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A GEOMETRICAL DESCRIPTION OF FILAMENTATION AND SELF MODULATION OF ELECTRO-MAGNETIC WAVES IN A PLASMA

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A GEOMETRICAL DESCRIPTION OF FILAMENTATION AND SELF MODULATION OF ELECTRO-MAGNETIC WAVES IN A PLASMA

by

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Abstract

Self modulation and filamentation of an electro-magnetic wave is considered as a problem of the non-linear interaction between electro-magnetic and ion waves. A simple geometrical model is given of filamentation when the non-linearity is due to the ponderomotive force. A new electro-magnetic modulational instability is obtained whose threshold is the same as the oscillating two stream instability. The relationship between the filamentation and electro-magnetic modulational instabilities and other parametric instabilities is considered. In particular, it is shown that both electro-magnetic modulational and filamentation instabilities can occur at the critical density where they have the same threshold as the modulational instability of a Langmuir wave. Finally, a conservation relation (a generalization of the Manley-Rowe relation) for the wave action density is obtained for the filamentation instability. This shows clearly that this instability results from a four wave interaction.

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1. INTRODUCTION

The phenomena of self-focusing, filamentation and self-modulation of electromagnetic waves in plasma are all closely related to one another. Self-focusing is the tendency of an electromagnetic beam (of definite width) to reduce its own transverse spatial dimension through interaction with the plasma medium. Filamentation, on the other hand, is the plane wave analogue of this phenomenon and results in a spatially periodic distribution of the energy of the incident (uniform) plane wave. Self-modulation is again due to the interaction of the electromagnetic wave with the plasma medium resulting in a modulation of the amplitude of the incident wave in the direction of propagation. The first description of this type of effect in a plasma was by Volkov in 1958 [1]. More recently, effects due to the self-action of electromagnetic waves in plasma have been described by a number of authors [2 - 7]. In references [2 - 5], the non-linearity producing the self-action was the ponderomotive force of the electromagnetic wave, in reference [6] it was the wave induced heating and in reference [7] the self-action was produced by relativistic effects.

In this paper we shall consider filamentation and self modulation due to the ponderomotive force although we shall not use this concept explicitly. The aims of the paper are the following. First, we will give an alternative geometrical description of filamentation. Second, we shall derive a new modulational instability of an electromagnetic wave in the vicinity of the critical density and indicate the relationship of this instability to the modulational instability of a Langmuir wave. Finally, we shall relate the phenomena of filamentation and self modulation to the parametric effects which can occur in a plasma.

2. THE MODEL

A finite amplitude electromagnetic wave may interact with other waves in the plasma. We shall describe filamentation and self modulation as non-linear wave interaction phenomena. These effects are, of course, three dimensional. However, the basic process can be described in two dimensions.

We shall consider an infinite, uniform, unmagnetized plasma in which a plane polarized, finite amplitude, electromagnetic wave propagates in the x-direction. The non-linear wave interaction we shall analyze is shown in figures 1(a,b). We are led to consider this particular configuration by analogy with the oscillating two stream instability. ω_o, k_o is the incident electromagnetic wave and figure 1a shows its interaction with an ion wave propagating at right angles (in the y-direction, say) and another electromagnetic wave propagating at a small angle to the incident wave, such that the wave vectors are conserved. This means that the electromagnetic wave ω_1, $\underline{k}_o - \underline{k}_s$ (equivalent to the Stokes wave) has a higher frequency than the incident wave. This interaction will not therefore give rise to the decay instability. Since this interaction is essentially mismatched in frequency we will include both ion waves (propagating in the $+y$ and $-y$ directions) in the interaction. Now, an exactly equivalent interaction is shown in figure 1(b). In this interaction, the same two ion waves are involved but a different electromagnetic wave (ω_2, $\underline{k}_o + \underline{k}_s$), equivalent to the anti-Stokes wave) is coupled. These two interactions are not independent because the low frequency ion waves take part in both cases. The incident electromagnetic wave is therefore coupled to four unperturbed waves. The frequencies of the two electromagnetic waves ω_1 and ω_2 are equal[1] since $|\underline{k}_o - \underline{k}_s| = |\underline{k}_o + \underline{k}_s|$. Figures 1(a,b) show clearly the possibility of a focusing of the incident wave. We shall show below that this process can be unstable resulting in the incident electromagnetic wave being converted to a converging electromagnetic wave in which the energy becomes more concentrated. By analogy with the oscillating two stream instability, which can be shown [8] to result from a similar process, we expect the ion waves to be purely growing and the Stokes and anti-Stokes waves to be shifted to the frequency of the incident electromagnetic wave.

