ON THE STRUCTURE OF OBLIQUE HYDROMAGNETIC SHOCK WAVES

by

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ABSTRACT

A phenomenological, computer-aided approach to the structure of fast oblique shock waves in a magnetized plasma is presented These shocks are preceded by "whistler" waves, which increase in amplitude towards the shock, and finally "break" to produce the shock front proper. We have used a two-fluid theory, the electrons and ions having distinct temperatures The main problem is how to choose appropriate values of the transport coefficients, for owing to the inevitable presence of turbulence, classical kinetic theory is inadequate. These coefficients - principally resistivity and viscosity - are greatly changed by the small-scale turbulence in the shock front, and as no satisfactory theory for their values in turbulent plasmas has yet been developed, the best one can presently do is to assume for them such values as to bring the computed shock profile into agreement with experiment.

Experimental work on these shocks at the University of Texas in Austin has been used to estimate the transport coefficients, but it must be admitted that there is a paucity of experimental results on which to base a phenomenological treatment. Still our limited objective to show what can be achieved by the approach, and what questions remain to be answered, has been attained.

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1. INTRODUCTION

MHD shock waves in which the magnetic field $\underline{B}$ is inclined at an angle α to the shock front, occur both in the earth's magnetic environment and in certain laboratory studies. Such oblique shocks ($0^{o} < \alpha < 90^{o}$) have not received the attention that has been devoted to the limiting cases $\alpha = 0^{o}$ and $\alpha = 90^{o}$, and it is our object to present some typical calculations and to make comparisons with the few experimental results available. Calculations similar to those given below have been published by Bickerton, Lenamon and Murphy (1971), but the particular set of fluid equations used by these authors lack some terms included by us.

The equations we shall employ are the two-fluid equations, with distinct ion and electron temperatures. The difficulty that haunts most calculations based on plasma fluid equations lies in choosing appropriate values for the transport coefficients. Classical (collisional) values of viscosity, resistivity and thermal conductivity are not appropriate for the low density plasmas that occur in the solar wind or in most laboratory experiments, and it remains an outstanding problem in plasma physics to calculate the so-called 'anomolous' values of these quantities appropriate for 'collisionless' shocks. If the dissipation mechanisms involve instabilities on much finer time and length scales than those arising in the shock structure, then the fluid equations are certainly valid but the actual values of the transport coefficients will depend somewhat on which instabilities are supposed to be effective. This problem is not solved in this paper, but as in the work just cited by Bickerton et al (1971) the hope is that in matching the computer calculations with experiment by modifying these dissipative quantities, one can at least arrive at estimates of their values useful in more basic studies of the wave-particle kinetics of saturated instabilities.

2. THE BASIC EQUATIONS

The shock is assumed to be at rest in the yz-plane with the fluid velocity $\underline{V}$ having a positive component V_x along the x-axis; conditions at the singular points upstream ($x = -\infty$) and downstream ($x = \infty$) are denoted by subscripts '1' and '2' respectively. The coordinate system is chosen so that the magnetic field $\underline{B}_1$ has no B_z component, i.e.

$$\underline{B}_1 = (B_x,\ B_{y_1},\ 0),\quad B_1 = |\underline{B}_1| = \surd(B_x^2 + B_{y_1}^2), \tag{2.1}$$

where the subscript '1' is omitted from B_x as the integral of $\nabla\cdot\underline{B} = 0$ along Ox gives $B_{x_1} = B_{x_2}$. Also if the z-component of the velocity of the coordinate frame is chosen to make $V_{z_1} = 0$, we have

$$\underline{V}_1 = (V_{x_1},\ V_{y_1},\ 0),\quad V_1 = |\underline{V}_1| = \surd(V_{x_1}^2 + V_{y_1}^2). \tag{2.2}$$

In the uniform states at $x = \infty$ and $x = -\infty$), Ohm's law and the integral of $\nabla \times \underline{E} = 0$ give

$$E_y = V_x B_z - V_z B_x = 0 \tag{2.3}$$

$$E_z = V_y B_x - V_x B_y = \text{const.}, \tag{2.4}$$

where the constant in (2.3) is zero by (2.1) and (2.2).

Equation (2.4) shows that provided $B_x \neq 0$ one may select the y-component of the coordinate frame velocity to reduce E_z to zero, in which case it follows from (2.3) and (2.4) that $\underline{V}$ and $\underline{B}$ will be parallel in both uniform states.

As $B_{x1} = B_{x2}$ and $V_{z2} = B_{z2}\, V_{x2}/B_x$, the complete upstream state is defined by $(p_1, \rho_1, B_{y_1}, \alpha_1, T_{e1}/T_{i_1})$ where T_{e_1}/T_{i_1} is the electron:ion temperature ratio and p is the total pressure

$$p = \frac{k}{m_i}\rho(T_i + T_e), \tag{2.5}$$

ρ is the density, k is Boltzmann's constant and m_i is the ion mass. The downstream state is defined by $(p_2, \rho_2, B_{y_2}, \alpha_2, T_{e2}/T_{i_2})$. Given the upstream state and the velocity V_{x1} we require six scalar jump relations across the shock to deduce the downstream state. These are provided, as usual, by the conservation of mass, momentum and energy plus a separate energy equation for either the ions alone or the electrons alone.

The basic equations are equations (2.4) and (2.5) plus the following (in Mks units)

$$\rho V_x = \rho_1 V_{x1} \tag{2.6}$$

$$\rho V_x^2 + p + \frac{1}{2\mu}(B_y^2 + B_z^2 - B_x^2) - \rho\nu V_x' = \rho_1 V_{x1}^2 + p_1 + \frac{1}{2\mu}(B_{y1}^2 - B_x^2) \tag{2.7}$$

$$\rho V_x V_y - \frac{1}{\mu}B_x B_y - \rho\nu V_y' = \rho_1 V_{x1} V_{y1} - \frac{1}{\mu}B_x B_{y1} \tag{2.8}$$

$$\rho V_x V_z - \frac{1}{\mu}B_x B_z - \rho\nu V_z' = 0 \tag{2.9}$$

$$V_x\{\tfrac{1}{2}\rho(V_x^2 + V_y^2 + V_z^2) + \tfrac{5}{2}p\} - \rho\nu(V_x V_x' + V_y V_y' + V_z V_z') -$$

$$K_e T_e' - K_i T_i' - \frac{1}{\mu}E_z B_y = V_{x1}\{\tfrac{1}{2}\rho_1(V_{x1}^2 + V_{y1}^2) + \tfrac{5}{2}p_1\} - \frac{1}{\mu}E_z B_{y1} \tag{2.10}$$

$$\tfrac{3}{2}V_x T_i' + T_i V_x' = \frac{2\nu}{R}\{\tfrac{2}{3}(V_x')^2 + \tfrac{1}{2}(V_y')^2 + \tfrac{1}{2}(V_z')^2\} \tag{2.11}$$

$$\frac{-\eta}{\mu}B_z' = V_z B_x - V_x B_z - \frac{B_x B_y'}{en\mu} \tag{2.12}$$

$$\frac{\eta}{\mu}B_y' = E_z + V_x B_y - V_y B_x - \frac{B_x B_z'}{en\mu}. \tag{2.13}$$

Here η, ν, K_e and K_i are resistivity, viscosity and thermal conductivities of the electrons and ions respectively, and R is the gas constant in $p = R\rho T$. As remarked above we may chose the coordinate system so that E_z is zero.

