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MICROWAVE PROPAGATION IN A PLASMA WITH A SHEARED MAGNETIC FIELD

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MICROWAVE PROPAGATION IN A PLASMA WITH A SHEARED MAGNETIC FIELD

by

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A B S T R A C T

The propagation of a plane electromagnetic wave through a Lorentz plasma in a sheared magnetic field is considered. The characteristic waves in a uniformly sheared, but otherwise homogeneous, medium are found, and their properties presented graphically. Some effects introduced by the presence of shear are investigated by means of elementary applications of the uniform-shear theory. By suggested extensions of the analysis, refinements may be made in the interpretations of microwave diagnostic experiments with controlled-fusion containment devices having complicated sheared fields.

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C O N T E N T S

1. INTRODUCTION

A common refinement to the Appleton-Hartree theory of electromagnetic wave propagation in a magnetized (anisotropic) plasma is consideration of spatial inhomogeneity of the plasma electron density (BACHYNSKI, 1960). In this paper, we investigate another type of inhomogeneity, namely, spatial variation of the orientation of the magnetic field. The problem is suggested by the importance of sheared magnetic fields in the plasma containment devices of controlled fusion research, such as the diffuse pinch, the multipolar Stellarator, and the Ioffe stabilized mirror configuration.

Initially assuming a simple method for the sheared field, we derive a dielectric tensor and the resulting wave equations. Then with further simplifying assumptions, we find the characteristic waves in a "uniform" sheared medium. The general problem, for an arbitrarily complicated model of magnetic field configuration and electron density profile, may be set up by a straightforward extension of the analysis given.

2. DIELECTRIC CONSTANT TENSOR

Consider the special case of propagation in the z direction with the magnetic field $\underset{\sim}{B}$ lying in the x-y plane. The magnitude B_o and the orientation of $\underset{\sim}{B}$ within the x-y plane, and the electron density n, may be arbitrary functions of z. If φ is the angle between $\underset{\sim}{B}$ and the x axis, then

$$B_x = B_o \cos \varphi(z)$$
$$B_y = B_o \sin \varphi(z). \tag{1}$$

The force equation for plasma electrons, neglecting collisions,* is

* The effect of short-range collisions may readily be included by inserting the Langevin damping term $-\nu m \underset{\sim}{v}$ on the right-hand side of equation (2), where ν is the effective collision frequency for momentum transfer. The result is simply to modify the dielectric coefficient elements (10)-(12) in a well-known manner (HEALD and WHARTON, 1964).

$$m \frac{d\underset{\sim}{v}}{dt} = -e\,(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}). \tag{2}$$

Assuming harmonic time dependence $\exp(+j\omega t)$ and neglecting the a.c. component of $\underset{\sim}{B}$, eq.(2) represents a linear relation between $\underset{\sim}{v}$ and $\underset{\sim}{E}$ with a complex tensor mobility. Alternatively the electron motion may be regarded as a current density

$$\underset{\sim}{J} = -ne\underset{\sim}{v}, \tag{3}$$

in which case (2) becomes Ohm's law,

$$\underset{\approx}{\sigma}^{-1} \cdot \underset{\sim}{J} = \underset{\sim}{E}. \tag{4}$$

For our assumed geometry the complex reciprocal-conductivity (resistivity) tensor is explicitly

$$\underset{\approx}{\sigma}^{-1} = \frac{j}{\omega_p^2 \varepsilon_o} \begin{bmatrix} \omega & 0 & j\omega_b \sin\varphi \\ 0 & \omega & -j\omega_b \cos\varphi \\ -j\omega_b \sin\varphi & j\omega_b \cos\varphi & \omega \end{bmatrix}, \tag{5}$$

where we have made the usual identification of the plasma frequency

$$\omega_p = \left(\frac{n e^2}{\varepsilon_o m} \right)^{\frac{1}{2}} \tag{6}$$

and cyclotron frequency

$$\omega_b = \left| \frac{eB_o}{m} \right|. \tag{7}$$

The tensor dielectric constant is related to the conductivity by

$$\underset{\approx}{\kappa} = \underset{\approx}{1} - \frac{j}{\omega\varepsilon_o}\, \underset{\approx}{\sigma}\,. \tag{8}$$

Inverting (5), we then obtain explicity

$$\underset{\approx}{\kappa} = \begin{bmatrix} \kappa_{\parallel} \cos^2\varphi + \kappa_{\perp} \sin^2\varphi & (\kappa_{\parallel} - \kappa_{\perp}) \sin\varphi \cos\varphi & j\kappa_{\times} \sin\varphi \\ (\kappa_{\parallel} - \kappa_{\perp}) \sin\varphi \cos\varphi & \kappa_{\parallel} \sin^2\varphi + \kappa_{\perp} \cos^2\varphi & -j\kappa_{\times} \cos\varphi \\ -j\kappa_{\times} \sin\varphi & j\kappa_{\times} \cos\varphi & \kappa_{\perp} \end{bmatrix}, \tag{9}$$

where

$$\kappa_{\parallel} = 1 - \omega_p^2/\omega^2 \tag{10}$$

$$\kappa_{\perp} = 1 - \omega_p^2/(\omega^2 - \omega_b^2) \tag{11}$$

$$\kappa_{\times} = \omega_p^2 \omega_b / \omega(\omega^2 - \omega_b^2). \tag{12}$$

The subscripts used in (10)-(12) refer to the orientation of the wave electric field with respect to the static magnetic field, in the well-known dielectric tensor for a homogeneous, cold magnetoplasma (HEALD and WHARTON, 1964).

3. WAVE EQUATION IN SHEARED MEDIUM

The Maxwell curl equations for exp(+jωt) time dependence and a medium described by a complex tensor dielectric constant $\underset{\approx}{\kappa}$ are, in MKS units,

$$\nabla \times \underset{\sim}{E} = -j\omega\mu_o \underset{\sim}{H} \tag{13}$$

$$\nabla \times \underset{\sim}{H} = j\omega\varepsilon_o \underset{\approx}{\kappa} \cdot \underset{\sim}{E}. \tag{14}$$

