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# THE FINITE GEOMETRY THEORY OF THE SLOWING DOWN OF FAST IONS IN A TOROIDAL PLASMA

by

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ABSTRACT

The effect of toroidal geometry upon the slowing down of fast ions in a tokamak plasma is considered. An appropriate bounce averaged Fokker-Planck equation which includes particle trapping in the toroidal field gradient is derived. The equation is solved by expressing the solution in terms of a series of 'finite geometry' eigenfunctions. This solution is then used to show that the trapping of the fast ions in the toroidal field gradient reduces the fast ion current by terms of order $(r/R)^{\frac{1}{2}}$. The radial transport of the fast ions as they slow down is calculated, and it is found that for counterinjection (coinjection) the ions diffuse outwards (inwards) by approximately a fast ion banana width. The diffusion is accompanied by a loss of fast ion toroidal momentum and a consequent reduction in the momentum transferred to the background plasma.

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# 1. INTRODUCTION

The heating of a plasma by the injection of fast ions has now been demonstrated in several experiments [1], and the possibility of enhancing the fusion power of a reactor by the injection of fast ions is being explored [2]. In both of these applications an accurate knowledge of the evolution of the fast ion distribution function through Coulomb collisions with the background plasma is required. Previous studies [3,4] of this slowing down process have neglected the effects of toroidal geometry. In particular, the variation of the magnetic field around a magnetic surface, which results in some of the fast ions being trapped in banana orbits, has been ignored. For well passing ions ($v_{\|}/v \sim 1$) the magnetic field geometry is unimportant; however, for trapped or just passing ions, the variation in magnetic field is clearly fundamental.

In an effort to remedy this situation Connor and Cordey [5] calculated the distribution function, for the two limiting cases, of all the ions being trapped or well passing. This treatment is of course only valid, if the ions are scattered through a small angle, in the time they take to slow down to thermal energies and thereby remain trapped or well passing. For the parameters of present and future beam heating or beam-fusion schemes a typical ion is usually scattered through $90^{\circ}$ during a slowing down time and thus the fast ion distribution will consist of both passing and trapped ions.

In this paper the bounce averaged Fokker-Planck equation is solved in both trapped and passing regions of velocity space and the complete fast ion distribution function obtained. The method of solution is as follows: since the Fokker-Planck is separable in angle $\xi(= v_{\|}(0)/v)$ and velocity $v$, a set of finite geometry eigenfunctions of the separated equation in $\xi$ can be found, and then by solving the subsidiary equation in $v$ the complete solution may be obtained. The properties of these finite geometry eigenfunctions are discussed in Section 2. Since the coefficients of the collision operator are elliptic integrals the eigenfunctions have to be obtained numerically; however in Section 2c the elliptic integrals are replaced by approximate forms, and then analytic eigenfunctions may be derived for this model collision operator. The two sets of eigenfunctions are found to be very similar, indicating that the model collision operator is a good approximation to the exact operator.

The structure of the remainder of the paper is as follows. The solution of the Fokker Planck equation obtained in Section 2 is first used in Section 3 to determine the effects of finite geometry upon the fast ion current. The trapping of the fast ions in the field gradient reduces the mean parallel velocity of the fast ions and hence the beam current [6] is reduced.

Then in Section 4 it is shown that the scattering of the fast ions by the background ions leads to radial diffusion of the fast ions. The diffusion is inward for coinjection (ie. beam and tokamak current are in the same direction) and outward for counter injection. The analysis takes into account the radial diffusion of the ions before they become trapped as well as their radial excursion on becoming trapped. This latter effect has recently been discussed by Callan et al. [4] where the authors consider the effect upon the distribution function of the loss of some of the trapped ions.

Finally in Section 5 the effect of toroidal geometry upon the energy and momentum transfer from the fast ions to the background plasma is calculated. Although one can show trivially that the energy transfer is unaffected by the geometry, the radial diffusion of the fast ions reduces the momentum transfer from the fast ions to the background plasma by a term of order $(r/R)^{\frac{1}{2}}$.

## 2. FOKKER PLANCK EQUATION AND SOLUTION

### (a) The bounce averaged Fokker-Planck equation

The bounce averaged Fokker Planck equation, describing the collisional behaviour of the fast ions in Tokamak geometry, has been previously derived by Connor and Cordey [5] and the analysis is repeated here for completeness. Starting with the collision operator in the Landau form we first make use of the inequality $v_i \ll v \ll v_e$ where $v_i$, $v_e$ are the thermal velocities of the plasma ions and electrons and $v$ is the velocity of the fast ions. This inequality will be satisfied in most injection heating schemes, apart from those in which the electron temperature is very low in comparison with the fast ion energy. The Fokker Planck equation for the fast ion distribution function $f$ in the guiding centre approximation can then be written in the following form for axisymmetric toroidal geometry with circular magnetic surfaces:

$$\frac{\partial f}{\partial t} + \sigma \frac{\Theta q}{r} \frac{\partial f}{\partial \theta} = \frac{1}{\tau_s} \left[ v^{-2} \frac{\partial}{\partial v} \{(v^3 + v_c^3) f\} + \beta \frac{B_o}{B} \frac{v_c^3}{v^3} \frac{q}{\xi v} \frac{\partial}{\partial \xi} \left\{ (1-\xi^2) \frac{q}{\xi v} \frac{\partial f}{\partial \xi} \right\} \right] + S \quad \ldots (1)$$

where $\xi = (1 - v_\perp^2 B_o/v^2 B)^{\frac{1}{2}} \equiv (1 - \mu B_o/\varepsilon)^{\frac{1}{2}}$ is an invariant, the local parallel velocity $q$ may be written in terms of the invariants $\xi$ and $v$ in the form $q = |v_{||}| = v \left[ 1 - \frac{B}{B_o} (1 - \xi^2) \right]^{\frac{1}{2}}$, the Spitzer slowing down time

$$\tau_s = 3\, m_e\, v_e^3\, m_f/(16\pi^{\frac{1}{2}}\, e^4\, z_f^2\, \ell n\, \Lambda\, n_e)$$