These equations are conservation of mass (2.6); momentum (2.7), (2.8) and 2.9); total energy (2.10); ion energy (2.11) and the transverse components of Ohm's law (2.12) and (2.13). The principal approximations involved here are

(i) neglect of electron inertia in Ohm's law

and (ii) neglect of thermo-electric effects;

(i) is valid if the length of the shock-structure and its attendant waves greatly **exceeds** c/ω_{pe} while (ii) is probably not a good approximation (see Woods 1969a) but in the absence of a satisfactory theory for anomolous dissipative effects is not readily avoided.

3. NON-DIMENSIONAL FORM OF EQUATIONS

We now introduce the non-dimensional values

$$\epsilon_y = \frac{B_y - B_{y1}}{B_1}, \quad \epsilon_z = \frac{B_z}{B_1}, \quad y = \frac{p - p_1}{p_1},$$

$$\omega = \frac{V_x}{V_{x1}}, \quad \omega_y = \frac{V_y - V_{y1}}{V_{x1}}, \quad \omega_z = \frac{V_z}{V_{x1}},$$

$$\theta = \frac{\mu n_1 k T_e}{B_1^2}, \qquad \phi = \frac{\mu n_1 k T_i}{B_1^2}, \qquad \nu^* = \frac{\nu}{V_{x1}\delta_{e1}}, \qquad \eta^* = \frac{\eta}{\mu V_{x1}\delta_{e1}},$$

$$s = \frac{5\mu p}{3B^2}, \qquad M = \frac{V_x(\mu\rho)^{\frac{1}{2}}}{B}, \qquad K_e^* = \frac{K_e}{n_1 V_{x1} k\delta_{e1}}, \quad K_i^* = \frac{K_i}{n_1 V_{x1} k\delta_{e1}},$$

where $\delta_e = c/\omega_{pe} = (\frac{m_e}{\mu e^2 n})^{\frac{1}{2}}$, $\delta_i = c/\omega_{pi} = (\frac{m_i}{\mu e^2 n})^{\frac{1}{2}}$.

The upstream state is now adequately defined by the values of s, α and T_e/T_i. In all the calculations we have taken $(T_e/T_i)_1 = 1$. To prescribe a given shock we also need the value of either the quantity ϵ_{y2} or the upstream Alfvén Mach number M_1.

We next define a new coordinate

$$X = x/\delta_{e1}$$

and let $(\dot{\ })$ denote differentiation with respect to X. So that

$$(\ )' = \frac{1}{\delta_{e1}}(\dot{\ }) .$$

Then the equations may be written

$$2M_1^2\,\nu^*\frac{\dot{\omega}}{\omega} = 2M_1^2(\omega-1) + \frac{6}{5}s_1 Y + \epsilon_y^2 + 2\epsilon_y \sin\alpha_1 + \epsilon_z^2 \tag{3.1}$$

$$M_1^2\nu^*\frac{\dot{\omega}_y}{\omega} = M_1^2\,\omega_y - \epsilon_y\cos\alpha_1 \tag{3.2}$$

$$M_1^2\,\nu^*\frac{\dot{\omega}_z}{\omega} = M_1^2\,\omega_z - \epsilon_z\cos\alpha_1 \tag{3.3}$$

$$\frac{\nu^*}{\omega}(\omega\dot{\omega} + \omega_y\dot{\omega}_y + \omega_z\dot{\omega}_z) + \frac{K_e^*}{M_1^2}\dot{\theta} + \frac{K_i^*}{M_1^2}\dot{\phi} =$$

$$\tfrac{1}{2}(\omega^2 - 1 + \omega_y^2 + \omega_z^2) + \frac{3s_1}{2M_1^2}(\omega[Y+1] - 1) + \frac{\epsilon_y\sin\alpha_1}{M_1^2} \tag{3.4}$$

$$-\eta^*\dot{\epsilon}_z = \omega_z\cos\alpha_1 - \omega\epsilon_z - \frac{\delta_{i1}}{\delta_{e1}M_1}\,\omega\,\dot{\epsilon}_y\cos\alpha_1 \tag{3.5}$$

$$\eta^*\dot{\epsilon}_y = (\omega-1)\sin\alpha_1 + \omega\,\epsilon_y - \omega_y\cos\alpha_1 - \frac{\delta_{i1}}{\delta_{e1}M_1}\,\omega\,\dot{\epsilon}_z\cos\alpha_1 \tag{3.6}$$

$$Y + 1 = \frac{5}{3s_1}\left(\frac{\theta+\phi}{\omega}\right) \tag{3.7}$$

$$\tfrac{3}{2}\,\omega\dot{\phi} + \phi\dot{\omega} = M_1^2\nu^*\left(\tfrac{4}{3}\dot{\omega}^2 + \dot{\omega}_y^2 + \dot{\omega}_z^2\right) \tag{3.8}$$

These are eight equations for the unknowns ω, ω_y, ω_z, ϵ_y, ϵ_z, Y, θ, ϕ.

There remains the problem of assigning values to the four transport coefficients η^*, ν^* K_e^* and K_i^* As stated in the Introduction, we shall choose values of these quantities to give agreement with experiment, and were there sufficient experimental data to determine them uniquely, we could subsequently use these values in different experimental circumstances with some confidence. This is, of course, the usual phenomenological approach to transport coefficients; some theory - in our case the two-fluid model equations - is unfortunately inescapable in arriving at the numbers.

A complication is that the transport coefficients are almost certain to be temperature dependent in a turbulent plasma. For example for η^* we might assume

$$\eta^* = \eta_1^* \left(\frac{\theta_1}{\theta}\right)^n + \eta_2^* \quad , \tag{3.9}$$

so that three constants, η_1^* , η_2^* and n need to be deduced from the comparison between the computer calculations and the experiments. (Classical resistivity has $n = \frac{3}{2}$, $\eta_2^* = 0$ and a known value for η_1^*). Similar assumptions could be made for ν^*, K_e^* and K_i^* ; but to reduce the complexity of the situation, especially necessary in view of the meagre experimental results available, we shall simply set $K_i^* = 0$ (K_e^* usually dominates heat transport), take K_e^* to be small, and use the classical value for ν^* i.e.,

$$\nu^* = \nu_1^* \left(\frac{\phi}{\phi_1}\right)^{\frac{5}{2}} , \tag{3.10}$$

We are then virtually ignoring thermal conductivity and supposing that the ion viscosity is unaffected by the plasma turbulence.