We shall now outline the derivation of the non-linear wave equations describing this process. It is sufficient to use the two fluid description of the plasma in which we assume $T_e \gg T_i$ so that the ion waves are weakly damped (this is not a necessary restriction). Starting from the equations

$$\nabla \times \underline{H} = \underline{J} + \epsilon_o \frac{\partial \underline{E}}{\partial t} \tag{1}$$

$$\nabla \times \underline{E} = -\mu_o \frac{\partial \underline{H}}{\partial t} \tag{2}$$

$$\frac{\partial \underline{v}_j}{\partial t} + (\underline{v}_j \cdot \nabla)\underline{v}_j + \frac{\kappa T_j}{n_j m_j} \nabla n_j + \nu_j \underline{v}_j = \frac{q_j}{m_j}(\underline{E} + \underline{v}_j \times \underline{B}) \tag{3}$$

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[1] The Stokes and anti-Stokes waves, which are usually defined as forced waves at different frequencies, have been identified with natural modes of the plasma. This is possible in this example since both forced waves are near to resonance under the same conditions.

where $\underline{J} = \sum_j n_j q_j \underline{v}_j$ and $j = i, e$, we obtain the following equation for a plane polarized electromagnetic wave $(\omega, \underline{k})$ propagating in the x, y plane

$$\left(k^2 + \frac{\omega_{pe}^2}{c^2} - \frac{\omega^2}{c^2}\right) E_z = - i \omega \mu_o e n_e v_z - \mu_o n_o e \nu_e v_z \tag{4}$$

where we have put the term due to collisions on the right hand side of the equation since we shall always treat the case of weak collisions. If all waves are infinitesimal in amplitude, the right hand side is negligible and we obtain the usual dispersion relation for electromagnetic waves in a plasma. However, when one (or more) of the waves has a finite amplitude, we must include the non-linear corrections, which are assumed to be small. In equation (4) we have only included the dominant non-linearity which comes from the current. We have also chosen the polarization of all electromagnetic waves (incident, Stokes and anti-Stokes) such that the electric field is in the z-direction.

We now assume that the wave fields (electro-magnetic and ion acoustic) are given by products of a slowly varying amplitude (on the time scale of the pump frequency) times the plane wave determined by the linear dispersion relation. The Stokes and anti-Stokes waves are then given by

$$E_{1,2}(\underline{x}, t) = \mathrm{Re}\, \mathcal{E}_{1,2}(\underline{x}, t)\, e^{i(\underline{k}_{1,2}\cdot\underline{x} - \omega_{1,2} t)}$$

where $\qquad \omega_{1,2}^2 = \omega_{pe}^2 + c^2 k_{1,2}^2$

and $\qquad \underline{k}_{1,2} = (k_o, \mp k_s, 0)$.

With the aid of the matching conditions

$$\underline{k}_o = \underline{k}_{1,2} + \underline{k}_s$$

$$\omega_o \approx \omega_{1,2} + \omega_s$$

where the second condition is only approximately satisfied, we can expand equation (4) about the point (ω, k) defined by the linear dispersion relation, making the identification $\delta\omega \to i\, \partial/\partial t$ and $\delta\underline{k} \to - i\, \partial/\partial\underline{x}$ to obtain equations for $\mathcal{E}_{1,2}$

$$\left(\frac{\partial}{\partial t} + \underline{v}_1 \cdot \frac{\partial}{\partial \underline{x}} + \gamma_T\right) \mathcal{E}_1(\underline{x}, t) = - i\, c_{so}\, \mathcal{E}_o \left\{ (N_s^+)^* e^{-i(\delta - \omega_s)t} + (N_s^-)^* e^{-i(\delta + \omega_s)t} \right\} \tag{5}$$

$$\left(\frac{\partial}{\partial t} + \underline{v}_2\cdot\frac{\partial}{\partial \underline{x}} + \gamma_T\right)\mathcal{E}_2(\underline{x},t) = -i\, c_{so}\, \mathcal{E}_o \left\{ N_s^+ e^{-i(\delta+\omega_s)t} + N_s^- e^{-i(\delta-\omega_s)t} \right\} \quad (6)$$

