CALCULATION OF THE GAIN FACTOR Q IN TWO COMPONENT PLASMA SYSTEMS

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ABSTRACT

We present calculations of Q including the effects of velocity diffusion, finite ion temperature, electric fields and charge exchange. We find that if all charge exchange fast ions are lost then Q is severely reduced for a neutral fraction $\frac{n_n}{n_e} \geqslant 7 \times 10^{-6}$. These losses are reduced by reionisation of the charge exchanged fast ions. Calculations for the case of a two component Tokamak show that the effective charge exchange rate is typically reduced by 2.5 → 4 times owing to reionisation. With allowance for this, we find that in a stationary system with a plasma loss rate $n\tau_p \sim 10^{12} \rightarrow 10^{13}$ cm^{-3}.s the charge exchange losses will restrict $Q \leqslant 1$.

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1. INTRODUCTION

In conventional fusion systems with thermal energy distributions the fusion release occurs through interactions of the high energy tail of the ion distribution function and a significant release is generally obtained only for temperatures in the 10 keV region. In the two component systems the high energy part of a cold plasma ion distribution is enhanced by the injection of a high energy neutral beam. The neutrals are trapped, primarily by ionisation on the background plasma and slow down by collisions. We require a high electron temperature $T_e \geqslant 5$ keV in order that the slowing down time will be long enough to give a high fusion probability but there is no constraint on the background ion temperature. We consider here the case of a deuteron beam, initial energy $\sim$ 100 keV injected into a tritium plasma. A two component Tokamak system of this type has been proposed by Dawson et al [1]. They define as a figure of merit for such a system

$$Q = \frac{\text{energy released in fusion products}}{\text{energy in the beam}} .$$

For a D-T system the local Q on a magnetic surface may be written

$$Q = \frac{\varepsilon_F}{S(r)\varepsilon_o} \iint d\phi \, d\phi' \, d\zeta \, d\zeta' \, v^2 dv \, v'^2 dv' f(\underset{\sim}{v}) f_T(\underset{\sim}{v}) \sigma(|\underset{\sim}{v}-\underset{\sim}{v}'|) |\underset{\sim}{v}-\underset{\sim}{v}'| \qquad (1)$$

where (σ) is the fusion cross section, [12] (f_T) is the background tritium distribution, (f) is the fast deuterium ion distribution, S is the number of injected particles per unit volume per second. We take the energy released per reaction as $\varepsilon_F = 17.6$ MeV. The attainment of $Q \geq 1$ is commonly taken as an objective.

The calculation of the average $\bar{Q}$ for an experiment, from the local Q, involves the radial profiles of electron and ion temperature, density, Z_{eff}, slow neutral density and injected beam. These should idealy all be treated in a self-consistent manner, this strictly requires a detailed knowledge of the transport coefficients and the solution, at the very minimum, of a set of one dimensional diffusion equations. Such work is currently in progress (11). However, some of the interacting factors may be decoupled and treated separately. In particular, the fast ion and slow neutral radial profiles can be obtained as a function of the radial density profile and then an average $\bar{Q}$ may be calculated which includes charge exchange losses but with the profile dependent correction for reionisation.

In section 2 of this paper we obtain the fast ion distribution function on a magnetic surface from an appropriate Fokker-Planck equation which includes a finite ion temperature $T_i = T_e$, the effects of velocity diffusion, an electric field (there is a toroidal electric field in a Tokamak) and of charge exchange. We find that the local Q, for a beam energy $\varepsilon_o \sim 150$ keV, is increased by 5% → 10% by velocity diffusion and that an accelerating electric field of 5.7×10^{-4} V cm^{-1} gives a further increase of 7%. With zero electric field and no charge exchange for $\varepsilon_o \sim 150$ keV and $T_e \geq 4.4$ keV we have $Q \geq 1$. If we assume that all charge exchanged ions are lost then $Q \leqslant 1$, for all temperatures, if the neutral fraction $\frac{n_n}{n_e} \geqslant 7 \times 10^{-6}$.

In a real experiment we recover some of the charge exchanged beam ions because they are reionised before they leave the plasma, we therefore define an effective charge exchange loss rate. In calculating this for a particular system we must treat the radial deposition of the injected fast particles and slow neutral population in a self-consistent fashion. We find, in section 3, that for a two-component Tokamak the effective loss rate is typically less by 2.5 → 4 times.

The result is insensitive to the plasma density profile because the penetration of the injected beam and of the slow neutrals have a similar dependence on the profile. In equilibrium the level of neutral density in the plasma depends on the particle containment time (τ_p) and we relate the average $\bar{Q}$ to $n\,\tau_p$. Finally in section 4 we treat the particular case of two-component operation in the presence of the trapped-ion instability, to illustrate the effects of these losses. (Note that the influx of slow neutrals may be reduced if injection is followed by rapid adiabatic compression [11]; this topic will not be discussed here).

2. SOLUTION OF THE FOKKER-PLANCK EQUATION AND DETERMINATION OF LOCAL Q.

We consider a stationary system and our first step is to solve the Fokker-Planck equation to obtain the distribution function of the injected deuterons. The method of solution is given in Appendix A and follows that of Cordey and Core [2] [3], except that in this paper the velocity diffusion corrections are obtained to first order in $\delta = \frac{T_e}{2\varepsilon_o}$ and we use the energy dependent charge exchange loss rate. The second step is to obtain Q from equation (1) using the distribution function given in Equations (A 12-15) of Appendix A. Equation (1) may be considerably simplified by expanding σ and $(\underline{v} - \underline{v}')$ as series in v/v' then completing the integration over ϕ, ϕ', ζ, ζ', and v' gives the following for Q local,

$$Q = \frac{\pi^{\frac{1}{2}}\varepsilon_F}{S\varepsilon_o}\int_o^\infty dv\; a_o(v)\; v^3\left[\sigma(v)(1 + v_i^2/2v^2) + \frac{v_i^2}{v}\frac{\partial\sigma}{\partial v} + \tfrac{1}{4}\,v_i^2\,\frac{\partial^2\sigma}{\partial v^2}\right]. \quad (2)$$

The above result is correct to first order in $\frac{v_i^2}{v^2}$ and the expression for $a_o(v)$ is obtained from equations (A13)-(A15). The remaining integration over (v) is completed numerically and the resulting Q is plotted against injection energy, neutral density, and electron temperature in Figures 1 → 5.

The effect of finite ion temperature and energy diffusion is shown in Figure 1. In curves A, the ion temperature is zero and the energy diffusion terms are omitted. These Q are the same as given by Dawson et al [1].
Curves B show the effect of finite ion temperature $T_i = T_e$ and curves C the additional effect of diffusion, these effects increase Q by ~ 5% at T_e = 5 keV and by ~ 10% at 10 keV for ε_o ~ 150 keV. Energy diffusion reduces the slowing down rate of the injected ions and also causes a high energy tail to be formed above the injection energy, typical fast ion distributions with and without diffusion are shown in Fig. 2. The effect of charge exchange on Q for $T_e = T_i$ = 5 keV is shown in Fig. 3 as a function of injection energy and neutral fraction. The expression given by Riviere [6] for the energy dependent charge exchange cross section is used (see also Figure 7):

$$(\sigma v)_{cx} = \frac{2.14 \times 10^{-7} \; \varepsilon^{0.5} \; (0.535 - 0.155 \log_{10}\varepsilon)}{(1 + 8.97 \times 10^{-5} \; \varepsilon^{3.3})} \; cm^3 s^{-1} \quad . (3)$$

It is assumed that all charge exchanged beam ions are lost. In Fig. 4, Q for a range of electron temperatures is plotted as a function of the neutral density. If all charge exchange ions are lost then Q < 1 for all T for $n_n/n_e > 7 \times 10^{-6}$; in practice 60 - 75% of neutrals are reionised and a higher neutral density can be tolerated.

Finally in Fig. 5 we see that the fusion rate can be increased by injecting parallel to an electric field.

The field has been taken as that which would be obtained with a driving voltage of 1 volt in a Tokamak of major radius 300 cm. For Q as defined in equation (1) this gives an increase of 7% for ε_o ~ 150 keV.

3. THE EFFECTIVE CHARGE EXCHANGE RATE

(a) The local Q of section 2 overestimate the charge exchange losses because they ignore reionisation of the fast neutrals, they also do not include radial profile effects.