$m_f$ and $z_f$ are the fast ion mass and charge respectively, $\beta = z_{eff}\, m_i/2\bar{z}\, m_f$, $z_{eff} = \sum_j n_j\, z_j^2/n_e$ is the effective plasma charge of the background plasma, $\bar{z} = \sum_j z_j^2\, n_j\, m_i/n_e\, m_j \sim 1$, $v_c = (3\sqrt{\pi}\, \bar{z}/4)^{1/3} (m_e/m_i)^{1/3}\, v_e$, $S$ is the source of fast ions. Geometrical effects enter through the dependence of $B$ on the poloidal angle $\theta$ in the form

$$B = B_o\, (1 - \varepsilon \cos\theta)/(1 - \varepsilon) \quad \ldots (2)$$

with $\varepsilon = r/R$. Finally $\Theta(r)$ is defined as the ratio of poloidal and toroidal fields. The r.h.s. of equation (1) (the collision operator) was derived from the local equation [ 3 ] by transferring from local variables $\hat{v}$, $\hat{v}_{||}/\hat{v}$ to the invariants **v**, $\xi$, using the transformation

$$\frac{\hat{v}_{||}}{\hat{v}} = \left[ 1 - \frac{B}{B_o} (1 - \xi^2) \right]^{\frac{1}{2}}, \quad \hat{v} = v \; .$$

Equation (1) is solved by expanding $f$ as a series in $\tau_B/\tau_s$ in the form $f = f_o + \tau_B/\tau_s\, f_1 + \ldots$, where $\tau_B$ is the bounce period of a hot ion and $\tau_s$ is the slowing down time. The zero'th order equation is

$$\frac{\partial f_o}{\partial \theta} = 0$$

which gives $f_o(v_o\, \xi, r)$. The function $f_o$ is then determined from the periodicity constraint in $\theta$ on the next order solution which is

$$\tau_s \frac{\partial f_o}{\partial t} = v^{-2} \frac{\partial}{\partial v}[(v_c^3 + v^3) f_o] + \frac{\beta\, v_c^3}{v^3\, \xi \left\langle \frac{v}{v_\parallel} \right\rangle} \frac{\partial}{\partial \xi}\left\{ \frac{(1-\xi^2)\left\langle \frac{v_\parallel}{v} \right\rangle}{\xi} \frac{\partial f_o}{\partial \xi} \right\} + \tau_s S \qquad \ldots (3)$$

where

$$\left\langle \frac{v_\parallel}{v} \right\rangle = \begin{cases} \dfrac{1}{2\pi\, B_o^{\frac{1}{2}}} \oint [\xi^2 B - (B - B_o)]\, d\theta & \text{for passing ions} \\[2ex] \dfrac{1}{2\pi\, B_o^{\frac{1}{2}}} \displaystyle\int_A^B [\xi^2 B - (B - B_o)]^{\frac{1}{2}}\, d\theta & \text{for trapped ions} \end{cases} \qquad \ldots (4)$$

(A, B are the turning points)

and a similar definition for $\left\langle \frac{v}{v_\parallel} \right\rangle$, with $B$ being a function of $\theta$ and given by equation (2). After substituting for $B$ from Eq.(2) these functions may be expressed in terms of the complete elliptic integrals $E$ and $K$ in the form

$$\left\langle \frac{v_\parallel}{v} \right\rangle = \begin{cases} \dfrac{2\xi}{\pi}\, E(2\varepsilon/\xi^2) \quad , & \xi^2 > 2\varepsilon \quad \text{(passing ions)} \\[2ex] \dfrac{2\sqrt{2\varepsilon}}{\pi}\left[ E(\xi^2/2\varepsilon) - \left(1 - \dfrac{\xi^2}{2\varepsilon}\right) K(\xi^2/2\varepsilon) \right], & \xi^2 < 2\varepsilon \quad \text{(trapped ions)} \end{cases} \qquad \ldots (5)$$

$$\left\langle \frac{v}{v_\parallel} \right\rangle \begin{cases} \dfrac{2}{\pi}\, K\left(\dfrac{2\varepsilon}{\xi^2}\right) & \xi^2 > 2\varepsilon \quad \text{(passing ions)} \\[2ex] \dfrac{2}{\pi \xi_t}\, K(\xi^2/2\varepsilon) & \xi^2 < 2\varepsilon \quad \text{(trapped ions)} \end{cases} \qquad \ldots (6)$$

where in deriving the above, terms of order $\varepsilon$ have been neglected. The expressions given in Eq.(5), (6) $\langle v_\parallel/v \rangle$ and $\langle v/v_\parallel \rangle$ are plotted in Fig.1 as a function of $\xi$. The dotted curves are the approximations to those functions which will be used in part (c) of this section.

The boundary conditions for Eq.(3) at the transition points $\xi^2 = \xi_t^2$ are:

f continuous

$$\left. \frac{\left\langle \frac{v_\parallel}{v} \right\rangle}{\xi} \frac{\partial f}{\partial \xi} \right|_{\xi_t^+} - \left. \frac{\left\langle \frac{v_\parallel}{v} \right\rangle}{\xi} \frac{\partial f}{\partial \xi} \right|_{-\xi_t^-} = \left. \frac{2\left\langle \frac{v_\parallel}{v} \right\rangle}{\xi} \frac{\partial f}{\partial \xi} \right|_{\xi_t^-} . \qquad \ldots (7)$$

This latter condition expresses the conservation of flux of particles between the passing and trapped regions. The solution in the mirror trapped region must be symmetric in $\xi$ ie. $f(-\xi) = f(\xi)$ this means that in general $\frac{\partial f}{\partial \xi}$ will be discontinuous at the transition points. At the transition points the function $\langle v/v_{\parallel} \rangle$ has a logarithmic singularity and the derivative of the function $\langle v_{\parallel}/v \rangle$ is also singular, and close to these points the expansion of $f$ as a series in $\tau_B/\tau_s$ is no longer possible since $\tau_B \to \infty$. Thus in this transition region equation (1) has to be solved and then the solution matched to the solutions in the trapped and passing regions of equation (3).

A similar problem occurs in the determination of the distribution function in the throat of a mirror machine which was discussed by Baldwin, Cordey and Watson [7] and also in the calculation of transport coefficients of a toroidal plasma in the low-to-intermediate collision frequency regime where Hinton and Rosenbluth [8] obtained the solution in the transition region. The width of this transition region $\delta\xi$ is of order $(\tau_{bo}/\tau_S)^{\frac{1}{2}}$ and this parameter is usually $\sim 10^{-3}$ for typical injection energies. Thus the transition region is very narrow indeed and since the solution in that region will only contribute terms of the order $(\tau_{bo}/\tau_S)^{\frac{1}{2}}$ to the averages derived in Sections 3 and 4, it will not be derived here.