Further assuming that the only spatial variation is in the $\underline{z}$ direction and denoting the operation d/dz by a prime, we obtain from (13)-(14) with the aid of (9) the four coupled equations

$$E_x' = -j\omega\mu_o H_y \tag{15}$$

$$E_y' = j\omega\mu_o H_x \tag{16}$$

$$H_y' = -j\omega\varepsilon_o(p\, E_x + r\, E_y) \tag{17}$$

$$H_x' = j\omega\varepsilon_o(r\, E_x + q\, E_y), \tag{18}$$

where

$$p = \kappa_{ord} \cos^2 \varphi + \kappa_{ex} \sin^2 \varphi \tag{19}$$

$$q = \kappa_{ord} \sin^2 \varphi + \kappa_{ex} \cos^2 \varphi \tag{20}$$

$$r = (\kappa_{ord} - \kappa_{ex}) \sin\varphi \cos \varphi, \tag{21}$$

and where

$$\kappa_{ord} = \kappa_{\parallel} = \frac{\omega^2 - \omega_p^2}{\omega^2} \tag{22}$$

and

$$\kappa_{ex} = \frac{\kappa_\perp^2 - \kappa_x^2}{\kappa_\perp} = \frac{(\omega^2 - \omega_p^2)^2 - \omega^2\omega_b^2}{\omega^2(\omega^2 - \omega_p^2 - \omega_b^2)} \tag{23}$$

are the squares of the refractive indices for, respectively, the <u>ordinary</u> ($\underset{\sim}{E}_{rf} \parallel \underset{\sim}{B}$) and <u>extraordinary</u> ($\underset{\sim}{E}_{rf} \perp \underset{\sim}{B}$) waves propagating across a shear-free magnetic field (HEALD and WHARTON, 1964). Equation (14) also yields the condition for the longitudinal $\underset{\sim}{E}$ component

$$E_z = j \frac{\kappa_x}{\kappa_\perp} (E_x \sin\varphi - E_y \cos\varphi), \tag{24}$$

which incidentally ensures the vanishing of the divergence $\nabla \cdot \underset{\sim}{D} = \nabla \cdot (\underset{\approx}{\kappa} \cdot \underset{\sim}{E})$.

Differentiating (15)-(16) and using (17)-(18), we may obtain two coupled second-order equations in $\underset{\sim}{E}$ components alone,

$$E_x'' = -\frac{\omega^2}{c^2}(pE_x + rE_y) \tag{25}$$

$$E_y'' = -\frac{\omega^2}{c^2}(rE_x + qE_y). \tag{26}$$

By two further differentiations, one can obtain a single fourth-order differential equation in E_x or E_y alone. Similar manipulations provide the corresponding, but more cumbersome, equations for $\underset{\sim}{H}$ components. This familiar line of attack is not very useful here because the field shear couples the cartesian components awkwardly. For example, the coefficients of the fourth-order equations in a single field component contain terms of the form tan φ and cot φ, which blow up when $\varphi = 0, \pi/2$, etc.

Our basic assumptions, of one-dimensional spatial variations and a purely transverse magnetic field, nevertheless permit realistic treatment of a large class of possible microwave-diagnostic ray paths in complicated experimental devices for plasma containment. By postulating the $\underline{z}$ dependence of the field orientation φ and of the dielectric coefficients $\kappa(n, B_0)$, one may obtain solutions to (15)-(18) by initial-value numerical computation. In practice, however, it may be difficult to guess a priori the proper initial conditions corresponding to a desired situation (e.g., a pure forward-travelling wave). As discussed further below, the wave impedance (ratio of E to H) is altered in the sheared medium, and reflections from the plasma boundary include a wave component with polarization orthogonal to that of the incident wave.

If one obtains solutions to the first-order simultaneous differential equations (15)-(18) by numerical integration, assuming certain initial values for the four field components, it is most convenient to present the results in terms of the parameters of elliptical polarization, rather than the cartesian field components directly. This conversion from cartesian components to the alternative variables (inclination, magnitude, and phaseshift of major axis, and ellipticity) is given in the Appendix. An example of this description is given in Section 6.

4. CONVERSION TO SHEARED COORDINATES

Because of the assumed structure of the magnetic field, it is useful to take a new sheared coordinate system in which the transverse axes, replacing x and y, are respectively parallel ($\parallel$) and orthogonal ($\perp$) to the local magnetic field direction. Thus for instance the $\underset{\sim}{E}$-field components in the two systems are related by

$$E_x = E_{\parallel} \cos \varphi - E_{\perp} \sin \varphi \tag{27}$$

$$E_y = E_{\parallel} \sin \varphi + E_{\perp} \cos \varphi \tag{28}$$

The coupled first-order Maxwell equations (15)-(18) then become:

$$E_{\parallel}' = \varphi E_{\perp} - j\omega\mu_o H_{\perp} \tag{29}$$

$$E_{\perp}' = -\varphi E_{\parallel} + j\omega\mu_o H_{\parallel} \tag{30}$$

$$H_{\perp}' = -\varphi H_{\parallel} - j\omega\varepsilon_o \kappa_{ord} E_{\parallel} \tag{31}$$

$$H_{\parallel}' = \varphi H_{\perp} + j\omega\varepsilon_o \kappa_{ex} E_{\perp} \tag{32}$$

5. CHARACTERISTIC WAVES IN A UNIFORM SHEARED MEDIUM

We now consider the special case of a uniformly sheared but otherwise homogeneous medium; that is, we take the shear $d\varphi/dz$ and the dielectric coefficients κ_{ord} and κ_{ex} to be constants, independent of position. Incidentally, a uniform shear of this sort is generated by a volume current density

$$\nabla \times \underset{\sim}{B} = - \underset{\sim}{B}\ d\varphi/dz, \tag{33}$$

which is everywhere aligned with the magnetic field and hence force-free. It is useful to normalize all variables to those for an isotropic medium of dielectric constant κ_{ord}; we define $\not{}$

relative anisotropy
$$a = \frac{\kappa_{ord} - \kappa_{ex}}{\kappa_{ord}} = \frac{\omega_p^2 \omega_b^2}{(\omega^2-\omega_p^2)(\omega^2-\omega_p^2-\omega_b^2)} \tag{34}$$

shear rate
$$s = \left(\frac{c}{\kappa_{ord}^{1/2} \omega}\right) \frac{d\varphi}{dz} \tag{35}$$

$$\left.\begin{aligned} {}^*H_{\parallel} &= \left(\frac{\mu_o}{\kappa_{ord}\,\varepsilon_o}\right)^{1/2} H_{\parallel} \\ {}^*H_{\perp} &= \left(\frac{\mu_o}{\kappa_{ord}\,\varepsilon_o}\right)^{1/2} H_{\perp} \end{aligned}\right\} \tag{36}$$