We now calculate the effective charge exchange loss rate coefficients, and find the interesting result that because of the constraints set by the need for good beam penetration ($\bar{n}a$) and long slowing down times τ_s (T_e) the effective rates are not very dependent on the radial profiles. We now define an effective charge exchange rate $\overline{\sigma v}_{cx}$ for the energy ϵ which includes reionisation,

$$\overline{\sigma v}_{cx} = (\sigma v)_{cx} \frac{\int_0^a n_n N_f P_L \, dr.}{\bar{n}_n \int_0^a N_f dr} \qquad cm^3 \; s^{-1} , \qquad (4)$$

where $n_n(r)$ is the radial profile of slow neutrals (equation B.8); and $\bar{n}_n$ is the equivalent uniform neutral density(equation (B.10)); $N_f dr$ is the number of injected ions trapped in the radial range $r \rightarrow r \rightarrow dr$; and $P_L(r)$ is the probability that a charge exchanged ion leaves the plasma(equation (B.3)). The integrals are evaluated in Appendix B and the solution is given in equation (B.11) and it is plotted in Fig. 7, For most of the range $60 < \epsilon < 200$ keV where the fusion cross section is high, and for a wide variety of density profiles $\overline{\sigma v}_{cx} \sim (0.25 \rightarrow 0.4) \; (\sigma v)_{cx}$.

This result is not unexpected because if we simply assumed that half the particles went directly towards the near wall and were lost and half went further into the plasma and were reionised then we would obtain $\overline{\sigma v}_{cx} = 0.5 \; (\sigma v)_{cx}$. The smaller factor occurs because some neutrals moving out are also reionised.

Now we may apply this to the Q calculations of Section 2. In figure 4, Q is plotted against $\frac{\bar{n}_n}{n_e}$. We now see that because of reionisation the actual neutral density which leads to a given Q may be 2.5 → 4 times greater. We may also relate Q to $\bar{n} \tau_p$ using equation (B.10) and the result (B.11).

$$\bar{n} \tau_p = (0.25 \rightarrow 0.4) \frac{f_n n}{(\sigma v)_e \bar{n}_n} \quad cm^{-3} \; s \qquad (5)$$

and for $100\ \text{eV} \lesssim T_e \lesssim 10^4\ \text{eV}$, $(\sigma v)_e \simeq 2.5 \times 10^{-8}\ \text{cm}^3\ \text{s}^{-1}$

this is shown in Fig.4. It must be remembered that for some loss mechanisms (e.g. radiation losses) the particle confinement time can be long compared to the energy confinement time. On the other hand for the trapped-ion mode, which has been predicted to occur in Tokamaks operating in the collisionless regime [10], the particle and energy confinement times are similar.

4. Q IN THE PRESENCE OF THE TRAPPED-ION MODE

The trapped-ion mode [10], if it comes up to the worst expectations, can severely limit the temperatures in a Tokamak, so that ignition conditions $n\tau_E \sim 10^{14}\ \text{cm}^{-3}\ \text{s}$ would be hard to achieve. Two-component Tokamak operation has been proposed [11] to permit sizeable fusion gains $Q \gg 1$, when $n\tau_E \sim 10^{13}\ \text{cm}^{-3}\ \text{s}$.

We see below that if we assume that $\tau_p = \tau_E$ for the trapped ion mode then to obtain $Q > 1$ we need very large Tokamaks in order to combat charge exchange losses.

For the trapped ion mode (when $n\tau_p = n\tau_E$)

$$n\tau_p \simeq \frac{0.9 \times 10^{-24}\ (B.R.)^2\ (n\ a)^2\ Z_{eff}\left(\frac{R}{a}\right)^{0.5}}{T^{3.5}}\ \text{cm}^{-3}.\text{s} \qquad (6)$$

where B(kG), ε_o, T, (keV), n(cm^{-3}), a, R(cm).

We substitute for na from B.3 ignoring the difference between n and $\bar{n}$, and obtain

$$n\ \tau_p = \frac{7.1 \times 10^2\ (B.R.)^2\ F_1^2\ \text{Sin}^2\ \theta\ \varepsilon_o^2 \left(\frac{R}{a}\right)^{0.5}}{T^{3.5}\ Z_{eff}}\ \text{cm}^{-3}.\text{s} \quad . \qquad (7)$$

Alternatively using (33) we see that

$$\frac{n_n}{n} = \frac{(1.4 \rightarrow 2.3) \times 10^4\ f_n\ Z_{eff}.\ T^{3.5}}{(B.R)^2\ .\ F_1^2\ \text{Sin}^2\ \theta\ \varepsilon_o^2} \left(\frac{a}{R}\right)^{0.5} \quad . \qquad (8)$$

To obtain $Q \geq 1$ we need, (using Fig. 4), with $\left(\frac{R}{a}\right) = 3$

$$T_e = \begin{Bmatrix} 5 \text{ keV} \\ 7.5 \text{ keV} \\ 10 \text{ keV} \end{Bmatrix} \qquad B.R \geq \begin{Bmatrix} 0.8 \rightarrow 1.1 \times 10^4 \\ 1.1 \rightarrow 1.4 \times 10^4 \\ 1.3 \rightarrow 1.7 \times 10^4 \end{Bmatrix} \times \left(\frac{f_n Z_{eff}}{F_1^2 \sin^2\theta}\right)^{\frac{1}{2}} \text{kG.cm} .$$

The lower limit is associated with the lower value of F_1 (see Fig. 7). We see that typically for $Z_{eff} = 2$, $f_n = 1$ $\sin\theta = 0.7$, to obtain $Q \geq 1$ we need $B.R \geq 2 \times 10^4$ kG.cm, or $B \geq 100$ kG for $R = 200$ cm. These parameters are outside the range of presently conceived experiments.

5. CONCLUSIONS

The calculation of the beam fusion gain Q for an experiment involves the radial profiles, of T_e, T_i, n_e, n_i, Z_{eff}, n_n and of the injected beam, and these should strictly be treated in a self-consistent fashion. We have treated parts of the calculation as a first step towards tackling the complete problem on a computer. This is a valid exercise because the calculations of local Q, charge exchange losses and radial profile effects may to a good approximation be decoupled. We find,

1. The local gain factor Q for deuteron injection energies $\epsilon_o \sim 150$ keV in a tritium plasma, can be greater than unity for $T_e > 4.4$ keV and it is enhanced by velocity diffusion effects, 5 - 10%. An accelerating electric field of 0.5 mV/cm gives a 7% increase at $\epsilon_o \approx 150$ keV.

For lower energies there is a significant increase when the ion temperature is large i.e. $T_i \simeq T_e$, because there is an increase in the number of reactions involving relative velocities in the region of the peak fusion rate.

2. In the presence of neutral gas, charge exchange losses of the fast ions can significantly reduce Q. If we assume that all charge exchanged deuterons are lost then

$$Q < 1 \quad \text{for all } T_e \quad \text{for } n_n/n_e > 7 \times 10^{-6} \quad .$$

3. In a real experiment the losses are reduced owing to reionisation of these fast neutrals. For the case of a two component tokamak subject to constraints on the injection energy to ensure good beam penetration, and on temperature $\bar{T}_e \geq 4.4$ keV to permit $\bar{Q} \geq 1$, we find that the effective charge exchange rate is reduced by 2.5 → 4 times.

4. It is interesting to see that high Z_{eff} is a disadvantage because of the reduction it causes in beam penetration.

5. We considered the implications for two component tokamak operation in the presence of trapped-ion mode losses, taking a pessimistic view of these losses. We find that typically we need

$$B.R \geq 1.0 \times 10^4 \left(\frac{f_n Z_{eff}}{F_1^2 \sin^2\theta}\right)^{\frac{1}{2}} \text{ kG.cm}$$

in order to obtain $\bar{Q} \sim 1$. It appears from this that near perpendicular injection ($\theta \sim 90^o$) would be advantageous, because of the Sin θ dependence and also because injected particles will be scattered into trajectories having an increased path to the wall.

6. It must be remembered that the attainment of $\bar{Q} = 1$ is a rather arbitrary goal, and that the more worrying feature of the charge exchange losses is the implied high level of wall bombardment. The local wall bombardment will be worse if the neutrals concentrate near the limiters as is commonly observed.

7. In our calculations we have not included the effect of impurity density, plasma temperature and density profiles in a self-consistent way. The effect of impurity density profile, whether it is peaked near the wall or on axis, will have a dramatic effect on $\bar{Q}$ because of the influence it has on fast particle ion deposition. The average $\bar{Q}$ taking into account the radial temperature profile will in general exceed the $\bar{Q}$ at the average temperature.