(b) The eigenfunction solution

Since equation (3) is separable in $\xi$ and $v$ and the equation in $\xi$ is self adjoint, the solution may be expressed as a series eigenfunctions in the form

$$f_o = \Sigma \; a_n(v) \; C_n(\xi) \qquad \ldots (8)$$

where the $C_n(\xi)$ are the eigenfunctions of the equation

$$\frac{1}{r}\frac{d}{d\xi}\{1 - \xi^2)\left\{\frac{t dC_n}{d\xi}\right\} + \lambda_n C_n = 0 \qquad \ldots (9)$$

with eigenvalue $\lambda_n$, satisfying the following boundary conditions

$$C_n \text{ continuous at } \xi = \pm \xi_t \qquad \ldots (10)$$

$$\left.\frac{t\partial C_n}{\partial \xi}\right|_{\xi_t^+} - \left.\frac{t\partial C_n}{\partial \xi}\right|_{-\xi_t^-} = \left.\frac{2t\partial C_n}{\partial \xi}\right|_{\xi_t^-} \qquad \dots (11)$$

$$C_n(-\xi) = C_n(\xi) \quad , \quad -\xi_t < \xi < \xi_t \qquad \dots (12)$$

$$C_n \text{ finite at } \xi = \pm 1, 0 \qquad \dots (13)$$

where in equation (9) the functions $r$ and $t$ are

$$r(\xi) \equiv \langle v/v_{\parallel} \rangle \xi$$
$$t(\xi) \equiv \langle v_{\parallel}/v \rangle /\xi \quad . \qquad \dots (14)$$

with the $\langle v_{\parallel}/v \rangle$ and $\langle v/v_{\parallel} \rangle$ given by Eqs.(5) and (6).

The $a_n$ are determined from the separated equation in **v**

$$v^{-2} \frac{d}{dv} [(v^3 + v_c^3) a_n] - \lambda_n a_n = S_o \, \delta(v - v_o) \, K_n \, \tau_s \qquad \dots (15)$$

where in equation (15) the source $S$ of equation (3) has been written in the

$$S(\xi, v) = S_o \, \delta(v - v_o) K(\xi) = S_o \, \delta(v - v_o) \Sigma \, K_n \, C_n(\xi).$$

The properties of the eigenfunctions $C_n$ are now determined. First the eigenfunctions are normalised in the form $C_n(1) = 1$ in the same manner as Legendre polynomials. Since Eq.(9) is invariant under the transformation $\xi' = -\xi$ and the solution must be finite at $\xi = -1$ (Eq.(13)), then

$$C_n(\xi) = A \, C_n(-\xi) \qquad \dots (16)$$

with $A$ a constant. Using the continuity of $C_n$ at the transition points $\xi = \pm \xi_t$ (condition (10)) and the symmetry of $C_n$ in the trapped region $-\xi_t < \xi < \xi_t$ (Eq.(12)) gives

$$C_n(-\xi_t) = C_n(\xi_t) \qquad \dots (17)$$

thus from the above equation and Eq.(16) either

$$A = 1 \quad \text{or} \quad C_n(\pm\xi_t) = 0 \; . \qquad \dots (18)$$

These two conditions each lead to a separate set of eigenfunctions. The subset given by the condition A = 1 are even in $\xi$ and henceforth will be referred to as the even set. The subset which vanish at $\xi = \pm \xi_t$ will be referred to as the odd set, since it can be trivially shown that this latter set are orthogonal to the even set $C_n(\xi) = - C_n(-\xi)$, that is the members of the set must be odd for $|\xi| > \xi_t$ and zero in trapped region $|\xi| < \xi_t$.

It can then be shown that together the even and odd set of eigenfunctions form a complete set for functions which are even in the trapped region and finite at $\xi = \pm 1$. The first few members of the set of eigenfunctions are shown in Fig.2. For r and t given by Eqs.(5-6) and (14) the functions of Fig.2 were obtained numerically using a simple shooting method; for the odd functions the integration was started from the point $\xi = 1$, and then $\lambda$ varied until $C(\xi_t) = 0$; for the even set the integration was started from the points $\xi = 1$ and $\xi = 0$ and $\lambda$ varied until the slopes of the two solutions matched at $\xi = \xi_t$.

To obtain the complete solution Eq.(15) has to be solved for the $a_n$, using the integrating factor method, and the following expression satisfies the boundary condition $a_n(v) = 0$ at $v = v_o$,

$$a_n(v) = S_o \tau_s K_n \, St\left(1 - \frac{v}{v_o}\right) \frac{(v_o^3 + v_c^3)^{\beta\lambda_n/3}}{(v^3 + v_c^3)^{1+\beta\lambda_n/3}} \left(\frac{v}{v_o}\right)^{\beta\lambda_n} \quad \ldots (19)$$

where St is the step function defined as $St(x) = 1$ for $x \geqslant 0$ and $St(x) = 0$ for $x < 0$. Thus from Eqs.(8) and (19) the complete solution may be written in the form

$$f_o = S_o \tau_s \sum \frac{K_n (v_o^3 + v_c^3)^{\beta\lambda_n/3}}{(v^3 + v_c^3)^{1+\beta\lambda_n/3}} \left(\frac{v}{v_o}\right)^{\beta\lambda_n} C_n(\xi). \quad v < v_o \quad \ldots (20)$$

In Eq.(20) the eigenfunctions $C_n$ and eigenvalues $\lambda_n$ have to be determined numerically for r and t of Eq.(14); however in the next part of this section analytic expressions are found for the $C_n$ and $\lambda_n$ for a model collision operator in which the functions r and t are replaced by approximate forms.