The relative anisotropy $\underline{a}$ is shown as a function of electron density and magnetic field magnitude in Fig. 1. The shear rate $\underline{s}$ is the fraction of a full 2π rotation of the magnetic field in the distance of a wavelength in an isotropic medium of dielectric constant κ_{ord}. The normalized $^*\underset{\sim}{H}$ components would be equal in magnitude to the corresponding $\underset{\sim}{E}$ components in a medium of dielectric constant κ_{ord}. Finally we assume that all four field components propagate with the spatial phase factor

$$E, H \propto \exp\left(-j\ {}^*\mu\ \frac{\kappa_{ord}^{1/2}\,\omega}{c}\ z\right) \tag{37}$$

where $^*\mu$ is the refractive index relative to a medium of refractive index $\kappa_{ord}^{1/2}$. With these notational assumptions and definitions, the differential equations (29)-(32) simplify to:

$$j\ {}^*\mu\ E_{\parallel} + s\ E_{\perp} - j\ {}^*H_{\perp} = 0 \tag{38}$$

$$-s\ E_{\parallel} + j{}^*\mu\ E_{\perp} + j\ {}^*H_{\parallel} = 0 \tag{39}$$

$$-j\ E_{\parallel} + j{}^*\mu\ {}^*H_{\perp} - s\ {}^*H_{\parallel} = 0 \tag{40}$$

$$j(1-a)\ E_{\perp} + s\ {}^*H_{\perp} + j{}^*\mu\ {}^*H_{\parallel} = 0 \tag{41}$$

$\not{}$ In the range $1<\omega_p^2/\omega^2<1+\omega_b/\omega$ the shear-free ordinary wave is cutoff ($\kappa_{ord}<0$) while the extraordinary wave is not. This situation is easily treated by interchanging the meanings of the subscripts $\parallel$ and $\perp$, ord and ex.

The set of homogeneous equations (38)-(41) is self-consistent only if the determinant of the coefficients vanishes. This constraint gives the refractive indices of the two characteristics waves in the uniformly sheared medium,

$$*\mu^2 = 1 + s^2 - \tfrac{1}{2}a \pm \left[4s^2(1-\tfrac{1}{2}a) + \tfrac{1}{4}a^2\right]^{\frac{1}{2}}. \qquad (42)$$

This relation is shown numerically in Fig. 2. Wave cutoff is displaced from $a = 1$ (i.e., $\kappa_{ex} = 0$) to $a = 1 - s^2$ and $s = 1$.

The relative magnitudes of the field components for the characteristic waves may be found from the co-factors of the determinant of coefficients of (38)-(41). The components are proportional to

$$*E_{\parallel} \propto 2s\,{*\mu} \qquad (43)$$

$$*E_{\perp} \propto j(1-s^2-{*\mu}^2) \qquad (44)$$

$$*H_{\perp} \propto s(1-s^2+{*\mu}^2) \qquad (45)$$

$$*H_{\parallel} \propto -j{*\mu}(1+s^2-{*\mu}^2), \qquad (46)$$

where the refractive index $*\mu$ is one or the other solution of (42). Thus the two characteristic waves are elliptically polarized, with electric-field ellipticities, $E_{\perp}/E_{\parallel}$ or $-E_{\parallel}/E_{\perp}$ as appropriate, shown in Figs. 3 and 4. The corresponding magnetic-field ellipticities are somewhat different, although of course they also go to zero and unity in the respective limits of no shear and no anisotropy.

The natural identification of the two characteristic waves is in terms of the sense (handedness) of elliptical polarization, which must be specified with care since a wave that is right-handed in <u>time</u> is left-handed in <u>space</u>, the handedness in both cases being defined with respect to the direction of wave travel. For $s < 1$, the space dependence of the wave with $*\mu < 1$ (Fig. 2) has the same sense as the field shear, and the wave with $*\mu > 1$ has the opposite sense. For very strong shear rates such that $s > 1$, the "$*\mu < 1$" wave becomes a backward wave, with Poynting vector in the opposite sense to the phase velocity, and the medium exhibits features normally associated with periodic structures. In the microwave plasma-diagnostic context the shear rates of interest are usually small, and in the remainder of our discussion we shall generally assume that s^2 is small compared with unity.

In practice, it is often preferable to label the characteristic waves as "ordinary" or "extraordinary" according to their limiting form as the shear rate $\underline{s}$ goes to zero, with resulting discontinuities at a = 0. In this latter special case of zero anisotropy, the characteristic waves are circularly polarized with refractive indices $*\mu = 1 \pm s$; that is, the mathematical formalism here implies that a linearly polarized wave experiences a "Faraday rotation" in the sheared coordinate system just so as to "unwind" the shear, and the wave remains properly unrotated in fixed coordinates.

It is also of interest to record the wave admittances (reciprocal impedances), i.e., the ratios of the $\tilde{H}$ components to the orthogonal $\tilde{E}$ components. For each of the two waves, we have both "parallel" and "perpendicular" wave admittances, defined by

$$*Y_{\parallel} = *H_{\perp}/E_{\parallel} \tag{47}$$

$$*Y_{\perp} = *H_{\parallel}/E_{\perp} , \tag{48}$$

where the asterisks recall the normalization (36) to the wave admittance $(\kappa_{ord}\varepsilon_o/\mu_o)^{\frac{1}{2}}$ of an isotropic medium of dielectric constant κ_{ord}. From a naive point of view one would expect $*Y_{\parallel} = 1$ and $*Y_{\perp} = (\kappa_{ex}/\kappa_{ord})^{\frac{1}{2}} = (1 - a)^{\frac{1}{2}}$. However, (47) and (48) differ from these expectations and, furthermore, differ between the ordinary and extraordinary waves, as shown in Figs. 5 and 6.

