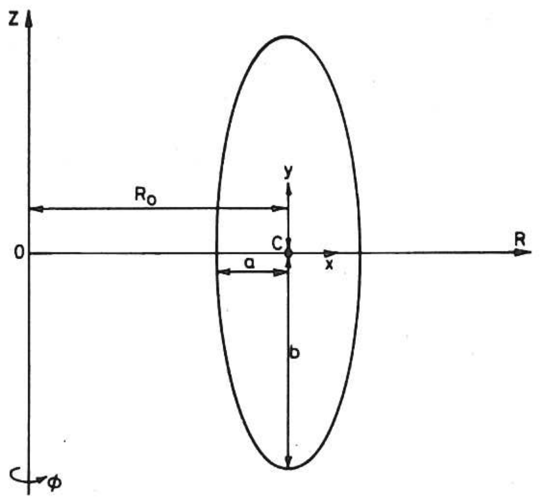

Fig.1 Coordinate systems. OZ is the axis of rotational symmetry.

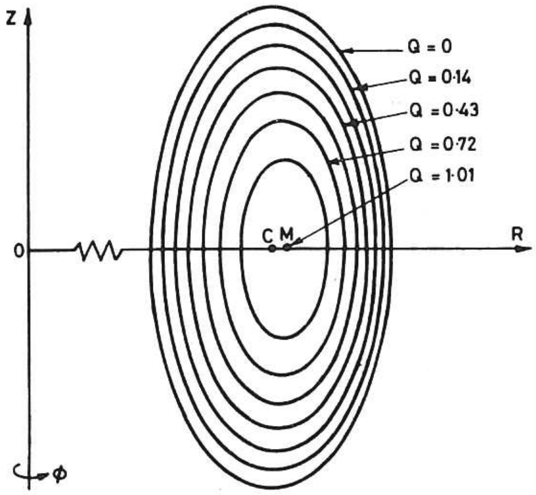

Fig.2 Flux surfaces for $\beta/\beta_c = 0.25$ and b/a = 2.0. Q is a dimensionless flux defined by $Q = -\dfrac{2\psi_1}{\psi_B}\dfrac{(1 + a^2/b^2)}{\eta - a^2}$.
Note the outward displacement of the magnetic axis M from the geometric centre C of the ellipse.

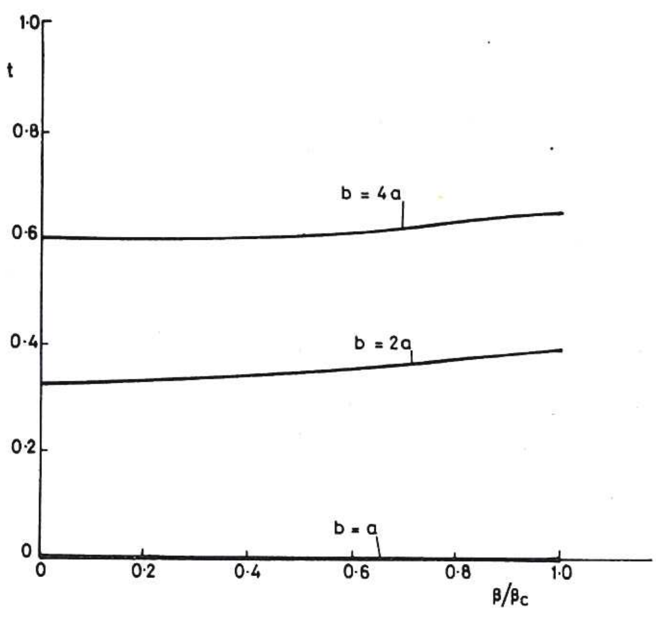

Fig.3 Marginal stability lines for an elliptic cross-section plasma with b = a, b = 2a and b = 4a. For stability, the effective wall distance t must be taken above the appropriate curve.