(c) Eigenfunctions for model operator

The functions r and t given by Eq.(14) are approximated in the trapped and passing regions by their limiting values near $\xi = 0$ and $\xi = 1$ respectively. These are

$$r \equiv \left\langle \frac{v}{v_{\parallel}} \right\rangle \xi = \frac{\xi}{\xi_t} \qquad |\xi| < \xi_t \quad \ldots (21)$$

$$t \equiv \left\langle \frac{v_{\parallel}}{v} \right\rangle \xi^{-1} = \frac{\xi}{2\xi_t}$$

$$t \simeq r \simeq 1 \qquad |\xi| > \xi_t \, . \qquad \dots (22)$$

With the above approximate forms for $r$ and $t$ Eq.(9) becomes the Legendre equation for passing ions

$$\frac{d}{d\xi}\left\{ (1-\xi^2)\frac{dC_n}{d\xi} \right\} + \lambda_n C_n = 0 \qquad |\xi| > \xi_t \qquad \dots (23)$$

and for trapped ions

$$\frac{1}{2\xi}\frac{d}{d\xi}\left\{ (1-\xi^2)\xi\frac{dC_n}{d\xi} \right\} + \lambda_n C_n = 0 \ , \quad |\xi| < \xi_t \qquad \dots (24)$$

This equation may be put in standard form by the transformation $x = 1 - 2\xi^2$ becoming Legendre's equation

$$2\frac{d}{dx}(1-x^2)\frac{dC_n}{dx} + \lambda_n C_n = 0 \, . \qquad \dots (25)$$

The odd eigenfunctions, ie. the set which vanish at $\xi = \xi_t$ are $C = P_{\nu_n}(\xi)$ with the $\nu_n$ given by

$$P_{\nu_n}(\xi_t) = 0. \qquad \dots (26)$$

For small $\xi_t$ $(\sim \varepsilon^{\frac{1}{2}})$ one can obtain an approximate value for $\nu_n$

$$\nu_n = n + \frac{4\xi_t}{\pi}\frac{\Gamma^2(1+n/2)}{\Gamma^2(\frac{1}{2}+n/2)} \qquad n \text{ odd} \dots (27)$$

and the eigenvalue

$$\lambda_n \equiv \nu_n(\nu_n + 1) = n(n+1) + \frac{4\xi_t(2n+1)}{\pi}\frac{\Gamma^2(1+n/2)}{\Gamma^2(\frac{1}{2}+n/2)} \, .$$

To obtain the even eigenfunctions the solution of Eq.(23) is

$$C_n = P_{\nu_n}(\xi) \qquad \xi > \xi_t \qquad \dots (28)$$

and the solution of Eq.(24) correct to order $\varepsilon$ is

$$C_n = A\left(1 - \nu_n\frac{(\nu_n+1)}{2}\xi^2\right) \qquad |\xi| < \xi_t. \qquad \dots (29)$$

Matching these two solutions at the transition points $\xi = \pm\xi_t$, by use of Eqs.(10) and (11), gives the equation for the eigenvalue

$$2P'_{\nu_n}(\xi_t) = -\xi_t\nu_n(\nu_n+1)P_{\nu_n}(\xi_t) \qquad \dots (30)$$

and the constant

$$A = P_{\nu_n}(\xi_t)\Big/\left(1 - \nu_n\frac{(\nu_n+1)}{2}\xi_t^2\right) \, . \qquad \dots (31)$$

The order $\nu_n$ of the Legendre functions may also be obtained correct to order $\xi_t$ $(\sim \varepsilon^{\frac{1}{2}})$ and is

$$\nu_n = n + \xi_t \, n(n+1)\, \Gamma^2(\tfrac{1}{2} + n/2)/\{2\pi\, \Gamma^2(1 + n/2)\}, \quad n \text{ even}$$

and thus ... (32)

$$\lambda_n \equiv \nu_n(\nu_n + 1) = n(n+1) + (2n+1)\, \xi_t \, n(n+1)\, \Gamma^2(\tfrac{1}{2} + n/2)/\{2\pi\, \Gamma^2(1 + n/2)\}$$

Thus the complete set of eigenfunctions may now be written down: they are

$$C_n = \begin{cases} P_{\nu_n}(\xi) & \xi_t < \xi < 1 \\ 0 & -\xi_t < \xi < \xi_t \\ -P_{\nu_n}(-\xi) & -1 < \xi < -\xi_t \end{cases} \quad \text{with } \lambda_n \text{ given by Eq.(25)} \qquad \text{... (33)}$$

and

$$C_n = \begin{cases} P\nu_n(\xi) & \xi_t < \xi < 1 \\ \dfrac{P\nu_n(\xi_t)\{2 - \nu_n(\nu_n + 1)\xi^2\}}{(2 - \nu_n(\nu_n + 1)\,\xi_t^2)} & -\xi_t < \xi < \xi_t \\ P_{\nu_n}(-\xi) & -1 < \xi \; -\xi_t \end{cases} \qquad \text{... (34)}$$

with $\nu_n$ given by Eq.(30).

The first few members of this set of functions are shown in Fig.3 and comparing them with the exact functions of Fig.2 it can be seen that the difference is less than 10%. Thus in the following sections where the various effects of tokamak geometry upon the beam current etc are calculated, use will be made of the solution of the model operator given by Eqs.(33) and (34).

Bringing together the results of this section Eqs.(32)-(34) the fast ion distribution function correct to $\varepsilon^{\frac{1}{2}}$ can be written in the form

$$f_o = S_o \tau_s \sum K_n \frac{(v_o^3 + v_c^3)^{n(n+1)\beta/3}}{(v^3 + v_c^3)^{1+n(n+1)\beta/3}} \left(\frac{v}{v_o}\right)^{n(n+1)\beta} P_n(\xi)$$

$$+ S_o \tau_s (2\varepsilon)^{\frac{1}{2}} \sum \frac{K_n R_n (v_o^3 + v_c^3)^{n(n+1)\beta/3}}{(v^3 + v_c^3)^{1+n(n+1)\beta/3}} \left(\frac{v}{v_o}\right)^{n(n+1)\beta} \left\{ \left.\frac{\partial P_\nu(\xi)}{\partial \nu}\right|_{\nu=n} \right.$$

$$\left. + (2n+1)\beta \log_e\left[\left(\frac{v_o^3 + v_c^3}{v^3 + v_c^3}\right)^{1/3} \frac{v}{v_o}\right] P_n(\xi) \right\}, \qquad |\xi| > \xi_t \qquad \text{... (35)}$$

for passing ions, and

$$f_o = S_o \tau_s \sum_{n \text{ even}} K_n \frac{(v_o^3 + v_c^3)^{n(n+1)\beta/3}}{(v^3 + v_c^3)^{1+n(n+1)\beta/3}} \left(\frac{v}{v_o}\right)^{n(n+1)\beta} \left[ P_n(0) + (2\varepsilon)^{\frac{1}{2}} R_n \left\{ \left.\frac{\partial P_\nu(0)}{\partial \nu}\right|_{\nu=n} \right.\right.$$

$$\left.\left. + (2n+1)\beta \log_e\left[\left(\frac{v_o^3 + v_c^3}{v^3 + v_c^3}\right)^{1/3} \frac{v}{v_o}\right] P_n(0) \right\} \right] \qquad |\xi| < \xi_t \qquad \text{... (36)}$$

for trapped ions. The first term in the large square brackets of both these terms is identical to that of references [3, 4] for the case of a uniform magnetic field, all the geometrical effects are in the second term which is proportional to $\varepsilon^{\frac{1}{2}}$. These geometrical effects which are mainly due to the trapping of the ions in banana orbits are illustrated in Fig.4 where the fast ion distribution is contoured as a function of velocity $v$ and angle $\xi$ for $\varepsilon = 0.1$ and $0$. The expression for $f_o$ given by Eqs.(35) and (36) is used in the next three sections where the important consequences of finite $\varepsilon$ are discussed.