For the index n we have taken the values $\frac{3}{2}$ (classical), 1 and $\frac{1}{2}$.

Note:

Equations (3.1) to (3.8) may describe a fast intermediate or slow shock depending upon the shock parameters s_1 , α_1 , ϵ_{y2} (or s_1 , α_1, M_1). (See for example Jeffrey and Taniuti 1964, Kantrowitz and Petschek 1966). With M_A defined by

$$M_A = M/\cos\alpha , \tag{3.11}$$

and C_f, C_s the non-dimensionalised fast and slow magneto-acoustic speeds; i.e.,

$$\begin{aligned} C_f^2 &= \tfrac{1}{2}\{1 + s + \sqrt{(1+s)^2 - 4 s \cos^2\alpha}\} \sec^2\alpha \\ C_s^2 &= \tfrac{1}{2}\{1 + s - \sqrt{(1 + s)^2 - 4 s \cos^2\alpha}\} \sec^2\alpha \quad , \end{aligned} \tag{3.12}$$

then these shocks can be categorised as follows:

(i)	fast	$M_{A1} > C_{f1}$;	$C_{f2} > M_{A2} > 1$
(ii)	intermediate	$M_{A1} > 1$;	$1 > M_{A2}$
(iii)	slow	$1 > M_{A1} > C_{s1}$;	$C_{s2} > M_{A2}$.

In the sequel (apart from some general remarks at the beginning of §4) we confine our attention to the fast shocks.

4. THE METHOD OF NUMERICAL SOLUTION

4.1 General remarks on the solution of shock wave equations

Since the boundary (or singular) points represent the steady upstream and downstream flows, the derivatives of all physical variables at these points are zero. It is thus impossible to make any headway in the integration of these equations by directly applying a numerical integration scheme from either end.

Instead, we first of all investigate the form of the solution near the singular points by linearising the equations about each point in turn, assuming that small perturbations in all quantities vary like e^{kX} close to the point. (Note that in order to carry out the linearisation at the downstream point we have first to determine the downstream values of all the variables.) If N is the number of shock wave equations, the eigenvalues $k = k_j$, $j = 1, 2, \ldots, N$ at each point are determined by solving the corresponding set of linearised equations. In the first instance it is not necessary to evaluate the eigenvalues but merely to determine their signs. Woods (1969b) has shown how this information will determine whether a structurally stable curve joining the two points exists, and in which direction it is possible to integrate numerically.

A solution curve can leave a singular point if there is at least one solution of the linearised equations which grows as X advances towards the other singular point, i.e. if there is at least one eigenvalue which corresponds to a growing exponential. Thus the solution curve can leave the singular point at $X = \pm\infty$ if one of the eigenvalues has $Re(k_j) \lesseqgtr 0$. If it happens that a curve can leave neither point in the required direction then it is impossible to obtain a solution to the shock wave equations.

A solution curve can enter a singular point if <u>all</u> the solutions of the linearised equations decay as the point is approached. (Theoretically it would be sufficient to set the amplitude of a solution which corresponded to a growing exponential to zero but in practice the computational inaccuracy would soon revive the unwanted solution which would then rapidly become dominant.) Thus the condition that a solution curve can enter the singular point at $X = \pm\infty$ is that <u>all</u> $Re(k_j) \lesseqgtr 0$.

If it is possible to leave one point and enter the other then this is the direction to choose for the numerical integration. If it is possible to leave both points but to enter neither then it is necessary to integrate from each point towards the other and to match the solutions at some intermediate position.

In order to determine whether a solution curve is structurally stable it is necessary to know the number n of eigenvalues, at each singular point, which have positive real part. If n_1 and n_2 are the numbers at the upstream and downstream points respectively, then Woods (1971) has shown that a necessary condition for a solution is

$$n_1 - n_2 = 1 .$$

Once the direction of integration has been determined, we proceed with the solution as follows. The variables at the initial singular point are perturbed by adding a small multiple of the growing linear solution or a linear combination of all growing solutions if more than one are present. (For this step we have first to determine all the eigenvalues and eigenvectors which correspond to growing solutions). The perturbed values define a point lying on the shock structure (either A_1 or A_2 of Fig. 2) where the derivatives of all variables are non-zero. The location of the shock needs to be assigned <u>a priori</u> as it is not determined by the boundary conditions; thus some convenient reference value X_o is assigned to the X-coordinate of the perturbed point.

The values of all derivatives at the perturbed point can be found from the shock wave equations so that now a numerical integration scheme can be used to integrate from this point through the shock.

When the solution curve begins to asymptote to its terminal value, this value is approached extremely slowly. Thus as soon as the variables are sufficiently close to their terminal values it is quicker to 'lead in' the solution to the terminal point by adding the appropriate multiple of the most slowly decaying linear solution at that point.

Stiffness

In many cases shock wave equations are stiff near the upstream singular point. If stiffness proves to be a problem during the numerical integration it is often worth applying one of the integration techniques recommended for this circumstance (e.g. Gear 1968).

4.2 Solution of the equations of §3

It can be shown (e.g. see Woods 1971) that in the case of fast shocks the equations of §3 give upstream eigenvalues which all have positive real part whilst at the downstream point there is one eigenvalue with negative real part. The solution will be structurally stable and the direction of integration is from the downstream to the upstream singular point.

The procedure for solution is summarised below:

(i) Evaluate all downstream variables (see §5).

(ii) Solve the linearised equations at the upstream singular point (§6).

(iii) Solve the linearised equations at the downstream singular point (§6).

(iv) Perturb the downstream variables by a small amount, using the eigenvector corresponding to the negative downstream eigenvalue. For example (see Fig.2) the amount Δ_2 of perturbation in the ϵ_{y_2} value can be specified as input data to the computer run, so that if $\hat{\underline{q}} = (\hat{\omega},\hat{\omega}_y,\hat{\omega}_z,\hat{\epsilon}_y,\hat{\epsilon}_z,\hat{\Upsilon},\hat{\theta},\hat{\phi})$ is the specified eigenvector, then

$$-\frac{\Delta_2}{\hat{\epsilon}_y}\hat{\underline{q}}$$

is the required perturbation.

The perturbed values define a point A_2 lying on the shock structure.