6. EXCITATION AND QUALITATIVE BEHAVIOUR OF WAVES

Two elementary applications of the results of the previous section provide insight into qualitative effects arising from shear in an anisotropic medium. First, consider two forward-travelling characteristic waves in the interior of a uniform sheared medium of infinite extent. Let these waves have amplitudes and phases such that at a particular reference plane their superposition yields an

$\underset{\sim}{E}$-field linearly polarized in the local direction of the sheared magnetostatic field. Because the ellipticities and wave impedances differ somewhat for the two characteristic waves (Figs. 3-6), it is impossible generally to have a resultant wave that is simultaneously linearly polarized in both $\underset{\sim}{E}$ and $\underset{\sim}{H}$. By constrast, in an anisotropic but shear-free medium, the wave $\underset{\sim}{E}$ and $\underset{\sim}{H}$ are simultaneously linearly polarized (although not in general in the same direction). Meanwhile the $\underset{\sim}{E}$-polarization of the resultant wave in the sheared medium, observed at subsequent planes, passes periodically from linear to elliptical and back to linear, etc. The maximum value of ellipticity (ratio of minor to major axis) may be computed from (42)-(46) and is given approximately by

$$(e_E)_{max} \approx \frac{4s}{|a|} \quad \text{or} \quad \frac{|a|}{4s} , \tag{49}$$

whichever is less than unity. The periodicity is the distance Δz required for the two characteristic waves to get 2π out of phase; that is $\Delta z = \lambda/\Delta\mu$, where $\Delta\mu$ is the magnitude of the difference between the two solutions of (42). This same periodic variation of ellipticity, sometimes called the Cotton-Mouton effect, occurs in a shear-free anisotropic medium only when the polarization is oblique to the anisotropy direction. With shear, if one follows the inclination ψ of the major axis of the elliptical polarization, with respect to the initial direction of linear polarization, then for $4s \lesssim |a|$ the major axis follows the shear on average, although "wobbling" somewhat within the periodicity. On the other hand, for $4s \gtrsim |a|$ the major axis "precesses" in one period by the angle

$$\Delta\psi = 2\pi \left(\frac{\lambda s}{\Delta\mu} - \tfrac{1}{2} \right) . \tag{50}$$

This precession (in fixed coordinates) is in the <u>opposite</u> sense to that of the magnetic-field shear, which meanwhile has turned through the angle $2\pi\lambda s/\Delta\mu$. That is, with respect to the magnetic field, the major axis has precessed backwards by π . The discontinuity between these two cases occurs where the maximum ellipticity is unity and the inclination ψ is undefined. In both cases, within one period, the inclination of the major axis wobbles in a more or less sinusoidal fashion about the average trend.

Numerical examples of the propagation of an initially linearly polarized wave are shown in Fig. 7. These curves were generated by numerical integration of (15)-(18) with the results expressed in the alternative coordinates defined in the Appendix. The examples illustrate the inclination "wobble" about an average trend, and the ambiguity arising in the special case where the ellipticity reaches unity.

As a second example, we consider the case of a uniform plasma slab of thickness L, within which the magnetic field shears uniformly from $\varphi = -\varphi_o$ through zero to $+\varphi_o$ ($\varphi_o < \pi/2$), a configuration crudely approximating a diffuse pinch. We also assume that the shear is small in the sense that $s = 2\varphi_o c/\kappa_{ord}^{1/2}\, \omega L$ is small compared to unity and to $\frac{1}{4}|a|$. In this limit the refractive indices and ellipticities of the two characteristic waves are approximately:

$$\mu_{ord} \approx \kappa_{ord}^{1/2} \left(1 + \frac{4-a}{2a} s^2\right) \tag{51}$$

$$\mu_{ex} \approx \kappa_{ord}^{1/2} \left[(1-a)^{1/2} - \frac{4-3a}{2a(1-a)^{1/2}} s^2\right] = \kappa_{ex}^{1/2}\left[1 - \frac{4-3a}{2a(1-a)} s^2\right] \tag{52}$$

$$e_{ord} \approx -\frac{2s}{a} \tag{53}$$

$$e_{ex} \approx \frac{2s}{a}(1-a)^{1/2} \tag{54}$$

That is, the waves are essentially linearly polarized, with the refractive indices of the corresponding unsheared medium; however, the polarization direction of the characteristic waves follows the shear of the magnetic field. An incident wave, linearly polarized with $\underset{\sim}{E}$ in the direction of $\varphi = 0$, will in general excite both characteristic waves inside the plasma slab and also return a reflected wave having linearly polarized components both parallel and orthogonal to the incident polarization. The respective amplitudes and phases of the four transmitted and reflected wave components could be found by application of the usual boundary conditions at the interface. However, for our present purposes, we shall take the crude approximation that the amplitude of the ordinary wave (major axis of $\underset{\sim}{E}_{rf} \parallel \underset{\sim}{B}$) in the sheared slab is proportional to $\cos\varphi_o$, the amplitude of the extraordinary wave (major axis of $\underset{\sim}{E}_{rf} \perp \underset{\sim}{B}$) is proportional to $\sin\varphi_o$, and the amplitudes of the reflected components are negligible. Then allowing the two characteristic waves to propagate through the slab, and making analogous simplifying assumptions for the recombination of the transmitted waves at the far surface, we obtain:

<u>transmitted component parallel to incident wave</u>

$$\text{amplitude} \quad [1 - \sin^2\varphi_o \cos^2 (\pi \, \Delta\mu \, L)]^{1/2} \tag{55}$$

$$\text{phaseshift} \quad \tan^{-1} \left[\frac{-\sin^2\varphi_o \sin (2\pi\Delta\mu L)}{1 - 2 \sin^2\varphi_o \cos^2(\pi\Delta\mu L)} \right] \tag{56}$$

<u>transmitted component orthogonal to incident wave</u>

$$\text{amplitude} \quad [\sin^2 2\varphi_o \cos^2(\pi\Delta\mu L)]^{1/2} \tag{57}$$

$$\text{phaseshift} \quad \tan^{-1} [\tan (\pi\Delta\mu L)] \tag{58}$$

In these formulas, $\Delta\mu = \mu_{ord} - \mu_{ex}$, and the phaseshift is reckoned with respect to an isotropic slab of refractive index μ_{ord}.

It should be noted that the level of treatment used in this second example is essentially that of geometrical optics; that is, the effect of shear, as distinct from anisotropy, enters only through the geometrical fact that the polarization of the characteristic waves follows the shear. The results are very similar to the case of an obliquely polarized wave incident on a shear-free anisotropic slab (Cotton-Mouton effect). Thus although the quantities (55)-(58) can be readily measured experimentally, it is difficult to extract the diagnostic information that they contain. In the shear-free case, the Cotton-Mouton complications may be avoided by alignment of the incident polarization parallel or orthogonal to the anisotropy direction. The analogous procedure in our sheared-slab example would be to launch the wave with the polarization aligned at $-\varphi_o$, and receive it with polarization at $+\varphi_o$. In the small-shear, reflectionless limit considered, the amplitude would then be essentially independent of plasma properties, while the phase would provide a direct measure of μ_{ord}, (51).

7. DISCUSSION

The preceding highly idealized examples suggest the sort of phenomena to be expected in practical cases where shear exists. In experimental situations where the gradients (in φ and κ) are weak relative to a wavelength, and where the geometry conforms to the rather special case considered here, considerable improvement over elementary shear-free theory can be expected from assuming that a linearly polarized wave outside the plasma gradually deforms through a succession of configurations, given by the locally evaluated characteristic wave for the uniform sheared medium of Section 5, in analogy with theorems on the adiabatic perturbation of quantum systems.