## 3. BEAM CURRENT

It was first pointed out by Ohkawa [6] that the fast ions constituted a current, and it was suggested that the tokamak current could be maintained by the injection of fast ions. The transfer of momentum from the fast ions to the electrons generates an electron current which tends to cancel out the beam current. The effect upon this electron current of electron trapping has been discussed previously by Connor and Cordey [5 ]. Here by using the results of Section 2 the reduction in beam current due to the trapping of the fast ions in the field gradient is calculated.

The average velocity of the fast ions is

$$\overline{n_f v_\parallel} = \int f_o \, v^3 \, \xi dv d\xi , \qquad \ldots (37)$$

note the volume element in velocity space $d^3v \equiv v^3 \xi d\xi dv/q\sigma$ where

$$\sigma = \begin{matrix} +1 & (\xi > 0) \\ -1 & (\xi < 0) \end{matrix} .$$

Substituting for $f_o$ from Eqs.(35) and (36) gives current to order $\varepsilon^{\frac{1}{2}}$

$$\overline{n_f v_\parallel} = S_o \tau_s v_o \Bigg[ \frac{2}{3} K_1 \int_0^1 \frac{du\, u^3 (1+u_c^3)^{2\beta/3} u^{2\beta}}{(u^3 + u_c^3)^{1+2\beta/3}} + 2\beta K_1 (2\varepsilon)^{\frac{1}{2}} \int_0^1 du\, u^3 \frac{(1+u_c^3)^{2\beta/3}}{(u^3+u_c^3)^{1+2\beta/3}} \log_e \left\{ \left( \frac{1+u_c^3}{u^3+u_c^3} \right)^{1/3} u \right\} + (2\varepsilon)^{\frac{1}{2}} \sum K_n J_n \int_0^1 \frac{du\, u^3 (1+u_c^3)^{n(n+1)\beta/3} u^{n(n+1)\beta}}{(u^3+u_c^3)^{1+n(n+1)\beta/3}} \Bigg] \qquad \ldots (38)$$

where $J_n = 2 P_n'(0)/\{n(n+1)-2\}$ , $n \neq 1$ and $J_1 = \frac{2}{3} P_1'(0)(\ln 2 - 1/3)$.

The geometrical corrections in the above equation, that is the second and third term inside the square brackets, are usually both negative so, as was expected, the trapping of the fast ions in the toroidal field gradient reduces the beam current by an amount of order $\varepsilon^{\frac{1}{2}}$.

## 4. DIFFUSION OF FAST IONS

In this section it is shown that the scattering of the fast ions by the background ions results in the radial diffusion of the fast ions. For the present generation of tokamaks with low ohmic current this transport process may give rise to a large fast ion loss, since the poloidal larmour radius is a significant fraction of the plasma minor radius.

In the remainder of this section the standard techniques of neoclassical diffusion are used to calculate the diffusion flux $\Gamma_f$ of the fast ions. Our starting point is the expression for the radial drift velocity which may be written in the form [10]

$$v_{Dr} = \frac{m_f}{ez_f} \frac{q}{r} \frac{\partial}{\partial\theta} \left(\frac{q}{B}\right) . \qquad \ldots (39)$$

Then the radial flux of fast ions is

$$\Gamma_f = \frac{m_f}{2\pi e\, z_f\, r} \int_0^{2\pi} d\theta \int v^3 \frac{\partial}{\partial\theta} \left(\frac{q}{B}\right) \left[f_o + f_1\right] dv\, \xi d\xi/\sigma \qquad \ldots (40)$$

Integrating the above equation by parts in $\theta$ gives

$$\Gamma_f = \frac{-m_f}{2\pi\, z_f\, er} \int_0^{2\pi} d\theta \int v\, \xi\, \frac{q}{B} \frac{\partial f_1}{\partial\theta} d\xi\, dv . \qquad \ldots (41)$$

Now the first order correction to $f_1$ in the expansion of $f$ in $\tau_B/\tau_s$ is given by (see Section 2)

$$\sigma \frac{\partial f_1}{\partial\theta} = \frac{r}{\tau_s\, \Theta\, q} \left[ v^{-2} \frac{\partial}{\partial v} \{(v_c^3 + v^3) f_o\} + \frac{\beta v_c^3}{v^3} \frac{q}{v^2\xi} \frac{\partial}{\partial\xi}\left( \frac{1-\xi^2}{\xi}\, q \frac{\partial f_o}{\partial\xi}\right) + S\tau_s \right] \qquad \ldots (42)$$

and then integrating by parts in $\xi$ to eliminate the singularities in $\partial^2 f/\partial\xi^2$ at $\xi = \pm\, \xi_t$ gives

$$\Gamma = \frac{-m_f}{2\pi e\, z_f\, \tau_s\, \Theta} \int_0^{2\pi} d\theta \int \frac{v^3}{B} \left[ v^{-2} \frac{\partial}{\partial v} \{(v_c^3 + v^2) f_o\} - \beta \frac{v_c^3}{v^3} \frac{(1-\xi^2)}{\xi} \frac{\partial f_o}{\partial\xi} + S\tau_s \right] \sigma\, \xi d\xi dv \qquad \ldots (43)$$

where the range of integration in $\xi$ is $[-1, -\xi_t \sin\theta/2]$ and $[\xi_t \sin\theta/2, 1]$. Since $f_o$ is symmetric for $|\xi| < \xi_t$ the contributions from the trapped particle regions cancel and the range of $\xi$ integration is reduced to $[-1, -\xi_t], [\xi_t, 1]$.