In order to obtain equations (5) and (6) from (4) we have selected the terms on each side of the equation having approximately the same phase and have taken the real parts of all complex variables. $\mathcal{E}_o$ is the amplitude of the incident electromagnetic wave which is taken to be much larger than all the other amplitudes and therefore assumed to remain constant. $N_s^{\pm}$ represents the slowly varying amplitude of the ion waves where + denotes the ion wave travelling in the +y direction and - denotes the ion wave travelling in the -y direction. $\underline{v}_{1,2}$ are the group velocities of the electromagnetic waves $(= c^2\underline{k}_{1,2}/\omega_{1,2})$, $\gamma_T (= \nu_e\, \omega_{pe}^2/2\omega_{1,2}^2)$ is the damping factor of the electromagnetic waves, $\delta \equiv \omega_0 - \omega_1$ where we have already noted that $\omega_1 = \omega_2$, ω_s is the unperturbed frequency of the ion acoustic waves ($\omega_s = k_s c_s$ and $\omega_s > 0$ since $k_s > 0$) and c_{so} is the coupling coefficient and is given by $c_{so} = e^2/4\epsilon_o m_e \omega_o$.

In order to obtain the corresponding equations for the ion waves we use equation (3) along with the two continuity equations for the ion and electron fluids and Poisson's equation. The unperturbed ion waves are assumed to propagate in the y-direction and we choose the perturbed electron density as the ion wave variable. We then obtain the equation

$$(\omega^2 - k^2 c_s^2) n_{es} = -i \frac{n_o e k}{m_i} (\underline{v}_e \times \underline{B}_1)_y - \nu_i \omega n_{is}$$

where the only non-linear interaction coupling the ion waves to the electromagnetic waves, is the $\underline{v} \times \underline{B}$ term.

If we now write the electron density perturbation as a product of the linear phase and an amplitude function describing the non-linear interaction

$$n_{es}^{\pm}(\underline{x},t) = N_s^{\pm}(\underline{x},t)\, e^{i(k_s y \mp \omega_s t)},$$

and then proceed as before, we obtain the non-linear equations for the ion waves[2]

[2] The non-linear wave equations for the ion waves can be derived rigorously by taking suitable linear combinations of the equations of motion, continuity and Poisson [8]. The above method gives the same result and has the advantage of being briefer.

$$\left(\frac{\partial}{\partial t} + c_s \frac{\partial}{\partial y} + \gamma_s \right) N_s^+(x,t) = -i\, c_{01} \left\{ \mathcal{E}_0\, \mathcal{E}_1^* e^{-i(\delta - \omega_s)t} + \mathcal{E}_0^* \mathcal{E}_2\, e^{i(\delta+\omega_s)t} \right\} \quad (7)$$

$$\left(\frac{\partial}{\partial t} - c_s \frac{\partial}{\partial y} + \gamma_s \right) N_s^-(\underline{x},t) = i\, c_{01} \left\{ \mathcal{E}_0\, \mathcal{E}_1^* e^{-i(\delta + \omega_s)t} + \mathcal{E}_0^* \mathcal{E}_2\, e^{i(\delta-\omega_s)t} \right\} \quad (8)$$

where $c_{01} = n_o e^2 k_s^2 / 4 m_e m_i \omega_s \omega_0 \omega_1$ and where the damping term arises from a phenomenological collision frequency. This term can be used to simulate Landau damping.

3. THE DISPERSION RELATION

(i) We must now solve equations (5) - (8). In order to calculate instability thresholds and initial growth rates we can linearize these equations in the usual way, i.e. by assuming that the incident wave is much larger than the Stokes, anti-Stokes and ion waves

$$\mathcal{E}_o \gg \mathcal{E}_{1,2} \quad , \mathcal{E}_o \gg \mathcal{E}_s^{\pm}$$

where $\mathcal{E}_s^{\pm}$ is the electric field corresponding to $N_S^{\pm}$. The equations can be put into a standard form by using the new amplitudes $\mathcal{E}_1^*$, $N_S^+ e^{i(\delta - \omega_s)t}$, $N_S^- e^{i(\delta+\omega_s)t}$ and $\mathcal{E}_2 e^{i2\delta t}$ and assuming these amplitudes vary as $\exp i\,(qx - \omega t)$. The following dispersion relation is then obtained

$$(\Omega - q v_{1x} + \delta + i\gamma_T)(\Omega - q v_{2x} - \delta + i\gamma_T)\,(\Omega + \omega_s + i\gamma_s)(\Omega - \omega_s + i\gamma_s) + K\,\delta\,\omega_s = 0 \quad (9)$$

where $\Omega \equiv \omega - \delta$, $\delta \equiv \omega_0 - \omega_1$ and $K \equiv 4 c_{so} c_{01} |\mathcal{E}_o|^2$. Equation (9) is a generalised form of Nishikawa's [9] dispersion relation and has unstable solutions both when $\delta > 0$ and when $\delta < 0$. In this case δ is necessarily negative since $k_{1,2}^2 > k_o^2$ and δ is given approximately by

$$\delta \approx \frac{c^2(k_o^2 - k_1^2)}{2\omega_o} = -\frac{c^2 k_s^2}{2\omega_o} \quad . \quad (10)$$