As mentioned in Section 3, one can handle quite complicated experimental situations by postulating the dependence of field orientation φ and the dielectric coefficients $\kappa(n,B_0)$ upon distance along the assumed ray path, and then solving (15)-(18) by numerical computation. Indeed, in most plasma devices the shear scale-length is necessarily comparable to the scale-lengths of electron density and field magnitude gradients, so that the electron-density inhomogeneity of the medium is as important as the shear, and numerical computation is the only effective way of treating a specific experimental situation. In this case, however, the four initial values of the respective field components take the place, mathematically, of the amplitudes of the two forward-travelling and two backward-travelling characteristic waves, which are available only when (15)-(18) can be solved explicitly. The initial-value point of view is usually more awkward in the sense that, for instance, a given set of initial conditions may implicitly presume a "reflected" wave, of prescribed amplitude and phase, in the space beyond the far side of the plasma sample. Thus trial-and-error methods may be required to find initial conditions that yield a physically acceptable solution.

The basic assumption of this paper, that the sheared magnetic field is always transverse to the direction of propagation, is of course not realistic for many experimental problems. This limitation can be removed by suitable generalization of the methods of Section 2 at the cost of considerable algebraic complexity.

I am very grateful to D.J.H. Wort for suggesting this problem and collaborating on several details of the work.

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APPENDIX

With time dependence $\exp(+j\omega t)$ understood, the complex amplitudes of the cartesian components,

$$E_x \equiv a + jb \equiv r \exp j\alpha \tag{A1}$$

$$E_y \equiv c + jd \equiv s \exp j\beta, \tag{A2}$$

represent, in general, an elliptically polarized field. We seek a set of parameters, equivalent to (a, b, c, d) or (r, α, s, β), that describes the field configuration directly in terms of the elliptical polarization. The <u>inclination</u> ψ of the major axis with respect to the <u>x</u> axis is given by

$$\psi = \tfrac{1}{2} \tan^{-1} \frac{[2(ac + bd)]}{[a^2 + b^2 - c^2 - d^2]}, \tag{A3}$$

where the brackets signify that the four-quadrant inverse tangent is to be used. The <u>magnitude</u> t and <u>phase</u> Υ of the semi-major axis are then

$$t = [(a \cos\psi + c \sin\psi)^2 + (b \cos\psi + d \sin\psi)^2]^{\frac{1}{2}} \tag{A4}$$

$$\Upsilon = \tan^{-1} \frac{[b \cos\psi + d \sin\psi]}{[a \cos\psi + c \sin\psi]}. \tag{A5}$$

Similarly for the semi-minor axis,

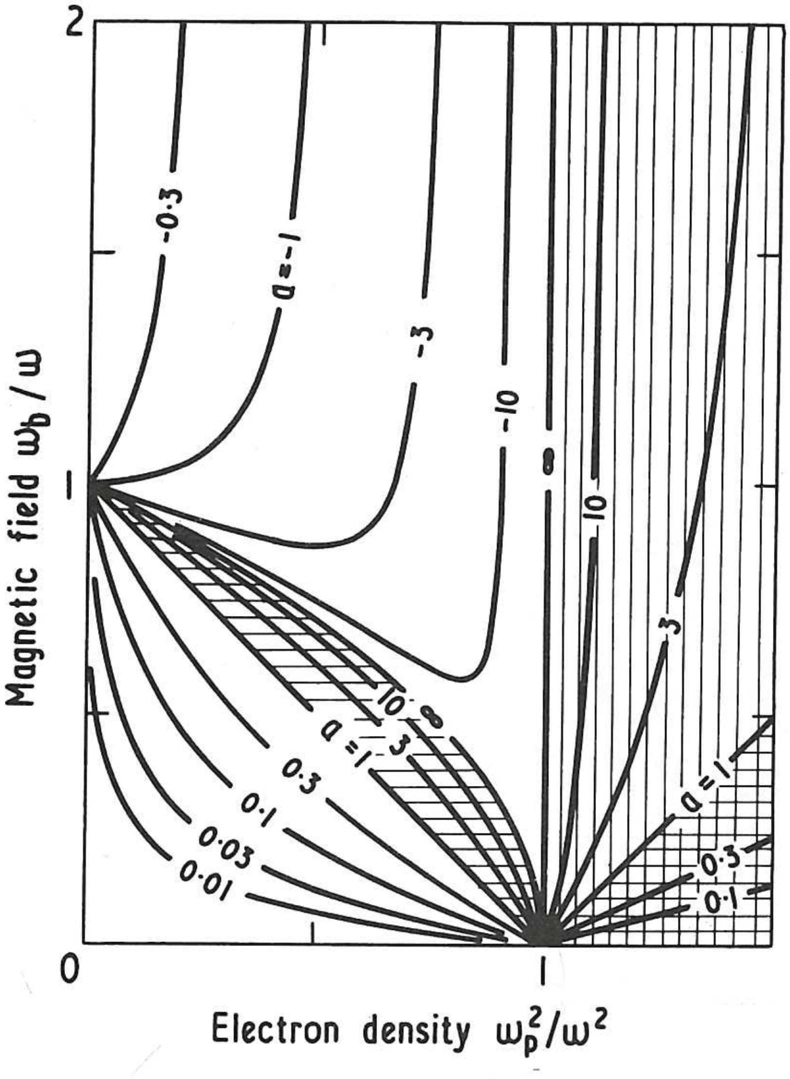
$$u = [(-a \sin\psi + c \cos\psi)^2 + (-b \sin\psi + d \cos\psi)^2]^{\frac{1}{2}} \tag{A6}$$

$$\delta = \tan^{-1} \frac{[-b \sin\psi + d \cos\psi]}{[-a \sin\psi + c \cos\psi]}. \tag{A7}$$