The first and third terms inside the integrand of Eq.(43) are now eliminated by use of Eq.(3) and then after integrating the new term by parts in $\xi$ and completing the $\theta$ integration Eq.(43) becomes

$$\Gamma = - \frac{m_f \; v_c^3}{e \; B_o \; z_f \; \tau_s \Theta} \int_o^\infty dv \left\{ \int \sigma(1-\xi^2) \left[ \frac{\partial}{\partial\xi} \left( \frac{1}{\langle \frac{v}{v_{\parallel}} \rangle} \right) \frac{\langle \frac{v_{\parallel}}{v} \rangle}{\xi} - 1 \right] \frac{\partial f_o}{\partial\xi} \, d\xi \right.$$

$$\left. + \left[ \frac{(1-\xi^2)}{\xi} \; \frac{\langle \frac{v_{\parallel}}{v} \rangle \; \sigma}{\langle \frac{v}{v_{\parallel}} \rangle} \; \frac{\partial f_o}{\partial\xi} \right]_{-\xi_t}^{\xi_t} \right\} \quad \ldots (44)$$

where the $\xi$ integration is $[-1, -\xi_t], [\xi_t, 1]$.

For the model operator Eq.(44) simplifies considerably giving

$$\Gamma = - \frac{2 \, \beta \, v_c^3 \; \xi_t}{\Omega_i \; \tau_s \; \Theta} \int_o^\infty dv \; \frac{\partial f_o}{\partial\xi_t} \quad \ldots (45)$$

where $\Omega_i = e \; B_o/m_f$ and $\frac{\partial f_o}{\partial\xi_t^+}$ is determined from Eq.(35). Now for co-injection it can be shown that $\frac{\partial f_o}{\partial\xi_t^+}$ is positive and for counter-injection negative, so for co-injection (counterinjection) the fast ions diffuse inwards (outwards). The magnitude of this diffusion is of order $\varepsilon^{\frac{1}{2}} \; S \; a_{i\theta}$ so typically in slowing down the fast ions diffuse radially one banana width.

For the actual operator by examining the integrand of Eq.(44) one can also show that the fast ions diffuse inwards for coinjection and outwards for counterinjection and that the order of magnitude of the diffusion is similarly $\Gamma \sim \varepsilon^{\frac{1}{2}} \; Sa_{i\theta}$.

As mentioned earlier the above analysis includes the change in radial position of a fast ion on becoming trapped, the effect discussed by Callen et al. [4]. This part of the fast ion radial transport can be obtained by evaluating the contribution to $\Gamma_f$ (using Eqs.(41) and (42)) from the transition region $[\xi_t^-, \xi_t^+]$. For the model operator the contribution to $\Gamma_f$ from the transition region alone is half of the total $\Gamma_f$ given in Eq.(45).

The radial diffusion of the fast ions results in a loss of their toroidal momentum and thereby changes the momentum balance from the uniform geometry case; this and the energy balance are evaluated in the next section.

## 5. ENERGY AND MOMENTUM TRANSFER TO THE BACKGROUND PLASMA

In this section the transfer rates of energy and momentum to the background plasma ions and electrons are obtained by evaluating the energy and momentum moments of the Fokker Planck Eq.(1). First the energy transfer rate: Multiplying Eq.(3) by $\frac{1}{2}mv^2$ and integrating over velocity space and $\theta$ gives,

$$-\frac{\Theta m_f}{2r}\int_0^{2\pi} d\theta \int \frac{\partial f}{\partial \theta} v^5\, dv\, \xi d\xi = \frac{m_f}{2\tau_s}\int_0^{2\pi} d\theta \int \Big[ v^{-2}\frac{\partial}{\partial v}\left\{(v_c^3 + v^3)f\right\} + \beta\frac{B_o}{B}\frac{v_c^3}{v^3}\frac{q}{\xi v}\frac{\partial}{\partial \xi}\left\{\frac{(1-\xi^2)q}{\xi v}\frac{\partial f}{\partial \xi}\right\} + S\,\tau_s\Big]\frac{v^5\, dv\, \xi d\xi}{q} \quad \ldots (46)$$

where the $\xi$ range of integration is $[-1, -\xi_t \sin\theta/2]$ and $[\xi_t \sin\theta/2, 1]$.

The term on the l.h.s. of the above equation vanishes after completing the $\theta$ integration. The first term on the r.h.s. is the rate of transfer of energy by friction to the background plasma, the second integral is zero, this can be shown by integrating over $\xi$ and using the boundary condition at the transition points; this is to be expected since scattering does not effect the energy transfer. The final term in Eq.(46) is the source of injected fast ions. Substituting expression (20) for $f_o$ into Eq.(46) and then completing the integrals over $\theta$ and $\xi$ gives the following expressions for the power transfer to the ions and electrons respectively :

$$P_i = \frac{m_f\, v_c^3\, S_o}{v_o}\int_0^1 \frac{u\, du}{u^3 + u_c^3} \;, \quad P_e = m_f\, v_o^2\, S_o \int_0^1 \frac{u^4\, du}{u^3 + u_c^3} \quad \ldots (47)$$

where $u_c = v_c/v_o$. Both these expressions are independent of $\varepsilon$, the inverse aspect ratio, and are identical to the transfer rates of the uniform geometry calculation [4]. This is to be expected since only the scattering operator is a function of the geometry and scattering does not give rise to energy transfer.

Similarly any $v$ dependent moment of Eq.(1) will be independent of $\varepsilon$ and so the $Q$ calculations of beam-plasma fusion systems [9] are unaffected by the effects of toroidal geometry considered here.