Further, one could conceive of carefully selected profiles, for example one with a region of low plasma temperature near the wall, that would reduce the slow neutral influx. However, it must be shown that these desirable profiles are self-consistent.

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APPENDIX A

Fast Ion Distribution Function The guiding centre Fokker-Planck equation on a magnetic surface, in non-dimensional form is (see Cordey [3])

$$\delta u^{-2} \frac{\partial^2}{\partial u^2} \left\{ \left(u^2 + \gamma/u \right) f \right\} + u^{-2} \frac{\partial}{\partial u} \left\{ \left(u^3 - 2u\delta + \alpha^3 + \delta\gamma/u^2 \right) f \right\}$$

$$+ (2u^3)^{-1} \frac{m_i}{m_f} \alpha^3 \frac{Z_{eff}}{\bar{Z}} (1 + b\delta) \frac{\partial}{\partial \zeta} (1 - \zeta^2) \frac{\partial f}{\partial \zeta} - f \frac{\tau_s}{\tau_{cx}}$$

$$+ S \frac{K(\zeta)}{v_o^3} \delta(u - 1) = \lambda \left[\frac{(1 - \zeta^2)}{u} \frac{\partial f}{\partial \zeta} + \zeta \frac{\partial f}{\partial u} \right] \qquad ..... (A1)$$

where $u = v/v_o$, $\zeta = v_{\parallel}/v$, v_o is the injection velocity,

$Z_{eff} = \Sigma n_j Z_j^2/n_e$ $\alpha^3 = v_c^3 \bar{Z}/v_o^3$, $\delta = 0.5\ m_e v_e^2/m_f v_o^2$,

$v_c^3 = 0.75\pi^{\frac{1}{2}}\ m_e\ v_e^3/m_i$, $\gamma = 0.75\pi^{\frac{1}{2}}\ v_i^2\ v_e\ \bar{Z}/v_o^3$

$\lambda = Z_f\ e\ E^*/m_f\ v_o\ \tau_s$, $\tau_s = 3m_e\ v_e^3\ m_f/(16\pi^{\frac{1}{2}}\ e^4\ Z_f^2 \log \Lambda\ n_e)$

and $b = -m_f T_i\ \bar{Z}/(2m_i\ T_e\ Z_{eff}\ u^2) + 0.75\ v_o\ u/(v_e\ \delta\ Z_{eff})$

$\bar{Z} = \sum_j \frac{n_j Z_j^2 m_i}{n_e m_j}$, S is the number of particles injected per unit volume/sec.

In obtaining equation (A1) the Rosenbluth potentials [4] were simplified by use of the approximation $v_i < v < v_e$ where $(v_{i,e})$ are the background ion and electron thermal velocities and (v) is the fast ion velocity. The terms on the left hand side have the following physical interpretation: they are in order, energy diffusion, friction, angular scattering, charge exchange and the final term is the source of injected fast ions (assumed to be mono-energetic with angular spread $K(\zeta)$). The term on the right hand side represents electric field acceleration. The boundary conditions that the solution must satisfy are, at $u = 1$, f continuous and

$$\delta\,(1+\lambda)\,\frac{\partial f}{\partial u}\Big|_{u=1^-}^{u=1^+} = -\,SK(\zeta)/v_o^3\,,$$

and $f \to o$ at $u = \infty$ and f is finite at $u = o$.

To solve the equation we expand (f) in a series of Legendre polynomials.

$$f = \sum_{n=o}^{\infty} a_n(u)\;P_n(\zeta), \qquad \text{(A2)}$$

and the equation for the coefficients $a_n(u)$ is

$$\delta u^{-2}\frac{d^2}{du^2}\left\{\left(u^2+\gamma u^{-1}\right)a_n\right\} + u^{-2}\frac{d}{du}\left\{\left(u^3-2u\delta+\alpha^3+\delta\gamma u^{-2}\right)a_n\right\}$$

$$-\left[(2u^3)^{-1}\frac{m_i}{m_f}u_c^3\;(1+b\delta)Z_{eff}\;n(n+1)+\frac{\tau_s}{\tau_{cx}}\right]a_n + S\,\delta(u-1)K_n/v_o^3$$

$$=\lambda\left\{n\left(\frac{da_{n-1}}{du}-(n-1)a_{n-1}u^{-1}\right)\Big/(2n-1)+(n+1)\left(\frac{da_{n+1}}{du}+(n+2)a_{n+1}u^{-1}\right)\Big/(2n+3)\right\}$$

(A3)

Use is now made of the two small parameters which appear in equation (A1),

$\delta = \dfrac{T_e}{m_f V_o^2}$ and $\lambda = Z_f e\,E^*\tau_s/m_f v_o$. The parameter δ multiplies the highest derivative in the equation and this suggests that a W K B solution of the form

$a_n = \phi_n(u)\,\exp\left(\int_1^u S_n\,du/\delta\right)$ is appropriate where the function ϕ_n is a power series in δ, $\phi = \sum_{m=o}^{\infty}\phi_n^m(u)\;\delta^m$.

The coefficients ϕ_n^m and S_n are then in turn expanded as power series in (λ) as follows

$$\phi^m_n = \sum_{t=o}^{\infty} \phi^m_{nt} \lambda^t , \qquad S_n = \sum_{t=o}^{\infty} S_{nt} \lambda^t \quad .$$

The zero'th order equation is

$$\left(u^2 + \gamma u^{-1}\right) u^{-2} S^2_{no} \phi^{(o)}_{no} + u^{-2} (u^3 + \alpha^3) S_{no} \phi^{(o)}_{no} = o \quad .$$

The above equation has two roots.

$$S_{no} = -\left(u^3 + \alpha^3\right) \Big/ \left(u^2 + \gamma u^{-1}\right) \qquad (A4)$$

and $S_{no} = o$. (A5)

The first root gives rise to a rapidly decaying solution as $u \to \infty$, $\left(e^{-u/\delta}\right)$ in contrast to the second root which leads to a solution varying as (u). The higher order equations in (δ) and (λ) are then solved and for the first root

$$\phi^{(o)}_{no} = A_n \left(\frac{u^3 + \alpha^3}{(1+\alpha^3)u^3}\right)^{\beta_n/3} \exp\left[- \tau_s \int_1^u \frac{x^2\,dx}{\tau_{cx}(x^3+\alpha^3)}\right] \qquad (A6)$$

$$S_{n1} = u^2 \left[n\, \phi^{(o)}_{n-1,o} \Big/ (2n-1) + (n+1)\, \phi^{(o)}_{n+1,o} \Big/ (2n+3)\right] \Big/ \left[(u^2 + \gamma u^{-1}) \phi^{(o)}_{n,o}\right] \qquad (A7)$$

with $\beta_n = \frac{m_i}{2m_f} u_c^3 \; (1 + b\delta) Z_{eff} \; n(n+1)$.

Where A_n is an arbitrary constant $\phi^1_{n_o}$ and $\phi^o_{n_1}$ have been derived but since they are not required in the remainder of the paper they are dropped from equations (A6) and (A7).

For the second root $S_n(o)$, the higher order solutions are

$$\phi^{(o)}_{no} = B_n u^{\beta_n} \left(\frac{1+\alpha^3}{u^3+\alpha^3}\right)^{1+\beta_n/3} \exp\; \tau_s \int_1^u \frac{x^2\,dx}{\tau_{cx}(x^3+\alpha^3)} \qquad (A8)$$

$$\phi_{no}^{(1)} = \phi_{no}^{(o)} \int_u^1 \frac{d}{du}\Big[(u^2 + \gamma u^{-1}) \frac{d\phi_{no}^{(o)}}{du} \Big] \Big/ \Big[\phi_{no}^{(o)} (u^3+\alpha^3) \Big] - \alpha^3 \beta_n b u^{-1} (u^3+\alpha^3)^{-1} \, du \tag{A9}$$

$$S_{ni} = o \tag{A10}$$

$$\phi_{n2}^{(o)} = \phi_{no}^{(o)} \int_1^u du \, u^2 \Big\{ n\Big[\frac{d\phi_{no}^{(o)}}{du} - (n-1)\phi_{n-1,o}^{(o)} u^{-1} \Big] \Big/ (2n+1) + (n+1)\Big[\frac{d\phi_{n+1,o}^{(o)}}{du} + (n+2)\, \phi^{o}_{n+1,o}\, u^{-1} \Big] \Big/ (2n+3) \Big\} \tag{A11}$$

where B_n is an arbitrary constant.