(v) Evaluate all derivatives at A_2 by substituting the perturbed values in the shock wave equations.

(vi) Integrate the shock wave equations from A_2 to some point A_1 where all variables are close to their upstream value. When the imaginary part of the upstream eigenvalue with the smallest positive real part is zero, A_1 can be defined in a similar manner to A_2 (see iv above) by the amount Δ_1 of perturbation of ϵ_y from its upstream value of zero. This case is shown in Fig.2.

When the upstream eigenvalue with smallest real part is complex, the solution oscillates at the singular point. In this case the relevant portion of Fig.2 appears as shown in Fig.3.

The position of A_1 in this figure could be defined here by Δ_1 and, say, a prescribed number of maxima of ϵ_y from the shock front.

(vii) Perturb the solution by adding the correct multiple of the eigenvector corresponding to the eigenvalue k_s which has the smallest positive real part. If k_s is real, $\hat{\underline{q}}$ the eigenvector and Δ_1 defined as above, then

$$-\frac{\Delta_1}{\hat{\epsilon}_y}\,\hat{\underline{q}}$$

is the required perturbation.

If k_s is complex it is necessary to use an appropriate linear combination of the eigenvector and its complex conjugate. The phase to be used in combining the two vectors can be determined from the phase of the numerical solution at A_1.
(Any deviation of the perturbed values from their upstream value will be an indication of the computational inaccuracy).

5. DETERMINATION OF THE DOWNSTREAM VALUES OF ALL VARIABLES

A full account of the method of determination of the values of the downstream variables for various types of shock wave is given in Dixon and Woods (1972) together with comprehensive tables and graphs of the values of downstream variables. For a particular shock the relevant values of the downstream variables in the tables are found by first referring to the upstream state, defined by s_1 and α_1, and then to the jump in the transverse magnetic field across the shock, ϵ_{y_2} (or by s_1, α_1, and $M_{A_1} = M_1/\cos\alpha_1$).

A brief description of the method now follows. At the downstream singular point, where all derivatives vanish, the equations of §3 become (NB $\omega_{z_2} = \epsilon_{z_2} = 0$),

$$2M_1^2(\omega_2 - 1) + \frac{6}{5}s_1 Y_2 + \epsilon_{y_2}^2 + 2\epsilon_{y_2}\sin\alpha_1 = 0$$

$$M_1^2\,\omega_{y_2} - \epsilon_{y_2}\cos\alpha_1 = 0$$

$$\tfrac{1}{2}(\omega_2^2 - 1 + \omega_{y_2}^2) + \frac{3s_1}{2M_1^2}(\omega_2[Y_2 + 1] - 1) + \frac{\epsilon_{y_2}\sin\alpha_1}{M_1^2} = 0$$

$$(\omega_2 - 1)\sin\alpha_1 + \omega_2\,\epsilon_{y_2} - \omega_{y_2}\cos\alpha_1 = 0$$

$$Y_2 + 1 = \frac{5}{3s_1}\left(\frac{\theta_2 + \phi_2}{\omega_2}\right)$$

$$\phi_2\,\omega_2^{\frac{2}{3}} = \phi_1\ .$$

We have thus six equations for the unknowns M_1^2, ω_2, ω_{y_2}, Y_2, θ_2, ϕ_2.
Let

$$\chi = \frac{3s_1 Y_2}{5\epsilon_{y_2}} + \tfrac{1}{2}\epsilon_{y_2} \qquad (5.1)$$

then after a little algebra it can be shown that χ satisfies the quadratic equation.

$$2\chi^2(\sin\alpha_1 - \tfrac{1}{3}\epsilon_{y_2}) - 2\chi(\tfrac{5}{6}\epsilon_{y_2}\sin\alpha_1 - 1 + s_1) - (\epsilon_{y_2} + 2s_1\sin\alpha_1) = 0 \qquad (5.2)$$

Since χ must be real and must have the same sign as ϵ_{y_2} (since $Y_2 \geq 0$) there may be two, one or no solutions of (5.2) depending upon the value of ϵ_{y_2}.

If we consider the case of fast shocks (i.e. $\epsilon_{y_2} > 0$) and let

$$Q = s_1 - 1 + \tfrac{5}{2}\sin^2\alpha_1\ ,$$

then if $Q \geq 0$ there is one solution if $\epsilon_{y_2} < 3\sin\alpha_1$ and no solutions if $\epsilon_{y_2} \geq 3\sin\alpha_1$. If $Q < 0$ there is one solution if $\epsilon_{y_2} < 3\sin\alpha_1$, two solutions if $3\sin\alpha_1 < \epsilon_{y_2} \leq \bar{\epsilon}$ and no solutions if $\epsilon_{y_2} > \bar{\epsilon}$, where

$$\bar{\epsilon} = \frac{6\sin\alpha_1(1+s_1) + 12\cos\alpha_1\sqrt{6(1-s_1)^2 + 25 s_1 \sin^2\alpha_1}}{24 - 25\sin^2\alpha_1}$$

Those shocks for which $Q \geq 0$ are termed 'type 1' shocks and those with $Q < 0$ are termed 'type 2' shocks.

The cases where there are two solutions show that it is not always possible to define a shock uniquely by the three parameters $s_1, \alpha_1, \epsilon_{y1}$. However s_1, α_1, M_{A1} always define a unique shock, provided M_{A1} is greater than the non-dimensionalised fast magneto-acoustic speed C_{f1}, where

$$C_{f1}^2 = \tfrac{1}{2}\left\{1 + s_1 + \sqrt{(1+s_1)^2 - 4 s_1 \cos^2\alpha_1}\right\}\sec^2\alpha_1 \ .$$

6. LINEARISATION AT THE SINGULAR POINTS

As we outlined in section 4, in order to investigate the nature of the solution near the singular points, it is necessary to linearise the equations of section 3 about these points. Let the subscript r denote values at points 1 and 2, and denote by $\hat{\omega}, \hat{\omega}_y, \hat{\omega}_z$, etc. the perturbations $\omega - \omega_r, \omega_y - \omega_{yr}, \omega_z - \omega_{zr}$, etc. then neglecting second order quantities, (3.1) becomes

$$\begin{aligned} 2M_1^2 \nu^* \frac{\dot{\hat{\omega}}}{\omega_r} &= \left\{2M_1^2(\omega_r - 1) + \frac{6}{5} s_1 Y_r + \epsilon_{yr}^2 + 2\epsilon_{yr}\sin\alpha_1 + \epsilon_{zr}^2\right\} \\ &\quad + 2M_1^2\hat{\omega} + \frac{6}{5} s_1 \hat{Y} + 2\epsilon_{yr}\hat{\epsilon}_y + 2\hat{\epsilon}_y\sin\alpha_1 + 2\epsilon_{zr}\hat{\epsilon}_z \\ &= 2M_1^2\hat{\omega} + \frac{6}{5} s_1 \hat{Y} + 2\hat{\epsilon}_y(\epsilon_{yr} + \sin\alpha_1) \ , \end{aligned}$$

since $\epsilon_{zr} = 0$ and the term in $\{\ \}$ is simply the RHS of (3.1) at the singular point and is zero since $\dot{\omega}_r = 0$.