The momentum transfer however is dependent upon the geometry as is shown in the following. Multiplying Eq.(1) by $\frac{1}{2}\, m_f\, \sigma q$ and integrating over velocity space and $\theta$ gives

$$\int \frac{\Theta q}{rB} \frac{\partial f}{\partial \theta}\, \xi\, v^3\, dv d\xi d\theta = \int \Bigg[ v^{-2} \frac{\partial}{\partial v}\left\{ (v^3 + v_c^3) f \right\} + \beta \frac{B_o}{B} \frac{v_c^3}{v^3} \frac{q}{\xi v} \frac{\partial}{\partial \xi} \left\{ (1 - \xi^2) \frac{q}{\xi v} \frac{\partial f}{\partial \xi} \right\} + S\tau_s \Bigg] . \qquad \ldots (48)$$

After integrating by parts in $\theta$ the l.h.s. of the above equation may be written in terms of the diffusion flux $\Gamma$ by use of Eq.(40). The second term on the r.h.s. of Eq.(46) may be simplified by integrating by parts in $\xi$, and then the $\theta$ integration of the r.h.s. may be completed, to give after some re-arrangement

$$\Theta\, \Gamma\, \Omega_i + \frac{1}{\tau_s} \int v^3 dv \int \xi d\xi \Bigg[ v^{-2} \frac{\partial}{\partial v} \left\{ (v^3 + v_c^3) f_o \right\} - (1 - \xi^2) \frac{v_c^3}{v^3 \xi} \frac{\partial f_o}{\partial \xi} + S\, \tau_s \Bigg] = 0 \ . \qquad \ldots (49)$$

The first term of Eq.(49) is the loss of momentum by the diffusion of the fast ions, (inward for coinjection, outward for counterinjection), the second term is the loss of momentum through friction with the background electrons and ions, the third term the loss of momentum to the background ions through scattering and the fourth term is the input of momentum from the source. Thus using the result of Section 4 it can be shown that diffusion of the fast ions in radius accounts for a reduction in the transfer of momentum to the background plasma by a term of order $\varepsilon^{\frac{1}{2}}\, m_f v_o$.

## 6. CONCLUSION

To summarise, in Section 2 the bounce averaged Fokker-Planck equation was derived for fast ions slowing down in an axisymmetric toroidal plasma. The equation was solved by expanding in a series of eigenfunctions which were obtained numerically. Analytic eigenfunctions were then found for a model collision operator and shown to be a good approximation to the eigenfunctions of the exact operator.

Then in Section 4 the scattering of the fast ions as they slow down was shown to result in a small radial diffusion; this was inwards for coinjection (the direction of injection in the same direction of the tokamak current) and outwards for counterinjection. The diffusion resulted in a reduction of the fast ions toroidal momentum and consequently a smaller transfer of momentum to the background plasma.

The technique used in this paper of obtaining the solution in terms of a series of eigenfunctions which contain the toroidal geometry explicitly could also be used to advantage in other toroidal problems. For example, the calculation of Spitzer resistivity with trapped electrons, where the conductivity for large $\varepsilon$ for the exact collision operator may be calculated.

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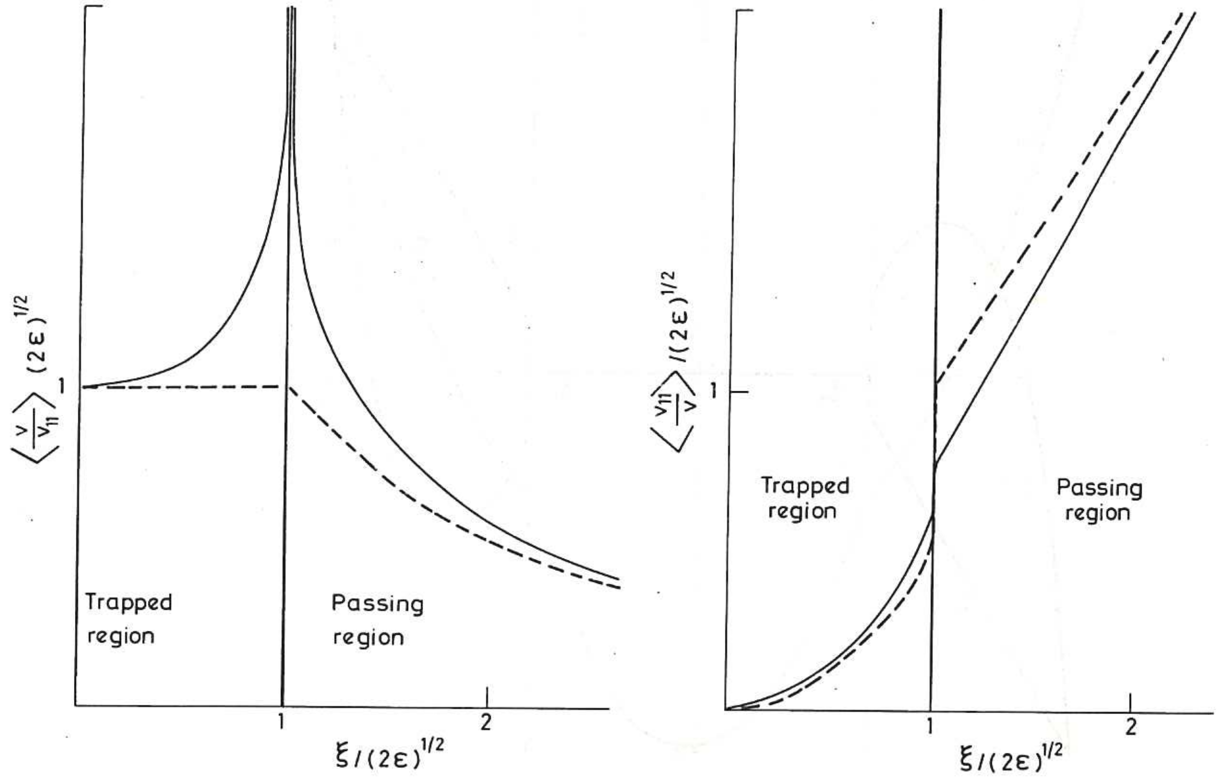


**Fig.1** The functions $<v/v_{||}>$ and $<v_{||}/v>$ as a function of $\xi$. The continuous curves are the expressions given by Eqs.(5) and (6), the dotted lines are the approximations used in the model operator and given by Eq.(21).