The boundary conditions at $u = 1$ are

a_n continuous

$$\text{and } \delta(1+\gamma) \left. \frac{da_n}{du} \right|_{u=1-}^{u=1+} = - S K_n / v_o^3 \tag{A12}$$

at $u = \infty$, $a_n \to o$ and at $u = o$, a_n is finite. The second solution, (equation (A5) with (A8)-(A11) , is not well behaved at $u = \infty$ so in the region $1 < u < \infty$ the appropriate solution is the exponentially decaying one, (equation (A4) with (A6) and (A7)). In the region $o < u < 1$ only the second solution is finite at $u = o$ so in this region the solution is given by equations (A8) -(A11). Using the boundary condition at the source $u = 1$ the constants A_n and B_n may be determined and the complete solution to first order in δ and ϵ is

$$f = \sum_{n=o}^{\infty} a_n(u) \quad P_n(\zeta) \tag{A13}$$

with

$$a_n = a_n^o(u) \cdot \Big\{ 1 + \delta \int_u^1 dx \Big([a_n^o (x^3 + \alpha^3)]^{-1} \frac{d}{dx}(x^2 + \gamma/x) \frac{da_n^o}{dx} - \frac{\alpha^3 \beta_n bx}{x^3 + \alpha^3} \Big)$$

$$+ \lambda \int_1^u dx \; x \; [(x^3 + \alpha^3) a_n^o]^{-1} \Big[n\Big(\frac{da_{n-1}^o}{dx} - (n-1) a_{n-1}^o \, x^{-1} \Big) / (2n-1)$$

$$+ (n+1) \Big(\frac{da_n^o}{dx} + (n+2) a_{n+1}^o \Big) / (2n+3) \Big] \Big\} \quad \text{for } u < 1 \quad \ldots\ldots. \tag{A14}$$

where

$$a_n^o(x) = A_n x^{\beta_n} \left((1+\alpha^3)/(x^3+\alpha^3)\right)^{1+\beta_n/3} \cdot \exp\left(-\int_u^1 (\tau_s/\tau_{cx}) x^2 (x^3+\alpha^3)^{-1} dx\right)$$

and for u > 1

$$a_n = A_n u^{-\beta_n} (u^3+\alpha^3)/(1+\alpha^3)^{\beta_n/3} \exp\left\{\int_u^1 (\tau_s/\tau_{cx}) x (x^3+\alpha^3)^{-1}\right.$$

$$+ (x^3+\alpha^3)(x^2+\gamma/x)^{-1}\delta^{-1} + \gamma \int_1^u x^2 [\delta(x^2+\gamma/x)A_n]^{-1}$$

$$\left[nA_{n-1}\left\{(x^3+\alpha^3)/x^3(1+\alpha^3)\right\}^{(\beta_{n-1}-\beta_n)/3} + (n+1)A_{n+1}\left\{(x^3+\alpha^3)/\right.\right.$$

$$\left.\left.(1+\alpha^3)x^3\right\}^{(\beta_{n+1}-\beta_n)/3}\right]\Bigg\} \qquad \ldots\ldots\ldots\ldots\ldots \quad \text{(A15)}$$

where

$$A_n = S\tau_s K_n \left\{1 + (3-2\alpha^3\beta_n - 2\tau_s/\tau_{cx})(1+\gamma)\delta(1+\alpha^3)^{-2}\right\}\left\{(1+\alpha^3)v_o^3\right\}^{-1}$$

$$+ S\,\tau_s\,\lambda\left\{n\,K_{n-1}/(2n-1) + (n+1)K_{n+1}/(2n+3)\right\}\left\{(1+\alpha^3)^2 v_o^{3/2}\right\}^{-2} \quad \ldots \text{(A16)}$$

Certain aspects of the above solution for the fast ion distribution function have been discussed elsewhere[2] (i.e. the effect of electric field, scattering etc.) and this solution has been compared with the measured fast ion distribution in the Cleo Tokamak[5]. In the calculation of Q only the n = o term of equations (13)→(16) is required but for completeness we have given the full solution.

APPENDIX B   Calculation of the Effective Charge Exchange Rate

This quantity $\overline{\sigma v}_{cx}$ is defined in equation 4. To evaluate it we need to obtain in a self consistent fashion the radial profiles of the injected fast particles, the probability of reionisation of a fast ion, charged exchanged at a given radius as function of energy, and the radial profile of the slow neutrals.

We will consider quasi-tangential injection, in the cylindrical approximation, see Fig. 6.

We will assume that the injected ion orbits are circular about the plasma centre, this is a reasonable approximation when

$\frac{140\left(\varepsilon_o\ (eV)\right)^{\frac{1}{2}}}{B_p\ (G)} \ll a$, where β_p (a) is the poloidal magnetic field

at the plasma edge.

Fast ion radial profile

The number of injected ions deposited in the range $r \rightarrow r + dr$, for the case of a pencil beam passing through the plasma centre is approximately

$$N_f(r)dr = 2\, I_o \frac{n_p\, a\, Z_{eff}}{L_o \operatorname{Sin}\theta} \frac{f(r)}{a} \exp\left[-\frac{n_p\, a\, Z_{eff}}{L_o \operatorname{Sin}\theta}\int_o^a \frac{dr'}{a} f(r')\right] .$$

$$\operatorname{Cosh}\left[\frac{n_p\, a\, Z_{eff}}{L_o \operatorname{Sin}\theta}\int_o^r \frac{dr'}{a}\ f\,(r')\right] dr \qquad (B1)$$

where I_o is the incident beam intensity, θ is the injection angle $n = n_p\ f(r)$ is the plasma density, and for deuterons [7] in a hydrogen plasma

$$L \cong 2.8 \times 10^{13}\ \varepsilon\ (keV)\ cm^2 \text{ for } \varepsilon = 40 \rightarrow 200 \text{ keV}. \qquad (B2)$$

Note that the trapping of fast injected deuterons for $\varepsilon_o > 80$ keV is predominantly through ionisation by plasma ions and the rate [8] is proportional to Z_{eff}. In this region we may rearrange (B2) as $$\bar{n}\, a = 2.8 \times 10^{13} \frac{F_1 \operatorname{Sin}\theta}{Z_{eff}} \epsilon_o \text{ (keV) cm}^{-2} \tag{B3}$$

where F_1 is the number of trapping lengths to the plasma centre, and $\bar{n} = \int_o^a n_p \, f\,(r)\, dr$.

Reionisation Probability

The probability that a charge exchanged ion leaves the plasma is

$$P_L\,(r) = \exp\left[-\frac{n_p\, a\, Z_{eff}}{L \operatorname{Sin}\theta} \int \frac{dr'}{a}\, f\,(r')\right] .$$

$$\operatorname{Cosh}\left[\frac{n_p\, a\, Z_{eff}}{L \operatorname{Sin}\theta} \int_o^r \frac{dr'}{a}\, f\,(r')\right] \tag{B4}$$

where we have assumed that a fast particle leaves the plasma along the same kind of trajectory as that on which it entered, i.e. we ignore angular scattering. This is a reasonable approximation for $\epsilon > \frac{15\, A_f\, T_e}{A_i^{2/3}} Z_{eff}$.

For perpendicular injection ($\theta = 90^o$) scattering will act to increase the path to the wall and will reduce the losses. For quasi-tangential ($\theta \lesssim 45^o$) and tangential injection we can expect to act to decrease the path.

Slow neutral profile The radial decay length of slow neutrals coming into the plasma when $\lambda s \ll a$ and the system is more or less stationary is for deuterium atoms (9)

$$\lambda_s \simeq \frac{10^6 \left[T \text{ (eV)}\right]^{\frac{1}{2}}}{\bar{\bar{n}} \left[(\sigma v)_T\right]} \left[\frac{(\sigma v)_{cx}}{(\sigma v)_e}\right]^{\frac{1}{2}} \text{cm} \tag{B5}$$

where $\bar{\bar{n}}$ is the mean density over the path, and $(\sigma v)_T = (\sigma v)_e + (\sigma v)_{cx}$ $(\sigma v)_e$ is the ionisation rate due to electrons, $(\sigma v)_{cx}$ is the charge exchange rate for ions of speed (v). Now for $T \leqslant 20$ eV the probability of ionisation is negligible compared to charge exchange and therefore a cold neutral from the wall will be charge exchanged many times before ionisation occurs and therefore in effect $T_{wall} \sim 20$ eV.

