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# ADIABATIC INVARIANTS FOR A TOROIDAL SLIGHTLY DISSIPATIVE PLASMA

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ABSTRACT

In a dissipative plasma the magnetic flux is not conserved. If, however, the dissipation is sufficiently slight, the helical flux in a slowly helically perturbed toroidal plasma is still conserved as an adiabatic invariant, up to first order in the inverse aspect ratio, while the toroidal and the poloidal fluxes are not.

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## 1. THE PHYSICAL MODEL

In the present paper we shall apply the theory of the adiabatic invariants to a toroidal plasma formed by slightly interacting particles. It will be shown that, if the interaction is sufficiently weak as to give rise only to a small resistivity, an adiabatic invariant exists up to first order in the inverse aspect ratio, which can be interpreted as a helical magnetic flux.

The motion of the system of particles with charge $e_j$ and mass $M_j$ is described by the Hamiltonian

$$H = \frac{1}{2} \Sigma_j \frac{1}{M_j} \left( \vec{p}(j) - \frac{e_j}{c} \vec{A} \right)^2 + \lambda U \tag{1}$$

where $\vec{A}$ is the vector potential related to the magnetic field $\vec{B}$ and $U$ is an electrostatic potential describing the coulomb interaction between the particles

$$U = \Sigma_{i\,j} \frac{e_i \, e_j}{|\vec{r}^{(j)} - \vec{r}^{(i)}|} \quad . \tag{2}$$

We assume that the temperature of the system is very high so that the average effect of the coulomb collisions is only a very small perturbation of the particle motion, characterized by the smallness parameter $\lambda$ . We consider an initial unperturbed magnetic configuration with axisymmetric toroidal geometry whose lines of force will be described using a curvilinear coordinate system in which they are straight lines:

$$\theta = \frac{1}{q(r)} \varphi + \text{const} \quad . \tag{3}$$

Here $\theta$ and $\varphi$ are the azimuthal and toroidal coordinates respectively and $q(r)$ is the safety factor on the magnetic surface $r = \text{const.}$ The unperturbed motion $(\lambda = 0)$ of the guiding centres of the circulating particles (neglecting the perpendicular drifts) is simply periodic on any rational magnetic surface.

Indeed in this case the trajectories of the guiding centres which are the lines of force, are exactly closed.

Neglecting the interaction between the particles, the Hamiltonian (1) is separable in the coordinates $\vec{r}^{(j)}$ of each particle. Under this circumstance, as known [1], an invariant $J^{(j)}$ exists for each particle, given by the expression

$$J^{(j)} = \sum_{1}^{3} {}_{s} \oint p_s^{(j)} \, dr_s^{(j)} \tag{4}$$

where the line integration is taken along a closed trajectory around the torus, including the Larmor gyration, which in a small Larmor radius expansion, contributes to $J^{(j)}$ only at first order. This first order contribution describes, as known [2] [3], the conservation of the magnetic moment $\mu$ of the particle. In the following we will neglect systematically the first order terms in the Larmor expansion and so we will be concerned only with the part of $J^{(j)}$ which is related to the rotation of the guiding centre along the closed line of force.

In the presence of particle interaction, the variables $\vec{r}^{(j)}$ are no longer separated in the Hamiltonian (1) and, as a consequence, the $J^{(j)}$ are no longer invariant. It is however a known result of the canonical perturbation theory that, if the interaction is sufficiently small (as is just our case), the $J^{(j)}$ are still invariant apart from terms of higher order of smallness than the order $\lambda$ of the interaction. The application of this result to the present case is discussed in the Appendix. We are then in a physical situation in which one can take into account all the effects of the coulomb interation at lowest order while the $J^{(j)}$ given by Eq (4) are still invariant. One of these effects is the existence of a small electrical resistivity which will be reflected in some kind of slow dissipation of the magnetic configuration. The resistivity is indeed proportional to the product $e_i e_j$ (which is of the same order as $\lambda$, compare Eq (2)) and is then of an order $O(\lambda)$ lower than that of the non-conserved terms of $J^{(j)}$, which are at least $O(\lambda^2)$ (see Appendix).