The <u>ellipticity</u> is given by

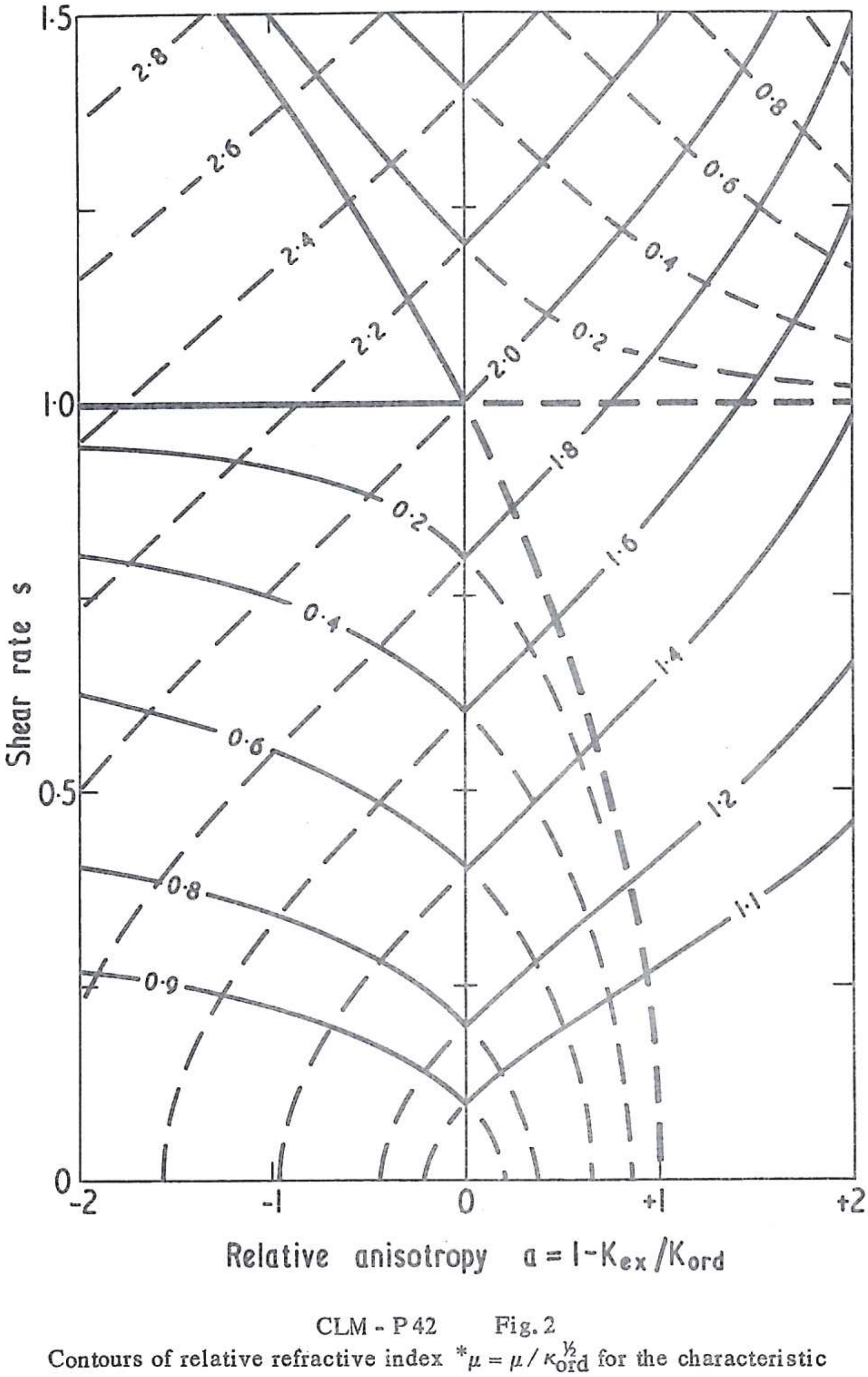
$$e = \frac{u}{t} \sin(\Upsilon - \delta), \tag{A8}$$

where the sine term has the value ±1 and denotes the right or left handedness, respectively, of the time dependence of the elliptical polarization with respect to the direction of propagation. Thus the alternative set of parameters (ψ, t, Υ, e) uniquely specifies the field and offers a somewhat more physical description than the original set (a, b, c, d).



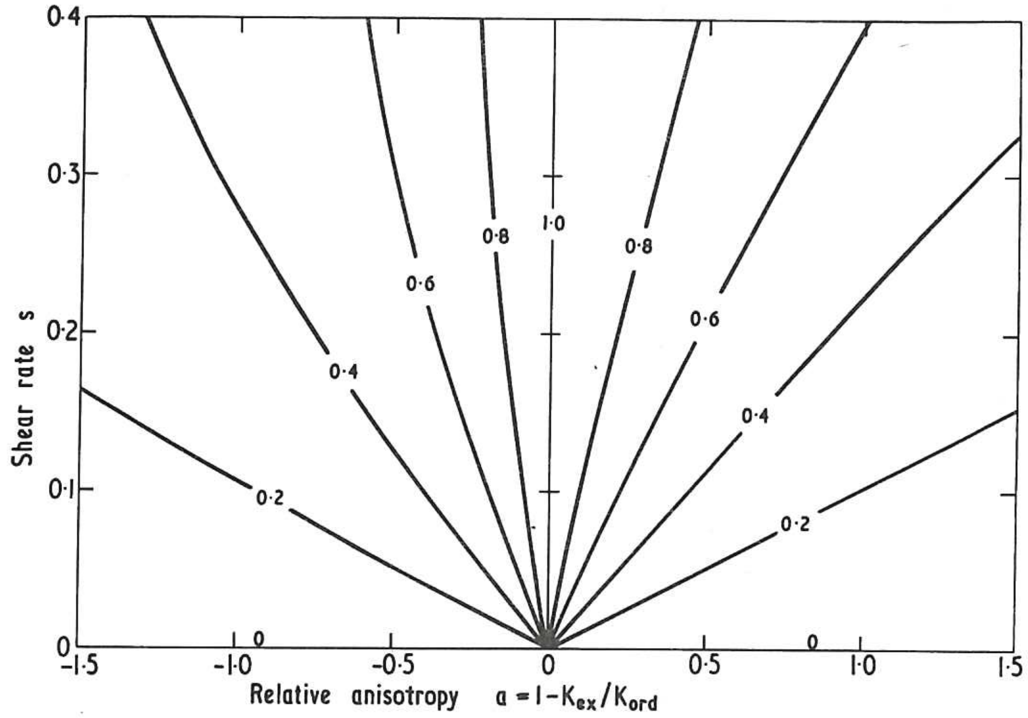
CLM - P 42 Fig. 1

Contours of relative anisotropy $a = 1 - \kappa_{ex}\kappa_{ord}$ as a function of normalised electron density and magnetic field magnitude for a collision-free Lorentz plasma in a shear-free magnetic field. The ordinary wave (κ_{ord}) is cut off in the region with vertical shading; the extraordinary wave (κ_{ex}) is cut off in the regions with horizontal shading