There is therefore only one unstable solution to equation (9). If we put $q = 0$ then we can immediately write down the instability threshold which is

$$K = -\frac{\omega_s}{\gamma}(\gamma_T^2 + \delta^2) \qquad (11)$$

which has the minimum value

$$K_m = 2\,\omega_s\,\gamma_T \qquad (12)$$

where $\delta = -\gamma_T$. Well above threshold the growth rate is given by

$$\gamma = (K\,\omega_s / |\delta|)^{\frac{1}{2}} . \qquad (13)$$

Substituting the expressions for K the minimum threshold can be written

$$\frac{v_o^2}{c_s^2} = \frac{8\,\omega_1 \gamma_T}{\omega_{pi}^2}$$

and

$$\gamma = \frac{\omega_{pi}}{\sqrt{2}} \frac{v_o}{c}$$

where $v_o \equiv e\,\mathcal{E}_o / m_e\,\omega_o$. These last two expressions agree exactly with the results in reference [5]. This instability (self-focusing or filamentation [3]) results in purely growing density perturbations and in the Stokes and anti-Stokes waves both being shifted to the frequency of the incident wave (provided $q = 0$). It is of the same type as the oscillating two stream instability and the modulational instability of a Langmuir wave, both of which can be derived in a similar way [10]. In the self-focusing instability the Stokes and anti-Stokes waves are of course electro-magnetic whereas in the latter two cases they are both Langmuir waves.

Finally, we can solve equation (9) for the spatial growth of the instability

$$q\,v_{1x} = \Omega + i\gamma_T - i\left[\frac{K\,\delta\,\omega_s}{\Omega^2 - \omega_s^2} - \delta^2\right]^{\frac{1}{2}} . \qquad (14)$$

When $\Omega = 0$ we obtain the spatial growth corresponding to the threshold given by equation (11).

The above analysis was carried out for the case where the low frequency density perturbation was in the y-direction. An exactly similar analysis can also be performed for the case where the incident wave couples to a density perturbation in the z-direction. In this case, the polarization of the Stokes and anti-Stokes waves are such that the magnetic field is in the y-direction. The threshold for instability is almost equal to that for the previous case but is larger by the factor k_1^2/k_0^2 where k_1 is again the magnitude of the wave number of either the Stokes or anti-Stokes wave. This is in agreement with the result of Kaw et al. [4] and shows that the incident wave, when it exceeds the instability threshold, will tend to distribute its energy periodically, transverse to the direction of propagation. This results in the energy being concentrated into a number of long threads or filaments (hence the name filamentation). The spacing of the threads is given by the wave number $\underline{k}_s$ of the density perturbation. A preferred $\underline{k}_s$ will be that corresponding to the threshold minimum.

3(ii) Now consider a generalization of the previous model. This is shown in figures 2(a,b). In this case we consider the coupling between electromagnetic waves and ion waves which propagate at an arbitrary angle (subject to the matching relations, of course) to the incident wave. It is clear from these diagrams that the ion wave in figure (2a) is unrelated to the ion wave in figure 2(b). The two scattered electromagnetic waves are therefore unrelated and so there will be no self-focusing. There will of course be the usual three wave decay, since for this case $\omega_{1,2}$ is not restricted to be less than ω_o. Including the off-resonant ion wave (the dotted line in figures 2(a,b) in the interaction the dispersion relation for the interaction shown in figure 2(a) is

$$(\Omega - q\ v_{1x} + \delta + i\gamma_T)(\Omega - q\,c_s + \omega_{s1} + i\gamma_s)(\Omega + qc_s - \omega_{s1} + i\gamma_s)$$
$$- 2\,c_{so}\,c_{o1}|\mathcal{E}_o|^2(\omega_{s1} - q\,c_s) = 0\,. \qquad (15)$$