For $100 \text{ eV} < T \leq 10{,}000$ eV, $\frac{1}{(\sigma v)_T}\left[\frac{(\sigma v)_{cx}}{(\sigma v)_e}\right]^{\frac{1}{2}} \simeq 0.2 \times 10^8 \text{ cm}^{-3} \text{ s}$. (B6)

for $T < 100$ eV, the factor is bigger.

We will consider profiles of the form

$$f(r) = \left(1 - \left(\frac{r}{a}\right)^m\right)$$

and for such profiles near the wall

$$\frac{\lambda_s}{a} \geq 10^{-1}\left[\frac{2.8 \times 10^2 \; Z_{eff}}{\left(\frac{m+2}{m}\right)\left(1 - \frac{(m-1)}{3}\frac{\lambda_s}{a} \cdots\right) F \sin\theta \; \varepsilon_o \; (\text{keV})}\right]^{\frac{1}{2}} \tag{B7}$$

where we have taken $T_w = 16$ eV, and assumed that the neutral is then penetrating a plasma with $T \geq 100$ eV. We have also substituted for $\bar{n}a$ from (B.3). Inspection of (B1) shows that for reasonable penetration of the incident beam, i.e. not too much power deposition near the wall we require $F_1 \leq 2$. For $Z_{eff} \leq 2$, $\varepsilon_o \leq 2$, $F_1 \leq 2$, $\sin\theta \leq 1$ and $m \leq 8$ $\frac{\lambda s}{a} \sim 0.05 \rightarrow 0.14$. Thus for most realisable temperature profiles where $T_p \geq 2.5$ keV we can expect the slow neutrals to enter regions where T exceeds 100 eV in one penetration length. We assume that $T_i \geq T_e/2$.

In the main body of the plasma $\bar{n} \sim \bar{n}$ and

$$\frac{\lambda_s}{a} \simeq \frac{22\;[T(\text{keV})]^{\frac{1}{2}} Z_{eff}}{\varepsilon_o(\text{keV}) \sin\theta \; F_1}$$

for $T \geq 2.5$ keV, $\frac{\lambda_s}{a} \geq 0.1$ for most conditions of interest.

Thus it seems reasonable to approximate the neutral density profile by

$$n_n = n_w \exp\left(-\frac{a}{\lambda_e}\int_o^a \frac{dr'}{a} f(r')\right) \cdot \exp\left(\frac{a}{\lambda_e}\int_o^r \frac{dr'}{a} f(r')\right) \tag{B8}$$

where the effective mean free for the slow neutrals path $\lambda_e \sim 0.1a$. In fact as we will see the charge exchange loss of fast ions does not vary much with $\left(\frac{\lambda_e}{a}\right)$ for a given neutral recycling rate so the choice is not critical.

Neutral density at the wall

The neutral density in the plasma is determined by the balance between the influx of neutrals from outside the plasma and their ionisation in the plasma. The neutral density outside the plasma is determined by the rate of loss of ions from the plasma and the probability that they return as neutrals (f_n). The factor (f_n) includes the probability that ions are trapped in the wall, return as ions, release neutrals from the wall, and that some neutrals do not go to the plasma. $f_n = 1$ is a stationary system, $f_n < 1$ means that the plasma and neutral density is decreasing in time, $f_n > 1$ that it is increasing. We may set

$$f_n \cdot \frac{\bar{n}}{2\tau_p} = \int_0^a \frac{dr}{a} \frac{r}{a} \cdot n_n \cdot n_e \cdot (\sigma v)_e \qquad \text{(B9)}$$

where f_n is determined by the balance between the rate of ionisation of neutrals in the plasma and the flux of neutrals from the wall.

$$n_w \bar{v}_w = f_n \frac{\bar{n}\, a}{2\tau_p}$$

where $\bar{v}_w \leq 10^6 (T_w(ev))^{\frac{1}{2}}$ is the net inward neutral velocity. We will normally be considering $f_n = 1$, and to see if a case is realistic we must check that the inequality holds. It is a convenient fact that for $100 \text{ eV} \leq T \leq 10^4 \text{ eV}$, $(\sigma v)_e \cong 2.5 \times 10^{-8} \text{ cm}^3 \text{s}^{-1}$ and may be taken outside the integral. The equivalent uniform neutral density which would lead to the same plasma production rate is given by

$$\bar{n}_n = \frac{f_n}{\tau_p (\sigma v)_e} \quad . \qquad \text{(B10)}$$

Effective charge exchange rate

We now substitute for n, n_n, N_f and P_L in (4) and (B9) complete the integrals and find

$$\overline{\sigma v}_{cx} = \frac{F_1 \exp\left(-\left(\frac{m+2}{m+1}\right)F_1 \frac{\varepsilon_o}{\varepsilon}\right)}{\operatorname{Sinh}\left(\left(\frac{m+2}{m+1}\right) F_1\right)} \quad \frac{G_1 + G_2 + G_3 + G_4}{8G} \quad (\sigma v)_{cx} \qquad (B.11)$$

where

$$G = \int_0^1 dx.x(1-x^m)\exp\left[\frac{a}{\lambda_e}\left(x - \frac{x^{m+1}}{m+1}\right)\right] \simeq \exp\left[\frac{a}{\lambda_e} \frac{m}{(m+1)}\right] \frac{\lambda_e}{a}\left[1-\sqrt{\frac{\pi\lambda_e}{2am}}\right]$$

and $G_s = \dfrac{\exp\left[\frac{m}{(m+1)} Z_s\right]-1}{Z_s}$ (s = 1,2,3,4,) where $Z_1 = \dfrac{a}{\lambda_e} + \left(\dfrac{m+2}{m}\right)F_1\left(\dfrac{\varepsilon_o}{\varepsilon} + 1\right)$

$Z_2 = \dfrac{a}{\lambda_e} + \left(\dfrac{m+2}{m}\right) F_1\left(\dfrac{\varepsilon_o}{\varepsilon} - 1\right)$, $Z_3 = \dfrac{a}{\lambda_e} + \left(\dfrac{m+2}{m}\right) F_1\left(1 - \dfrac{\varepsilon_o}{\varepsilon}\right)$ and

$Z_4 = \dfrac{a}{\lambda_e} - \left(\dfrac{m+2}{m}\right) F_1 \left(\dfrac{\varepsilon_o}{\varepsilon} + 1\right)$.

In Fig. 7 we plot values of $\overline{\sigma v}_{cx}$ for a range of values of m, $(2,4,\infty)$; F_1 (1,1.5,2.0): $\frac{a}{\lambda_e}$ (5,10,20) and we see that

$\overline{\sigma v}_{cx} \simeq (0.25 \rightarrow 0.4)\ \sigma v_{cx}$ for most of the range 60 → 200 keV where the fusion reaction rate is high.

SYMBOLS

r	(cm)	radial coordinate
n, n_e, n_i, n_n	(cm^{-3})	density, electron, ion, neutral
$f(r)$		radial density profile
T, T_e, T_i	(keV)	temperature, electron, ion
$f(v, \phi, \zeta)$		velocity distribution function
R, a	(cm)	Major and minor radii of plasma
B	(kG)	toroidal magnetic field
A_i, A_f		atomic mass number, plasma ions, fast ions

$$Z_{eff} = \frac{\Sigma n_j Z_j^2}{n_e}$$

$$\overline{Z} = \sum_j \frac{n_j Z_j^2 m_i}{n_e m_j}$$

τ_E, τ_p	(s)	energy and particle confinement times
τ_s	(s)	fast ion slowing down time
$Q, \overline{Q}$		local and average beam fusion gain factor
ϵ_F	(keV)	energy release in fusion reaction
ϵ_o	(keV)	injection energy
$S(r)$		number of injected particles per cm^3 per sec
$\sigma(v)$	(cm^2)	fusion cross section
$(\sigma v)_{cx}$	($cm^3 s^{-1}$)	charge exchange rate coefficient
$\overline{\sigma v}_{cx}$	($cm^3 s^{-1}$)	effective charge exchange rate coefficient
$(\sigma v)_e$	($cm^3 s^{-1}$)	electron ionisation rate coefficient
$N_f dr$		number of fast ions deposited in $r \to r+dr$
F_1		number of trapping lengths to plasma centre for fast ions
θ	(deg)	mean injection angle
f_n		probability that lost plasma ions return as neutrals