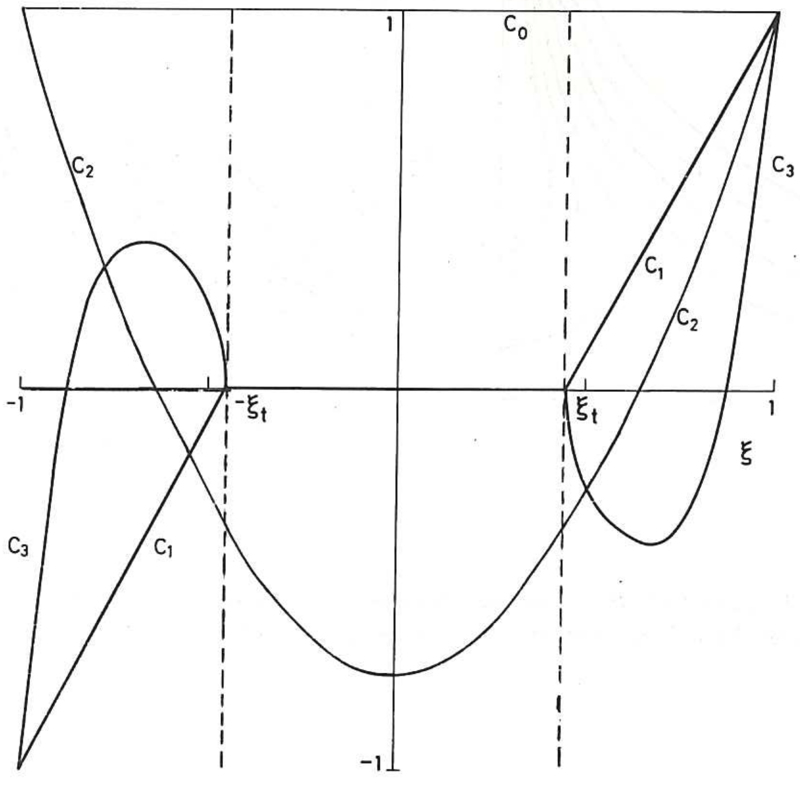


**Fig.2** The first four eigenfunctions versus $\xi$ for the exact operator (i.e. $<v_{||}/v>$ and $<v/v_{||}>$ given by Eqs.(5) and (6)).

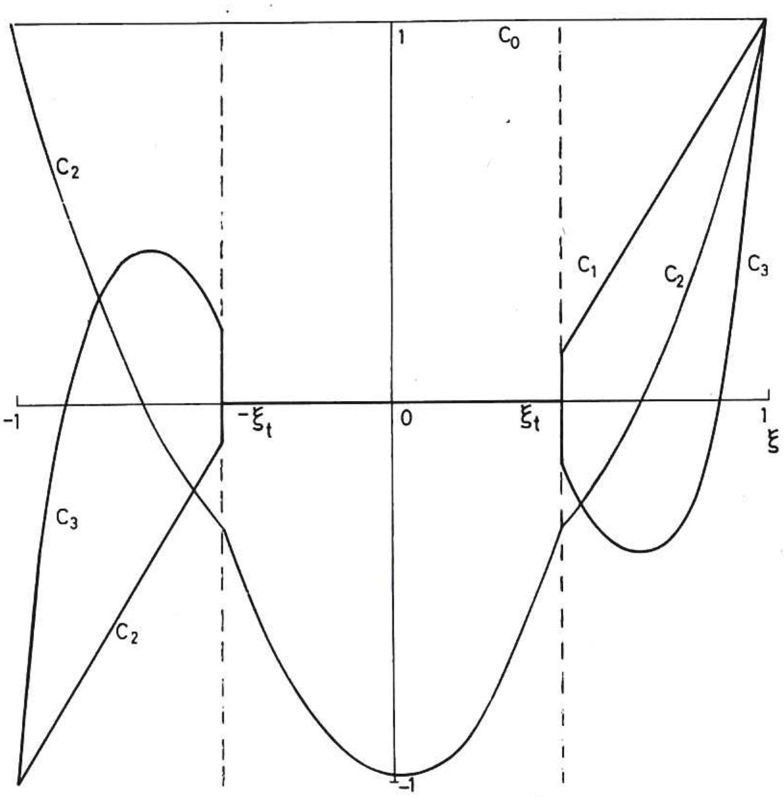


**Fig.3** The first four eigenfunctions versus $\xi$ for the model operator.

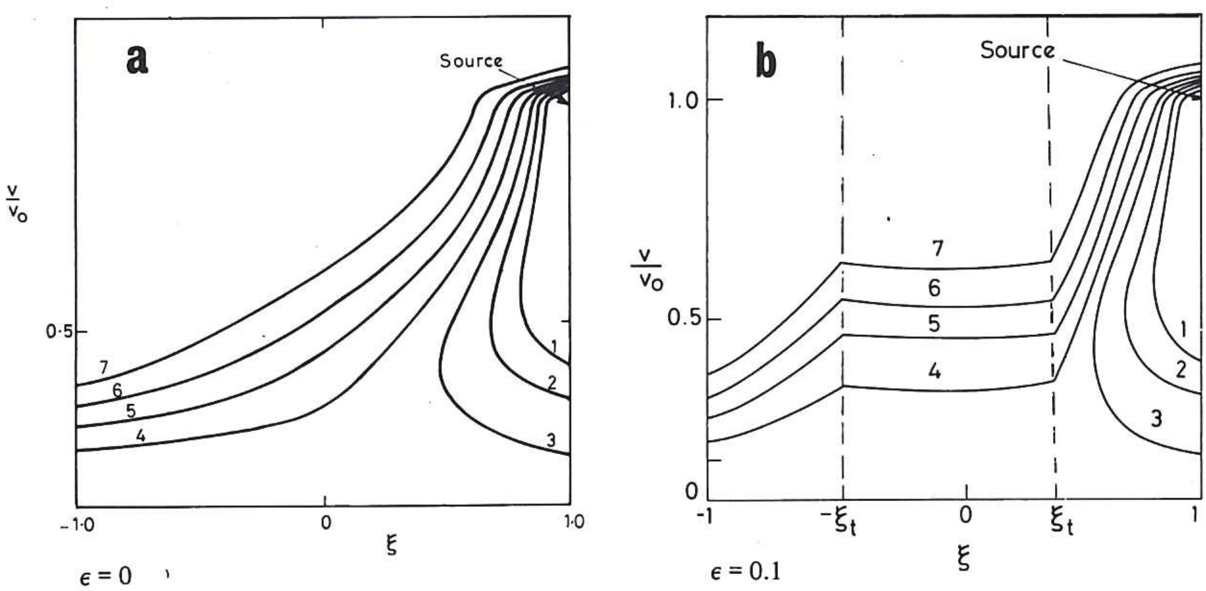


**Fig.4** Contours of the hot ion distributions in v, $\xi$ space (the contours decrease by 14% per contour, 1 is the highest). (a) $\epsilon = 0$; (b) $\epsilon = 0.1$.