An analogous dispersion relation describes the interaction shown in figure 2(b). This dispersion relation gives the usual decay instability when $\delta > 0$ ($\omega_o > \omega_1$ and therefore $\omega_o > \omega_2$) and also a modified decay instability [11] when $\delta < 0$. The threshold for modified decay is usually much higher than simple decay, except when $\gamma_T \gtrsim \omega_s$. Thus, the thresholds are given by

$$K = 4\,\frac{\delta}{\omega_s}\,\gamma_T\,\gamma_s \qquad \text{for} \quad \delta > 0 \qquad (16a)$$

$$K = 2|\delta|\,\omega_s \qquad \text{for} \quad \delta < 0 \tag{16b}$$

when $\gamma_T \ll \gamma_s$ and by

$$K = 4\,\gamma_T\,\gamma_s \qquad \text{for} \quad \delta > 0 \tag{17a}$$

$$K = 4\,\gamma_T\,\gamma_s \qquad \text{for} \quad \delta < 0 \tag{17b}$$

when $\gamma_T \gtrsim \omega_s$. (N.B. K was defined after equation (9)).

3(iii) Finally, we wish to point out an electromagnetic modulational instability for the back scattering problem [12]. This instability has not been noted before.

First, consider the one dimensional problem of a standing electromagnetic pump wave. The two travelling waves which make up the pump wave can each give rise to a back-scattered electromagnetic wave and an acoustic wave travelling in the direction of the travelling wave component of the pump. Treated as three wave decays, these two interactions are independent. However, since the acoustic frequency is so low compared with the pump frequency, the acoustic wave in the first decay must also be included in the second decay, as was done in the self-focusing problem. The resulting dispersion relation is then

$$(\Omega+\delta+i\gamma_T)(\Omega-\delta+i\gamma_T)(\Omega-\omega_s+i\gamma_s)(\Omega+\omega_s+i\gamma_s) + 4c'_{so}c'_{o1}|\mathcal{E}_o|^2\delta\,\omega_s = 0 \tag{18}$$

where the pump wave has been taken as $2\,\mathcal{E}_o \cos k_o x \cos \omega_o t$. For this case, the coupling coefficients are $c'_{so} = \omega_{pe}^{\,2}/4\,n_o\omega_o$; $c'_{o1} = n_o e^2\omega_s/4\,m_e^{\,2}\gamma_e v_{Te}^{\,2}\omega_o\omega_1$. Equation (18) gives the usual threshold field and initial growth rate for the Brillouin back scattering instability [12]. However, equation (18) is in the same form as Nishikawa's general dispersion relation [9] and therefore gives instability for $\delta < 0$. This instability is purely growing, i.e. the ion response is purely growing and the excited electromagnetic wave is a standing wave but of a shorter wavelength than the pump wave. The minimum threshold for this 'modulational' instability is

$$\mathcal{E}_o^{\,2} = 4\,\frac{m_e}{e^2}\,\gamma_e\,\kappa\,T_e\,\omega_1\,\nu_e \tag{19}$$

which is equal to the threshold for the oscillating two stream instability [13]. Well above threshold, the growth rate for this instability is given by

$$\gamma^2 = \frac{1}{2}\left\{\delta^2 + \omega_s^2 - \left[(\delta^2 - \omega_s^2)^2 - 4K\omega_s\delta\right]^{\frac{1}{2}}\right\} \tag{20}$$

where $K \equiv 4\, c'_{so}\, c'_{o1} |\mathcal{E}_o|^2$. For $\delta \gg \omega_s$ we obtain

$$\gamma = \frac{1}{\sqrt{2}}\,\omega_{pi}\,\frac{v_o}{c} \tag{21}$$

which is the same as the growth rate for the filamentation instability obtained by Drake et al. [5] and derived earlier in this paper. This electromagnetic modulational instability can also occur at the critical density i.e. when $k_o = 0$. Under these conditions the instability represents the resonant limit of the instability found by Volkov, i.e. the limit when both Stokes and anti-Stokes waves resonate with natural electromagnetic modes (NB Volkov did not consider the effect of damping and therefore did not find any threshold). In this region this electromagnetic modulational instability would be a competitor with the modulational instability of Langmuir waves for which the threshold is the same [10].

4. CONSERVATION OF WAVE ACTION DENSITY

We may derive an equation for the incident wave $\underset{\sim}{E}_o$ in the filamentation problem by an exactly similar procedure to the one used to obtain equations (5) and (6). Starting from equation (4) we pick out the non-linear terms which give a phase close to that of $\underline{E}_o(\underline{x},t)$. For simplicity, we restrict the analysis to be very close to threshold where $\gamma \ll \omega_s$. (The same results can be obtained without this restriction but this simplifies the analysis). The expression for the density perturbation is then simplified and is

$$n_{es} \simeq \frac{i\, n_o e}{k_s \kappa T_e}\, v_z\, B_x \tag{22}$$

Again, selecting the quadratic terms which give a phase close to that of n_s and substituting into the equation for E_o we finally obtain