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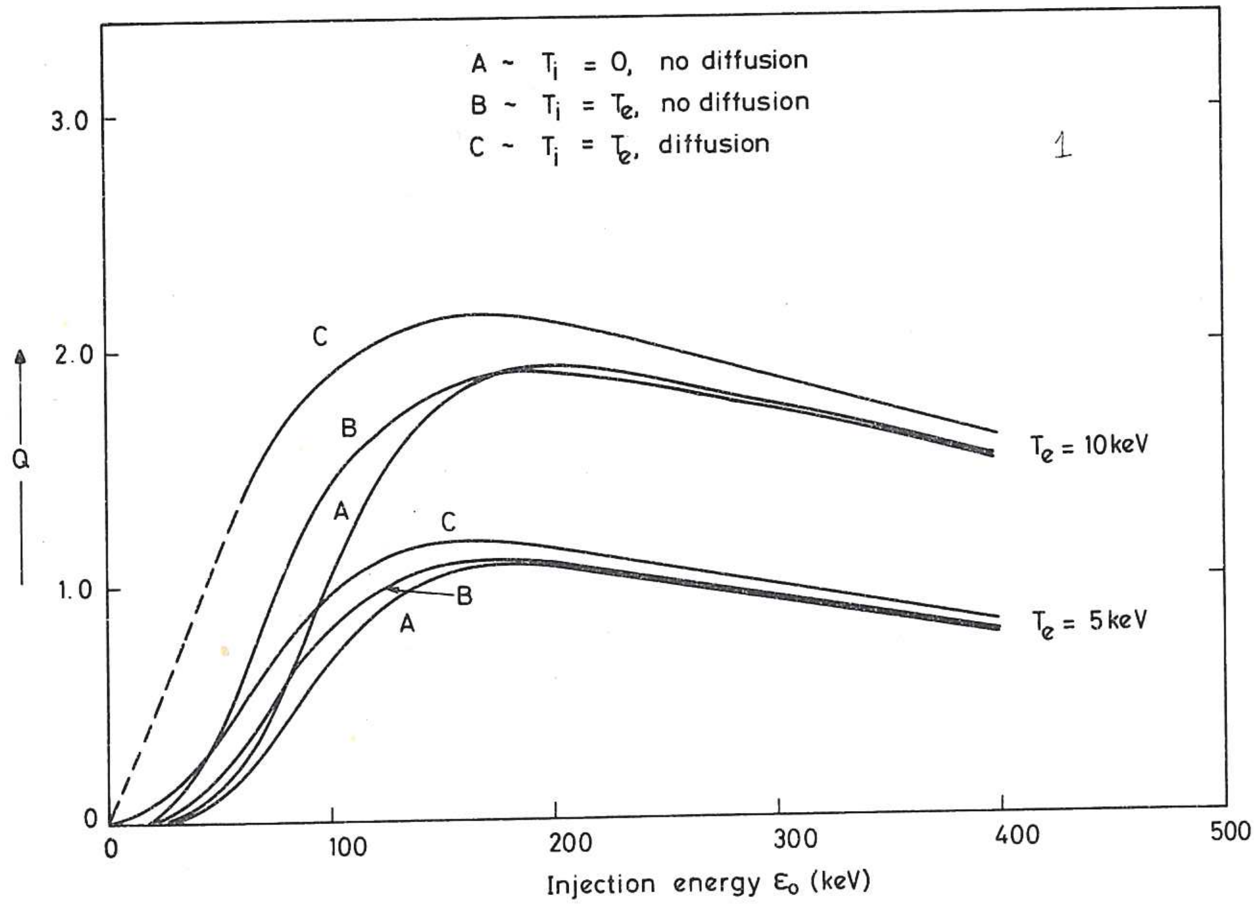

Fig.1 Gain factor Q for tritium plasma, as a function of injected deuteron energy.

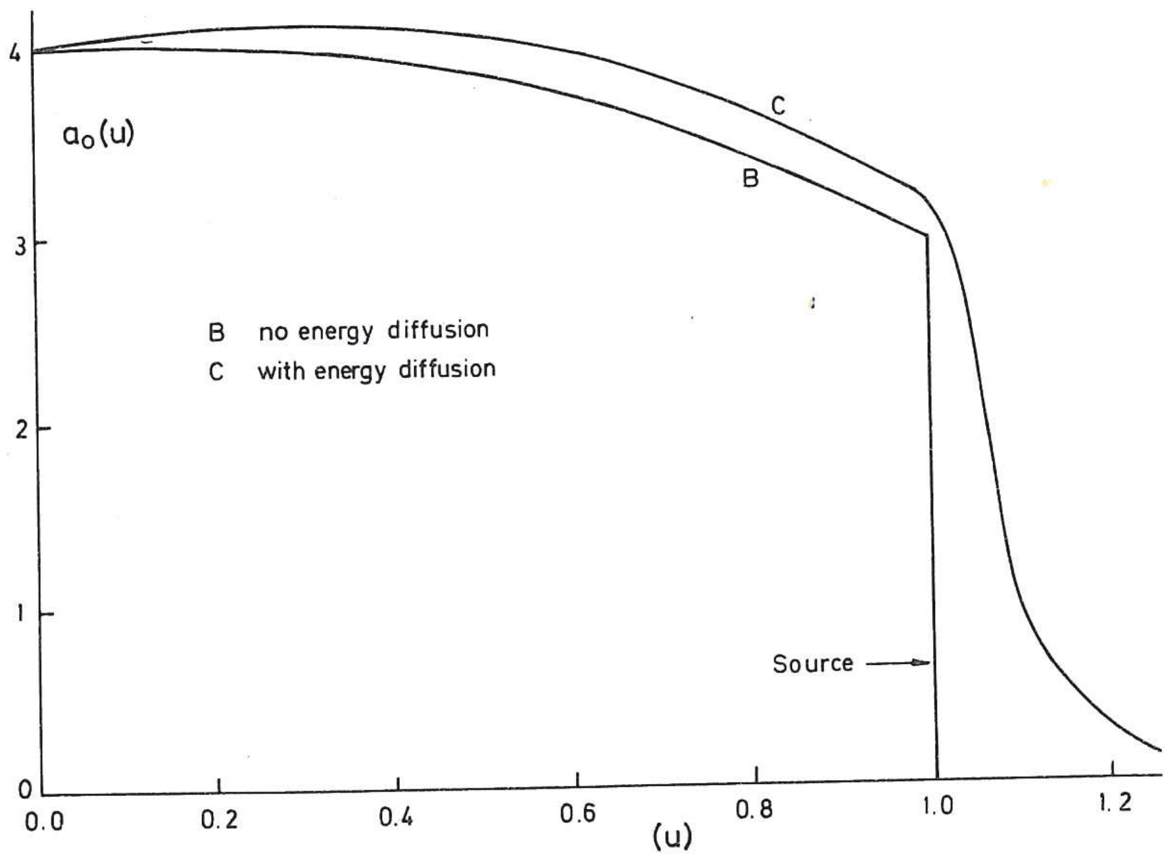

Fig.2 The effect of energy diffusion on the injected velocity distribution.