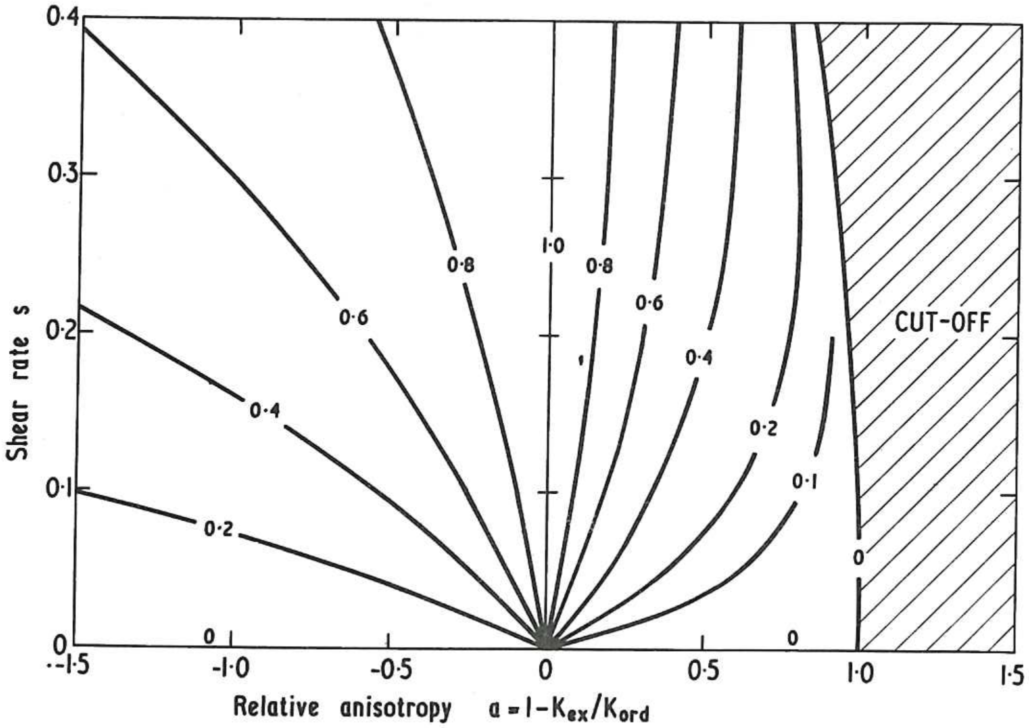


CLM - P 42 Fig. 2

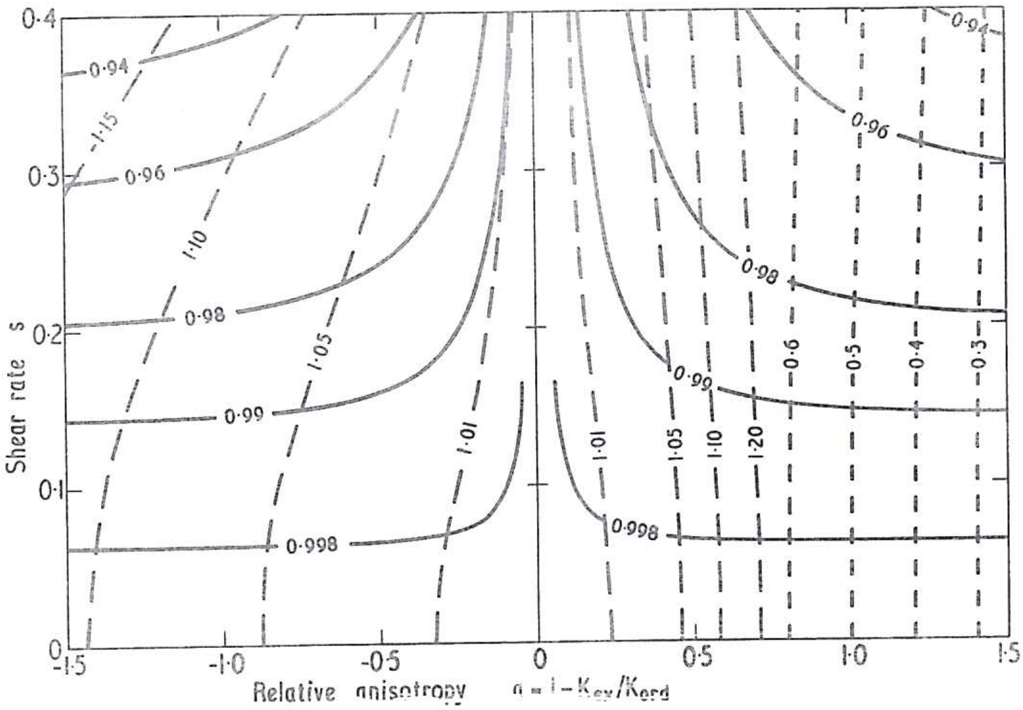
Contours of relative refractive index $^*\mu = \mu / \kappa_{ord}^{1/2}$ for the characteristic waves in a uniform sheared medium. Solid curves represent the 'ordinary' wave, which reduces to the usual ordinary wave in the absence of shear; dashed curves represent the 'extraordinary' wave. One wave is cut off in the regions at lower right and upper left.