$$\left(\frac{\partial}{\partial t} + c^2\frac{k_o}{\omega_o}\frac{\partial}{\partial x} + \gamma_T\right)\mathcal{E}_o(x,t) = i\,\frac{\Gamma}{\omega_1}\left\{|\mathcal{E}_1|^2\mathcal{E}_o + |\mathcal{E}_2|^2\mathcal{E}_o + 2\mathcal{E}_o^*\mathcal{E}_1\,\mathcal{E}_2\,e^{i2\delta t}\right\} \tag{23}$$

where $\Gamma \equiv e^2\omega_{pe}^2/4m_e^2\omega_o\omega_1 v_{Te}^2$ and where $\mathcal{E}_o(x,t)$ is the slowly varying amplitude of the incident wave $\underline{E}_o(\underline{x},t)$. Using equation (22) with equations (5) and (6) we can obtain the corresponding equations for $\mathcal{E}_1$ and $\mathcal{E}_2$ which are

$$\left(\frac{\partial}{\partial t} + \underline{v}_1 \cdot \frac{\partial}{\partial \underline{x}} + \gamma_T \right) \mathcal{E}_1(\underline{x},t) = i \frac{\Gamma}{\omega_o} \left\{ |\mathcal{E}_0|^2 \mathcal{E}_1 + \mathcal{E}_0^2 \mathcal{E}_2^* e^{-i2\delta t} \right\} \tag{24}$$

$$\left(\frac{\partial}{\partial t} + \underline{v}_2 \cdot \frac{\partial}{\partial \underline{x}} + \gamma_T \right) \mathcal{E}_2(\underline{x},t) = i \frac{\Gamma}{\omega_o} \left\{ |\mathcal{E}_0|^2 \mathcal{E}_2 + \mathcal{E}_0^2 \mathcal{E}_1^* e^{-i2\delta t} \right\} \tag{25}$$

In order to see the connection between the wave action densities in its simplest form we shall neglect the damping term and assume $\frac{\mathcal{E}}{\partial \underline{x}} = 0$. Let us introduce the new amplitudes defined by

$$\alpha_o = \left(\frac{\epsilon_o}{2} \right)^{\frac{1}{2}} \mathcal{E}_0; \quad \alpha_1 = \left(\frac{\epsilon_o}{2} \right)^{\frac{1}{2}} \mathcal{E}_1$$

and

$$\alpha_2 = \left(\frac{\epsilon_o}{2} \right)^{\frac{1}{2}} \mathcal{E}_2^* \; e^{-i2\delta t} .$$

The significance of the α_n's is that $|\alpha_n|^2$ is the total energy density of mode n <u>in the absence of the interaction</u>. Similarly, $|\alpha_n|^2/\omega_n$ is then the action density. Using equations (23) - (25) and these new amplitudes it is straightforward to derive the following conservation relation,

$$\frac{\partial}{\partial t} \left\{ \frac{|\alpha_1|^2}{\omega_1} + \frac{|\alpha_2|^2}{\omega_2} \right\} = - \frac{\partial}{\partial t} \frac{|\alpha_0|^2}{\omega_0} . \tag{26}$$

This may be contrasted with the corresponding conservation relation (Manley-Rowe) for parametric (or three wave decay) instability which is

$$\frac{\partial}{\partial t} \frac{|a_1|^2}{\omega_1} = \frac{\partial}{\partial t} \frac{|a_2|^2}{\omega_2} = - \frac{\partial}{\partial t} \frac{|a_0|^2}{\omega_0} \tag{27}$$

where a_0, a_1 and a_2 are the amplitudes of the pump and two excited waves and ω_0, ω_1 and ω_2 their frequencies respectively. Comparing (26) and (27) we see that the filamentation (and related phenomena) instability corresponds to a process whereby <u>two</u> pump 'quanta' are required to produce <u>two</u> excited 'quanta'. The decay interaction corresponds to <u>one</u> pump 'quantum' breaking up into two excited 'quanta'. In other words, the filamentation instability is a four wave effect.

The conservation of momentum is trivially satisfied since we have assumed the matching of the k-vectors from the outset. However, we have been unable to derive the corresponding conservation relation for the wave energies. This is evidently due to the fact that the excited waves do not satisfy the linear dispersion relation and therefore $|\alpha_{1,2}|^2$ does not represent the total energy of the excited electromagnetic waves.