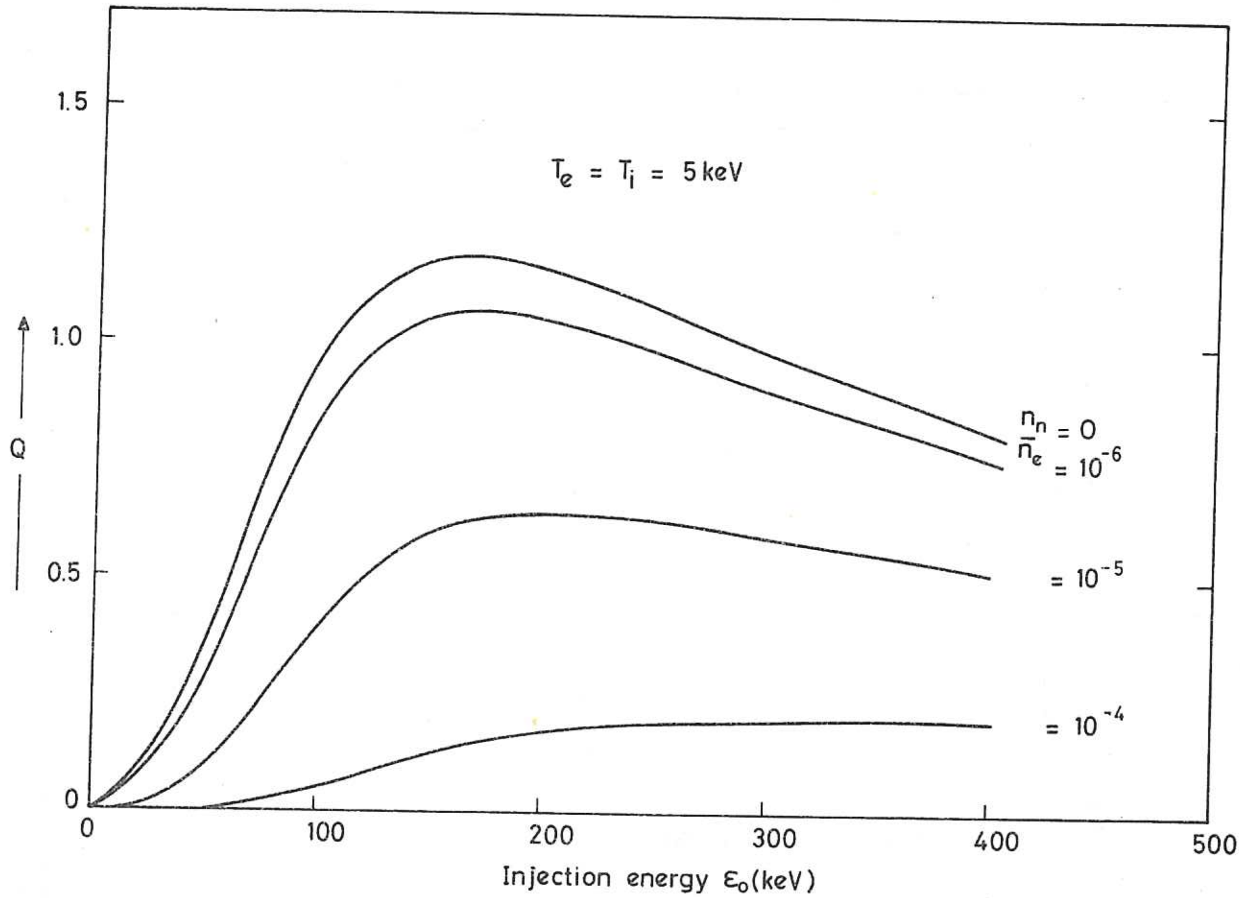

Fig.3 The effect of increasing neutral fraction on Q, for D → T.

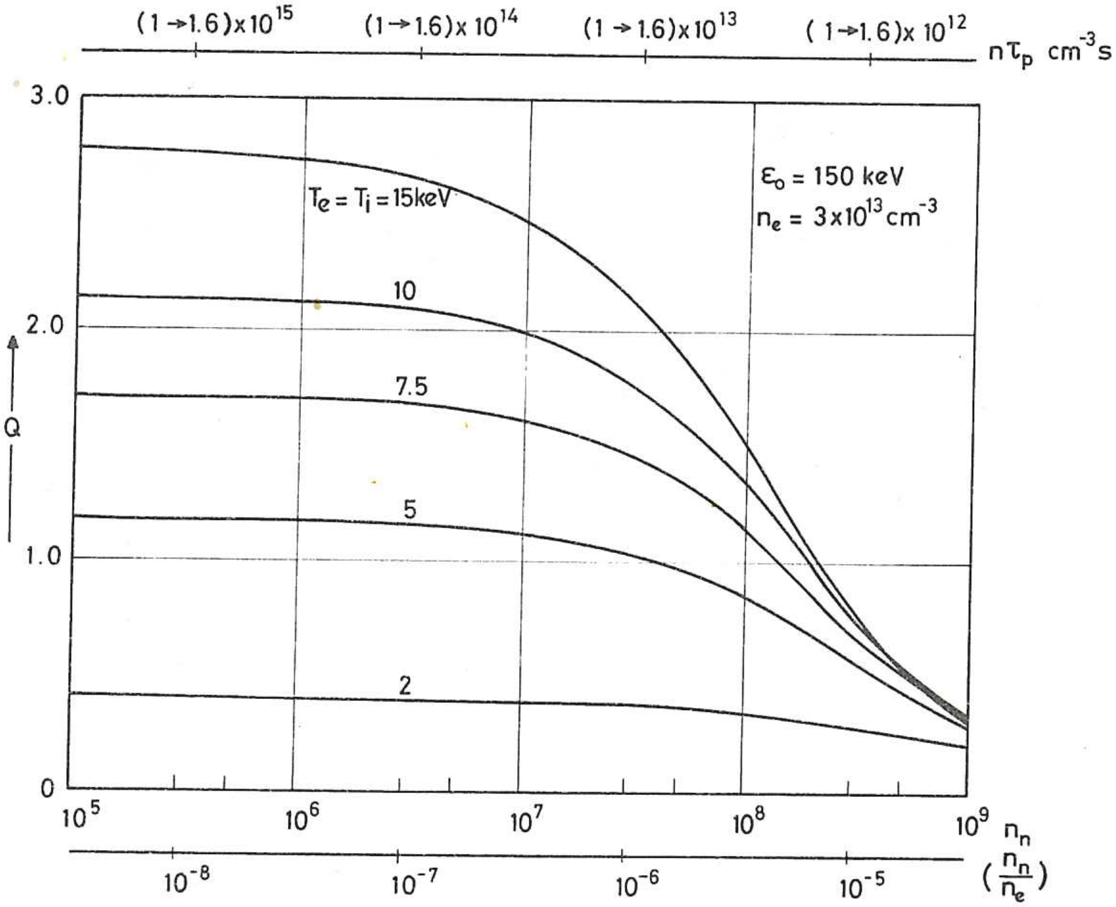

Fig.4 The effect of charge exchange losses on Q, D → T, ϵ_0 = 150 keV, as a function of $\overline{n}_n$ for $n_e = 3 \times 10^{13}$ cm^{-3}, and of $\frac{n_n}{n_e}$ and $n\,\tau_p$.