Equations (3.2) to (3.8) can be linearised in the same manner. The linearised versions of (3.7) and (3.8) can be used to express $\dot{\hat{\theta}}$ and $\dot{\hat{\varphi}}$ in terms of $\dot{\hat{\omega}}$ and $\dot{\hat{Y}}$. The expressions obtained enable $\dot{\hat{\theta}}$ and $\dot{\hat{\varphi}}$ to be eliminated from the linearised version of (3.4). Then the linearised equations (3.1) to (3.6) give a set of equations in the six unknowns $\hat{\omega}, \hat{\omega}_y, \hat{\omega}_z, \hat{\epsilon}_y, \hat{\epsilon}_z, \hat{Y}$ which can be expressed in matrix form, i.e.

$$\begin{bmatrix} M_1^2\frac{\nu^*}{\omega_r} & 0 & 0 & 0 & 0 & 0 \\ 0 & M_1^2\frac{\nu^*}{\omega_r} & 0 & 0 & 0 & 0 \\ 0 & 0 & M_1^2\frac{\nu^*}{\omega_r} & 0 & 0 & 0 \\ \alpha_r & \frac{\nu^*\omega_{yr}}{\omega_r} & 0 & 0 & 0 & \frac{3 s_1 K_e^*\omega_r}{5M_1^2} \\ 0 & 0 & 0 & \beta_r & -\eta^* & 0 \\ 0 & 0 & 0 & \eta^* & \beta_r & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{\omega}} \\ \dot{\hat{\omega}}_y \\ \dot{\hat{\omega}}_z \\ \dot{\hat{\epsilon}}_y \\ \dot{\hat{\epsilon}}_z \\ \dot{\hat{Y}} \end{bmatrix} =$$

$$\begin{bmatrix} M_1^2 & 0 & 0 & \epsilon_{yr}+\sin\alpha_1 & 0 & \frac{3}{5}s_1 \\ 0 & M_1^2 & 0 & -\cos\alpha_1 & 0 & 0 \\ 0 & 0 & M_1^2 & 0 & -\cos\alpha_1 & 0 \\ \gamma_r & \omega_{yr} & 0 & \frac{\sin\alpha_1}{M_1^2} & 0 & \frac{3 s_1\omega_r}{2M_1^2} \\ 0 & 0 & \cos\alpha & 0 & -\omega_r & 0 \\ \epsilon_{yr}+\sin\alpha_1 & -\cos\alpha_1 & 0 & \omega_r & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\omega} \\ \hat{\omega}_y \\ \hat{\omega}_z \\ \hat{\epsilon}_y \\ \hat{\epsilon}_z \\ \hat{Y} \end{bmatrix} \qquad (6.1)$$

where

$$\left.\begin{aligned}
\alpha_r &= \nu^* + \frac{1}{3M_1^2\omega_r}\left\{K_e^*(3\theta_r + 5\varphi_r) - 2K_i^*\varphi_r\right\} , \\
\beta_r &= \frac{\delta_{i1}\omega_r \cos\alpha_1}{\delta_{e1}M_1} , \\
\gamma_r &= \omega_r + \frac{3s_1}{2M_1^2}(Y_r + 1) ,
\end{aligned}\right\} \tag{6.2}$$

Let $\quad \underline{\hat{q}} = (\hat{\omega}, \hat{\omega}_y, \hat{\omega}_z, \hat{\varepsilon}_y, \hat{\varepsilon}_z, \hat{Y})$

then (6.1) can be written

$$A\dot{\underline{\hat{q}}} = B\underline{\hat{q}} , \tag{6.3}$$

where A and B represent the above matrices. We now look for solutions of the form

$$\underline{\hat{q}} = \underline{c}\, e^{kx} , \tag{6.4}$$

then (6.3) becomes

$$kA\underline{\hat{q}} = B\underline{\hat{q}} \quad \text{or} \quad A^{-1}B\underline{\hat{q}} = k\underline{\hat{q}} .$$

The quantities k are the solutions of the algebraic equation

$$\det(A^{-1}B - kI) = 0$$

and the vectors $\hat{q}$ are the eigenvectors of the matrix $A^{-1}B$.

7. WAVE LENGTH AND DAMPING OF THE PRECURSOR WAVE

For a shock in which resistivity provides the damping, it may be shown from the analysis of section 6 that the eigenvalues, λ, at the singular points are given by the quadratic

$$\lambda^2 - (a+b)\,\eta^*\lambda + ab\left(\eta^{*2} + \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \frac{\cos^2\alpha}{M^2}\right) = 0 \tag{7.1}$$

where

$$\lambda = (k\delta_e)^{-1} ,$$

$$a = \frac{M^2(M^2 - s)}{(M^2 - C_s^2\cos^2\alpha)(M^2 - C_f^2\cos^2\alpha)} ,$$

and

$$b = \frac{M^2}{M^2 - \cos^2\alpha} .$$

k is defined in (6.4) and C_f, C_s are the non-dimensionlised fast and slow magneto-acoustic speeds defined in (3.12).

In the example considered below, $M \gg C_f\cos\alpha$ so that a and b are each very close to unity. For this case (7.1) has the solution

$$\lambda = \eta^* \pm i\left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \frac{\cos\alpha}{M} ,$$

whence the non-dimensional damping length is η^*, and the wave length is

$$\ell = 2\pi\left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \frac{\cos\alpha}{M} . \tag{7.2}$$

Provided $\eta^* \ll \ell$, we may expect the thickness λ_s shown in Fig.3 to be about $1/4\ell$, i.e.

$$\lambda_s \simeq \frac{\pi}{2}\left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \frac{\cos\alpha}{M} . \tag{7.3}$$

8. COMPARISON OF COMPUTER SOLUTIONS WITH EXPERIMENT

Robson and Sheffield (1968) have given interesting results for an oblique shock; their experimental curves for $B_{\parallel}$ and B_{θ} (in our notation B_y and B_z) versus distance and for $B_{\parallel}$ versus B_{θ} through the shock front are given in Fig.5.