5. DISCUSSION AND CONCLUSIONS

In this paper we have considered those instabilities in which an electromagnetic wave generates an ion wave and another electromagnetic wave. The simplest case is, of course, the Brillouin back scatter instability which has been discussed by many authors. The generalization of this case is when the ion wave and electromagnetic wave are excited at an angle to the incident transverse wave. The decay type instability then includes all possibilities from back scatter to side scatter (where the excited transverse wave propagates at right angles to the incident wave). Since the acoustic frequency is so low, the ion acoustic wave propagating in the reverse direction to the resonant ion acoustic wave is only off resonant by a small mis-match. When this off resonant acoustic wave is included in the interaction the modified decay [11] instability is then obtained. In all these cases the Stokes and anti-Stokes waves are independent of each other.

There are two cases however when the inclusion of the off resonant ion wave results in the Stokes and anti-Stokes electromagnetic waves becoming coupled. This occurs when the acoustic wave vector is either parallel or perpendicular to the incident wave vector and in both cases a purely growing instability results. The purely growing instability for the parallel case has not been pointed out before. It can occur in an under-dense plasma provided the pump wave is a standing wave. At the critical density its threshold is the same as that for the oscillating two stream instability and its growth rate is equal to the filamentation instability which is the purely growing instability which occurs when the ion wave vector is perpendicular to the incident wave. The filamentation instability does not require the pump to be a standing wave. This instability can also occur at the critical density where its threshold is also equal to that for the oscillating two stream instability. At the critical density, the filamentation instability and the electromagnetic modulational instability are closely related. Both generate standing electromagnetic waves at the pump frequency. In the filamentation case the excited electromagnetic wave vectors are converging - hence the filamentation or self-focusing - whereas in the modulational case they are parallel to each other. It is worth pointing out the similarity between these two instabilities and the modulational instability of a Langmuir wave. Finally, we have derived an action density conservation relation for the filamentation case (in fact, the analysis would cover other cases of modulational instability since the equations are of similar form). This conservation relation is the analogue of the Manley-Rowe relations for a three wave process and shows clearly that filamentation is a four wave process where two pump 'quanta' produce two other 'quanta'.

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REFERENCES

[1] VOLKOV, T.F., Fizika plazmy i problema upravljaemyh termojadernyh reakcij, 4, 98 Izd. Akad. Nauk. SSSR Moscow (1958). [Plasma Physics and the Problems of Controlled Thermonuclear Reactions, 4, 114, Pergamon Press, London (1960)].

[2] LITVAK, A.G., Izvestiya Vuz. Radiofizika, 9, (1966) 675. [Sov. Phys. Radiophysics, 9, (1966) 404].

[3] DYSTHE, K.B., Phys. Letters 27A, (1968) 59.

[4] KAW, P., SCHMIDT, G., WILCOX, T., Phys. Fluids 16, (1973) 1522.

[5] DRAKE, J., KAW, P.K., LEE, Y.C., SCHMIDT, G., LIU, C.S., ROSENBLUTH, M.N., Phys. Fluids 17, (1974) 778.

[6] PERKINS, F.W., VALEO, E.J., Phys. Rev. Letters 32, (1974) 1234.

[7] MAX, C.E., ARONS, J., LANGDON, A.B., Phys. Rev. Letters 33 (1974), 209.

[8] LASHMORE-DAVIES, C.N., Plasma Phys. 17, (1975), 281.

[9] NISHIKAWA, K., J. Phys. Soc. Japan 24 (1968), 916.

[10] LASHMORE-DAVIES, C.N., Phys. Letters 49A (1974), 375.

[11] GALEEV, A.A., SAGDEEV, R.Z., Nuclear Fusion 13 (1973), 603.

[12] FORSLUND, D.W., KINDEL, J.M., LINDMAN, E.L., Phys. Rev. Letters 30 (1973), 739.

[13] NISHIKAWA, K., J. Phys. Soc. Japan 24 (1968) 1152.