CLM - P 42 Fig. 3
Ellipticity $E_{\perp}/E_{\parallel}$ for the ordinary wave

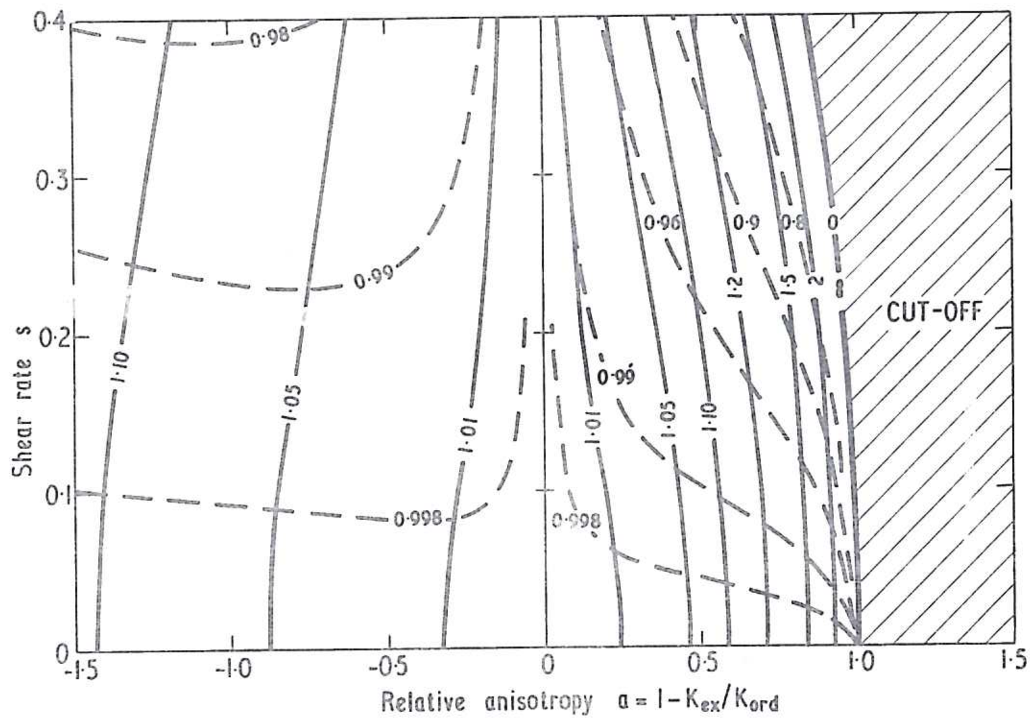


CLM - P 42 Fig. 4
Ellipticity $E_{\parallel}/E_{\perp}$ for the extraordinary wave



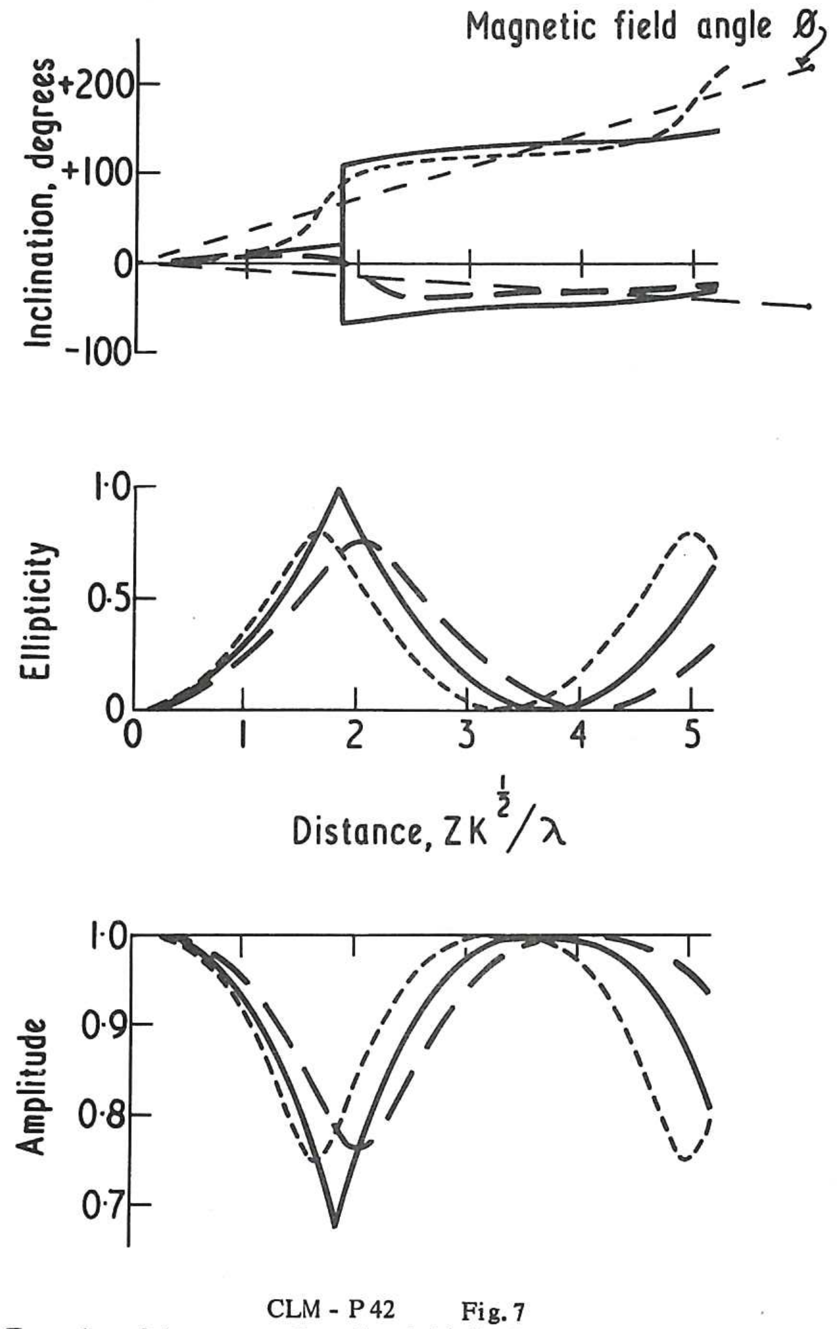
CLM - P 42 Fig. 5

Normalised wave admittances for the ordinary wave. 'Parallel' admittance $^*Y_{\parallel} = (\mu_o / \kappa_{ord}\,\epsilon_o)^{1/2}\, H_{\perp} / E_{\parallel}$, solid curves. 'Perpendicular' admittance $^*Y_{\perp} /(1 - a)^{1/2} = (\mu_o / \kappa_{ex}\epsilon_o)^{1/2}\, H_{\parallel} / E_{\perp}$, long-dashed curves. In the region where the extraordinary wave approaches cutoff ($a \geqslant 0.8$), the latter is given as $^*Y_{\perp} = (\mu_o / \kappa_{ord}\,\epsilon_o)^{1/2}\, H_{\parallel} / E_{\perp}$, short-dashed curves



CLM - P 42 Fig. 6

Normalised wave admittances for the extraordinary wave; $^*Y_{\parallel}$, solid curves; $^*Y_{\perp} /(1 - a)^{1/2}$, dashed curves



CLM - P 42 Fig. 7

Examples of the propagation of an initially linearly polarized wave. Shown are the inclination of the major axis of the electric field with respect to the initial polarization (fixed co-ordinates), the ellipticity (ratio of minor to major axis), and the amplitude of the semi-major axis. Shear rate s = 0.1; relative anisotropy a = -0.5 (short-dashed curves), -0.396 (solid curves), -0.3 (long-dashed). For the transition case (solid curves), the major and minor axes interchange as the wave passes through circular polarization