From the upstream conditions in this experiment one finds that the classical resistivity is given by

$$\eta^* = \eta_1^* \left(\frac{\theta_1}{\theta}\right)^n + \eta_2^* ,$$

with $\eta_1^* = 11$, $n = \frac{3}{2}$ and $\eta_2^* = 0$. Also the classical viscosity is given by

$$\nu^* = 0.0137 \left(\frac{\varphi}{\varphi_1}\right)^{\frac{5}{2}} .$$

With these values and negligible thermal conduction we obtained the curves for the non-dimensional variables ϵ_y and ϵ_z shown in Fig.6. Clearly the damping is insufficient.

Notice the formation of a subshock to the rear of the profile. This is a result of the small value of the viscosity and the fact that the flow goes from supersonic to subsonic across the shock. The subshock thickness provides the best indication of the magnitude of the viscous dissipation but without density profile measurements this important feature of the structure is not available in the experimental results.

The next step was to seek agreement with experiment via changes in η^* alone. We found good general agreement with the following choices of η_1^*, n and η_2^*. In all cases we retained the classical viscosity.

Model	(i)	(ii)	(iii)	(iv)	(v)
η_1^*	28	9	4	11	0
n	$\frac{3}{2}$	1	$\frac{1}{2}$	$\frac{3}{2}$	0
η_2^*	0	0	0	0.5	3

TABLE 1

COMBINATIONS OF η_1^*, n AND η_2^* WHICH GIVE GENERAL AGREEMENT WITH THE EXPERIMENTAL RESULTS

The curves for cases (ii) and (iv) are shown in Figs.7 and 8.

Of course, what is needed is a variety of experimental results with varying α, M, θ_1 and φ_1. All we can do at this stage is to show what could be achieved by a purely phenomological approach.

Even without a model from which n could be estimated we can draw some conclusions from our 'computer experiment', at least for the Texas experiments. Very likely some anomalous resistivity is present, although this is not as large as in the Culham laboratory experiments with perpendicular shocks (Woods, 1973). In fact, were the measured upstream temperature in error by a factor of about 2, one could increase η_1^* from 11 to 28, and so have an entirely adequate classical resistivity. Also, as classical resistivity must remain relatively important, the most reasonable modification is (iv) which requires only 0.5 additional η^* throughout the profile. Similarly, model (ii) is plausible, whereas models (iii) and (v) seem unlikely.

We have tried increasing ν^* and leaving η^* classical but the general effect was to damp the upstream oscillations in B_y and B_z rather more than in the experiment.

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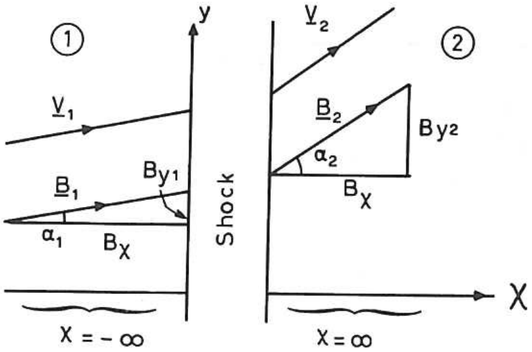


Fig.1 Shock wave field vectors

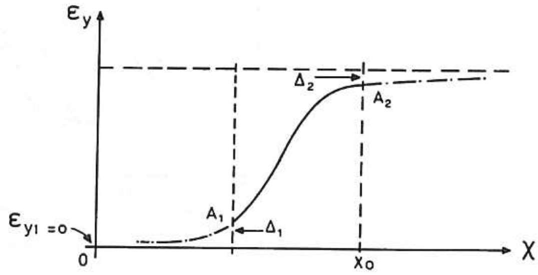


Fig.2 Typical variation of ϵ_y through the shock structure.

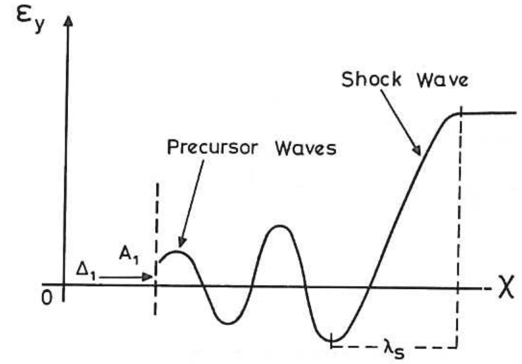


Fig.3 Solution near the upstream singular point with a dominant complex eigenvalue.

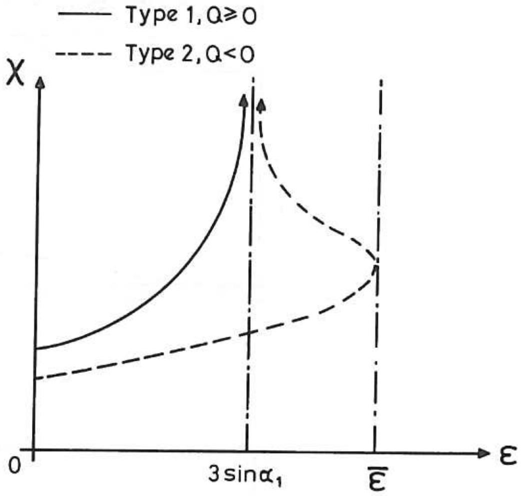
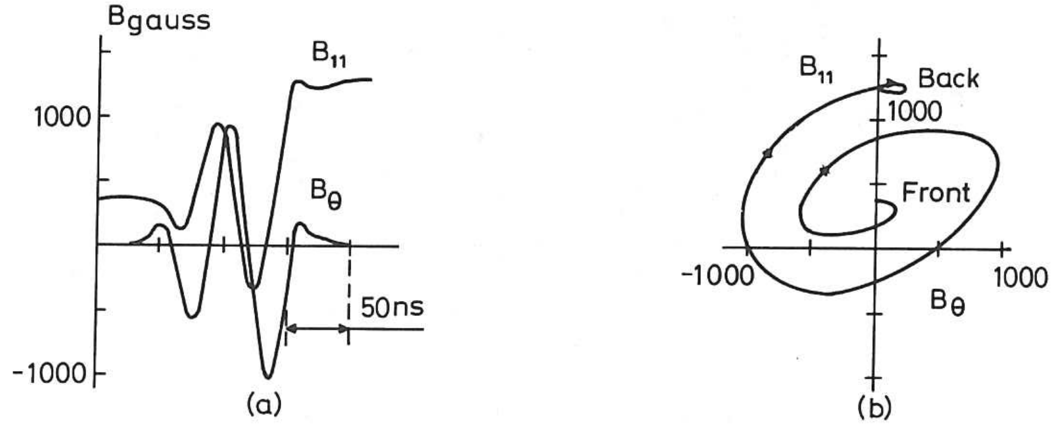