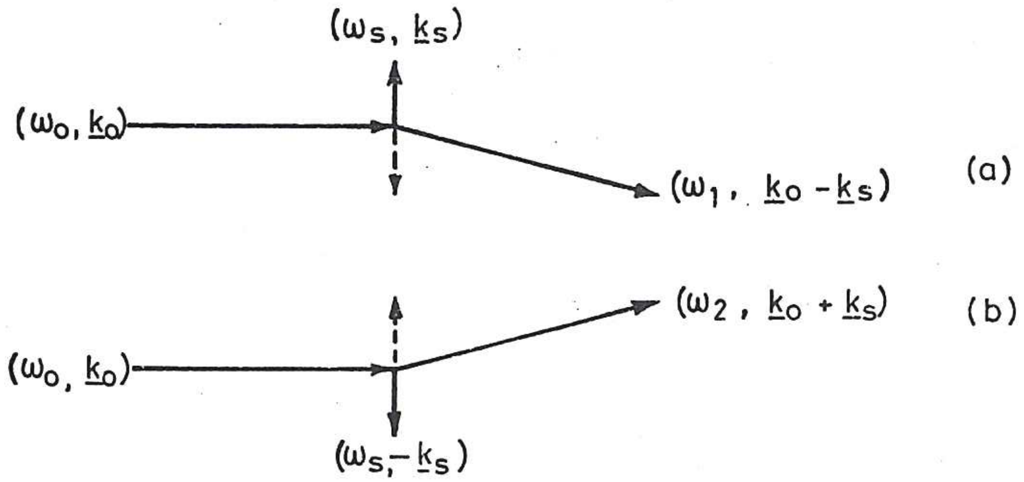

Fig.1 Wave number configuration for the filamentation instability. ω_s refers to the ion wave and ω_0, ω_1 and ω_2 to the incident and excited electromagnetic waves respectively. Configurations (a) and (b) are coupled.

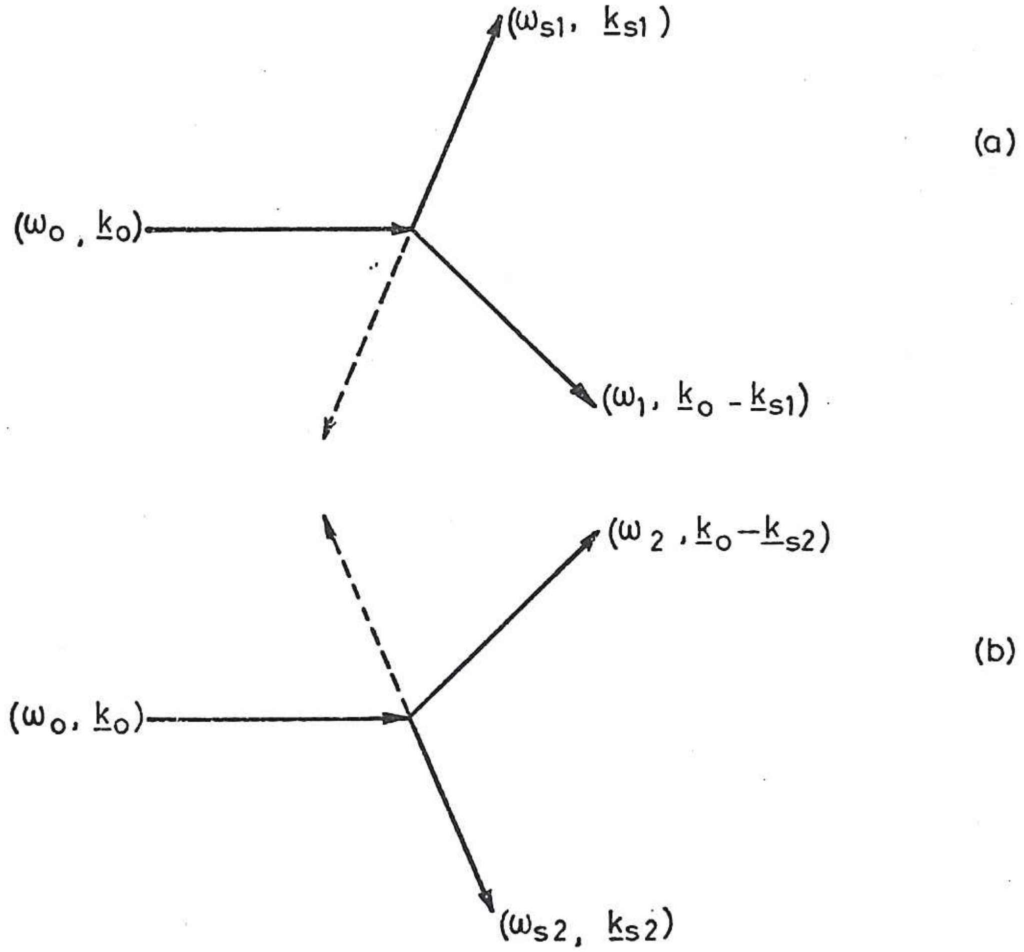

Fig.2 Wave number configuration for two possible decay instabilities. Configurations (a) and (b) are independent and the symbols have the same meaning as for figure 1.