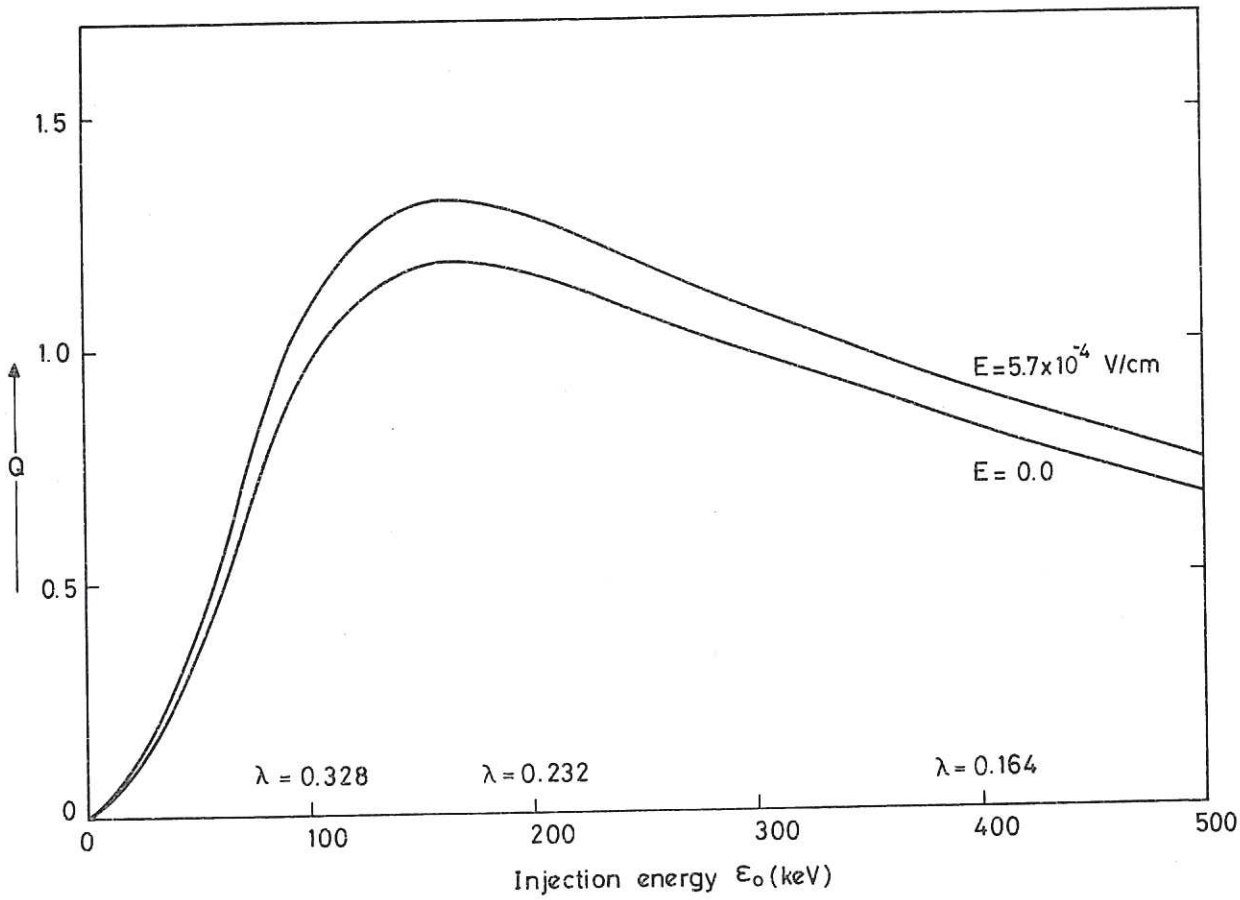

Fig.5 The effect of a weak electric field on Q.

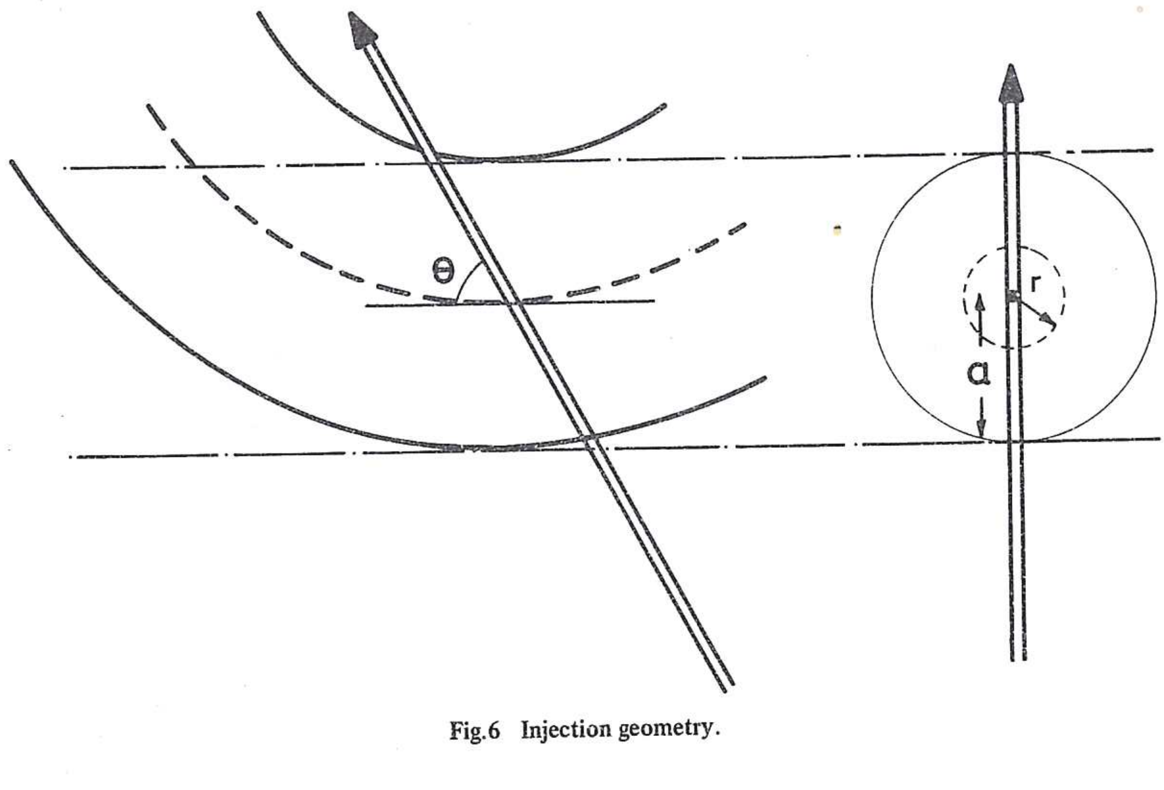

Fig.6 Injection geometry.

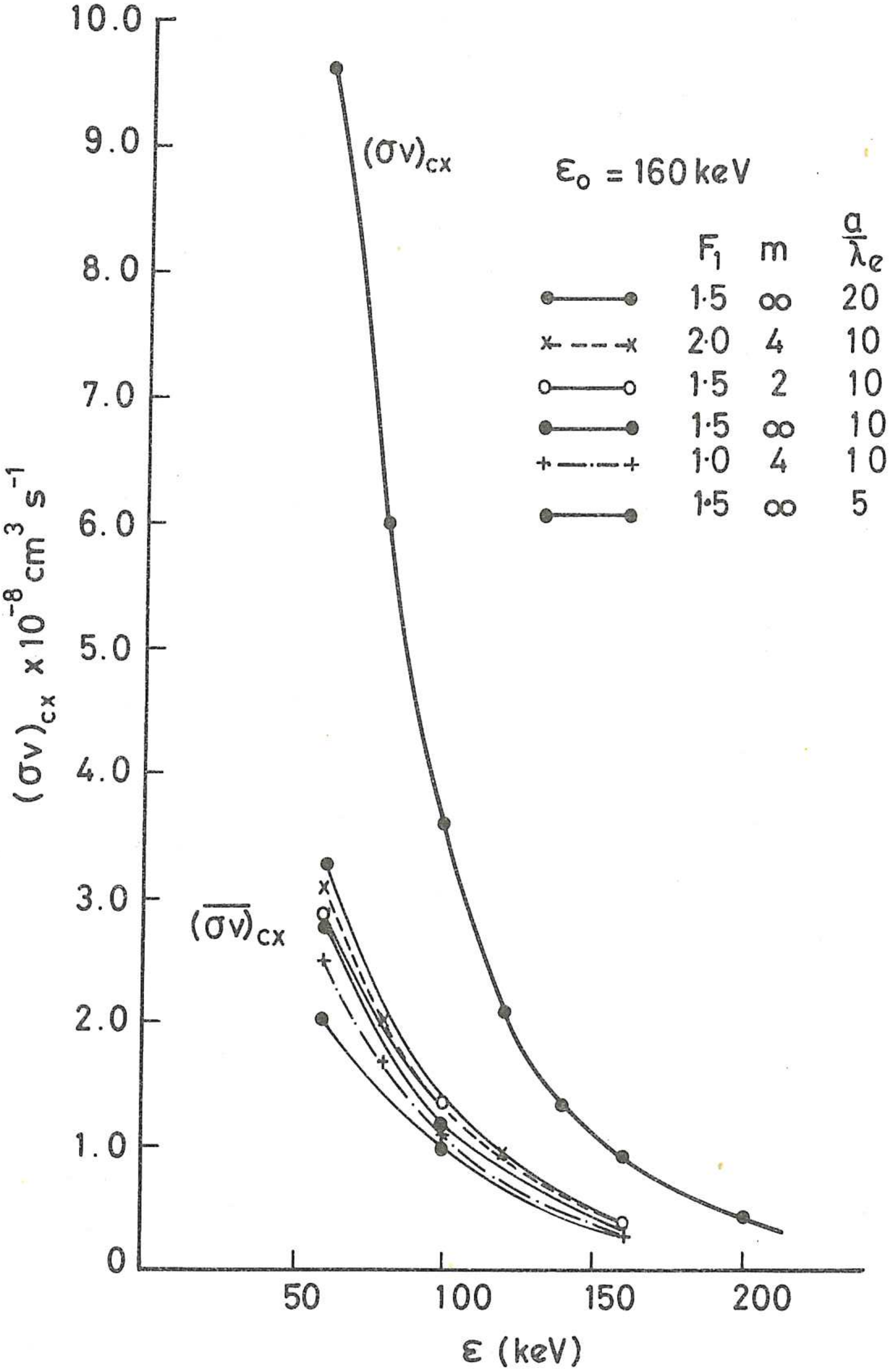

Fig.7 The local charge exchange rate and the effective charge rate for a Tokamak as a function of deuteron energy.