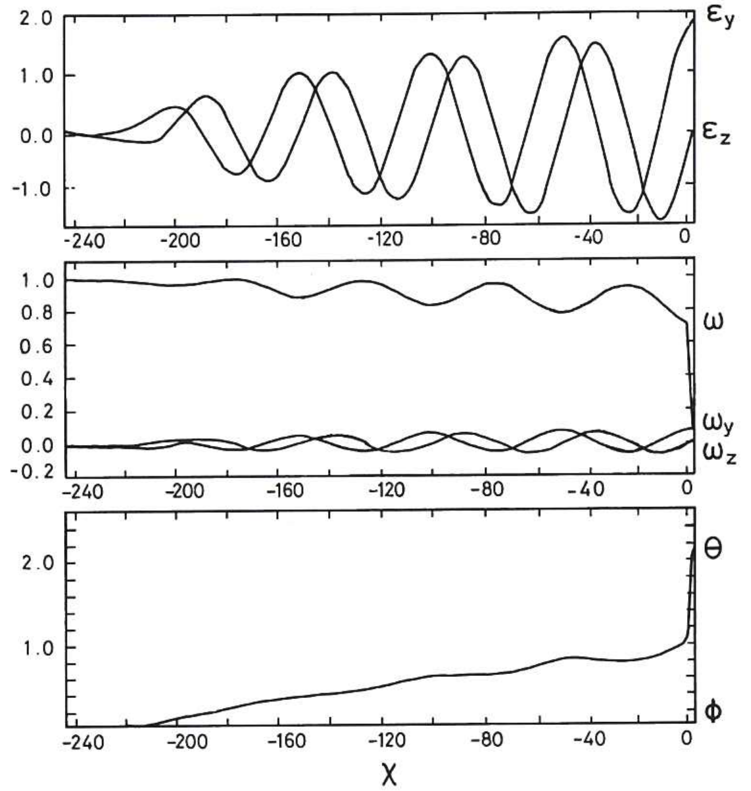
Fig.4 The variation of X with ϵ_{y2} for fast shocks.



Integrated magnetic probe traces at r = 10cm, z = 12cm.

$B_{\parallel}$ versus B_θ conditions, $a = 45°$, $B_\theta = 500G$, $n = 5 \times 10^{14} cm^{-3}$, $u_s = 2 \times 10^7 cm/s$ ($M_A = 4.0$).

Fig.5 Experimental profiles of Robson and Sheffield.



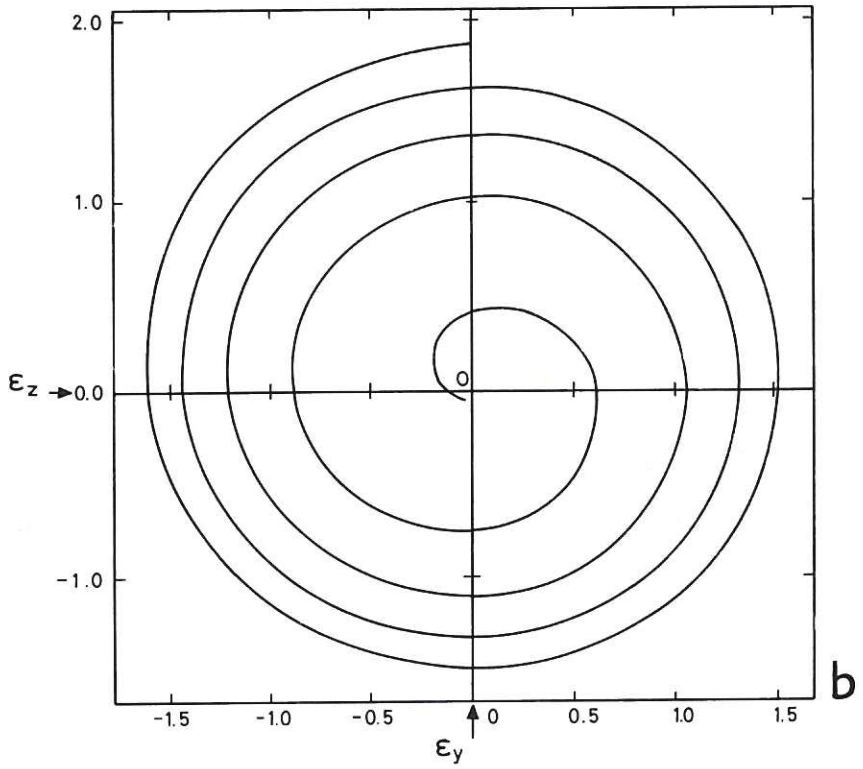
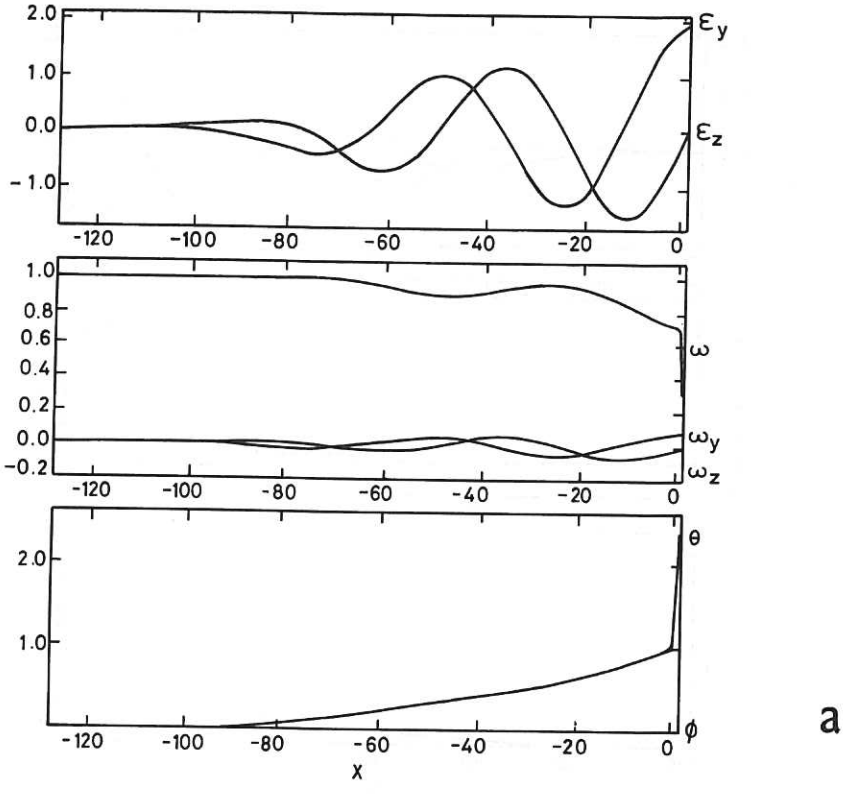


Fig.6 Non-dimensional profiles for the classical model. Upstream conditions: $a_1 = 45°$, $s_1 = 0.133$, $M_1 = 4.0$



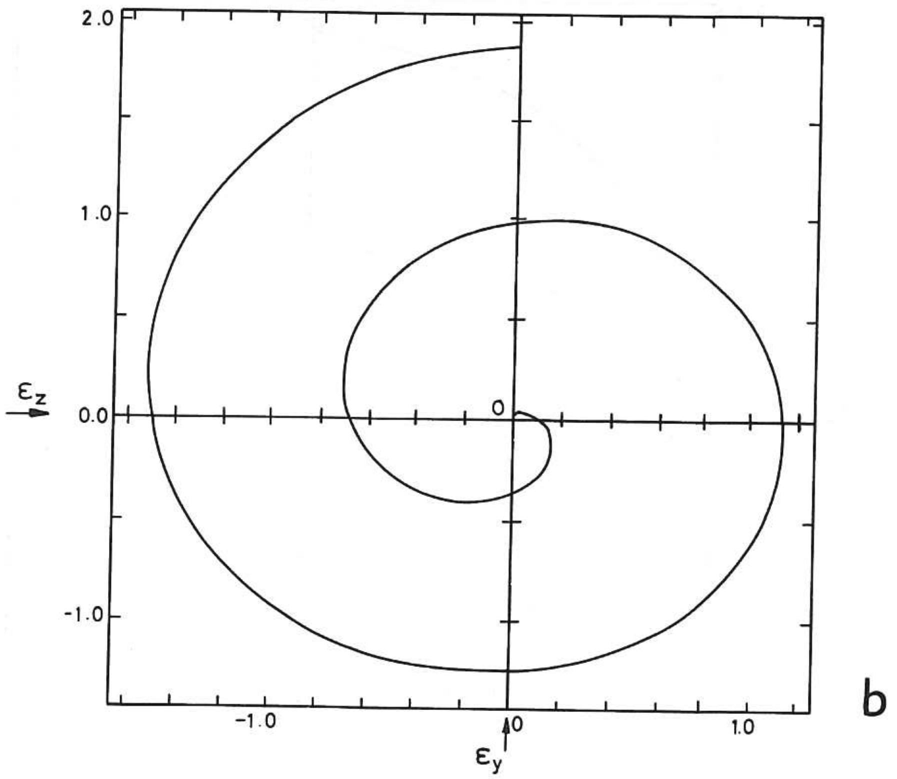


Fig.7 Non-dimensional profiles for model (ii). Upstream conditions: $\alpha_1 = 45°$, $s_1 = 0.133$, $M_1 = 4.0$

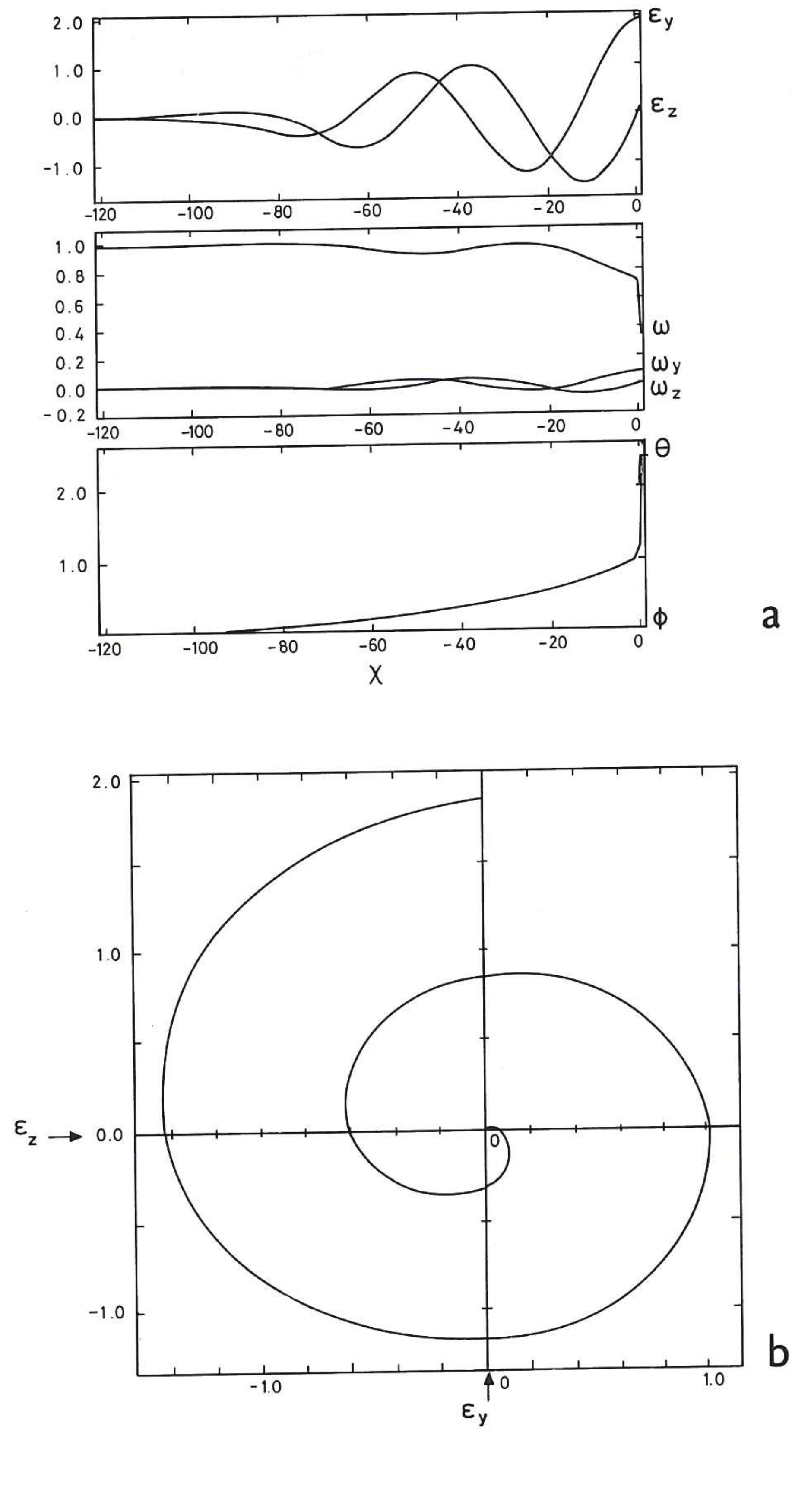


Fig.8 Non-dimensional profiles for model (iv). Upstream conditions: $a_1 = 45°$, $s_1 = 0.133$, $M_1 = 4.0$.