In the Hamiltonian (1) $\vec{A}$ describes the true magnetic field experienced by a single particle, including the part created by the other particles. The detailed form of this field is, of course, unknown, as the knowledge would imply the complete solution of the N-body problem. Nevertheless it is sufficient for our purposes to assume that the slow dissipation of the magnetic configuration, resulting, on the macroscopic scale, from the overall mutual interaction considered above, can be taken into account phenomenologically by considering that $\vec{A}$ depends explicitly on time through a suitable set of parameters $a(t)$, characterising the decaying magnetic configuration. This time dependence is supposed to be adiabatically slow, namely slow with respect to the period of rotation of a particle along an initially closed line of force. Under this circumstance the $J^{(j)}$ are still conserved as adiabatic invariants.

The $J$ (dropping henceforth the index $j$) represent useful constants of the motion when one is able to calculate the actual trajectory appearing in the line integral (4). Then, as will be seen, the $J$ can be related to the behaviour of the magnetic flux across suitably defined surfaces. For the calculation of the $J$ it is sufficient, as discussed above, to consider the Hamiltonian (1) at the order $\lambda = 0$, but considering the time dependence and the loss of axisymmetry of the dissipating magnetic field. Moreover, since we are only interested in the guiding centre motion, it is convenient to introduce a Hamiltonian such that the Larmor gyration is eliminated through a canonical transformation. This Hamiltonian will be introduced in the following section.

## 2. THE GUIDING CENTRE HAMILTONIAN

In our perturbed situation, as a consequence of the slow dissipation, the lines of force will gradually lose their individuality. Nevertheless an intermediate period of time will exist in which the particles which were originally following a closed line of force will continue to describe almost closed orbits around the torus, before the memory of their original equilibrium trajectory is completely lost. Indeed, if the <u>local</u> change of the magnetic field is sufficiently slow, the particle motion does not change essentially during a certain time. On the other hand, even a very small amount of dissipation is sufficient to produce immediately a change in the topology of the overall magnetic configuration which is not simply describable in terms of a displacement of the lines of force, frozen to the plasma. In such a situation the magnetic field cannot be characterised, even in the intermediate period considered above, by $\alpha,\beta$ coordinates which are constant on a line of force and related to $\vec{B}$ by $\vec{B} = \nabla\alpha \times \nabla\beta$ , as these coordinates do not exist, in general, as regular functions of time.

It is then not possible, even in a slightly dissipative system, to apply the canonical formalism for the guiding centre motion involving the $\alpha,\beta$ as canonical variables, as was developed by Northrop and Teller [3] and Taylor [4]. It is however still possible to describe the motion in a canonical formalism starting from the basic equation of motion for the guiding centre

$$M\ddot{\vec{x}} = \frac{e}{c}\,\dot{\vec{x}} \times \vec{B}(\vec{x},t) - \mu\,\nabla B(\vec{x},t) + eE(\vec{x},t) \qquad (5)$$

where $\vec{x}$ denotes the guiding centre position of a particle. The equation above can be derived from the rigorous Lorentz equation of motion in a number of ways [5] [6] [7], all essentially implying an average over the Larmor rotation. As is well known, the perpendicular component of the inertial term $M\ddot{\vec{x}}$ only contributes to the higher orders in the adiabatic iteration procedure [7].

The motion of the guiding centre is then described by the following Hamiltonian

$$\overline{H} = \frac{1}{2M}\,g^{ik}\left(p_i - \frac{e}{c}A_i\right)\left(p_k - \frac{e}{c}A_k\right) + \mu B \quad . \qquad (6)$$

Indeed the equation of motion (5) and the ordinary expression for the canonical momentum $p_i = M\dot{x}_i + eA_i/c$ are obtained from the canonical system:

$$\dot{p}_i = -\left(\frac{\partial \overline{H}}{\partial x^i}\right)_{p_i} , \quad \dot{x}_i = \left(\frac{\partial \overline{H}}{\partial p^i}\right)_{x^i} . \tag{7}$$

In Eq (6) $g^{ik}$ is the metric tensor appropriate to our curvilinear coordinates $x^i \equiv (r,\theta,\varphi)$ ; the $A_i$ are the covariant components of the vector potential. We suppose that $\overline{H}$ can be split into a zero order part $H_o(\theta)$ describing the guiding centre motion in the toroidal axisymmetric equilibrium, in the absence of dissipation, and a perturbed part $\eta H_1(\theta, n\varphi - m\theta, a(t))$ of order $\eta$ representing the dissipative effects which produce, at the same time, a slow time dependence of the magnetic configuration through the adiabatic parameters $a(t)$ and a helical magnetic deformation described by the variable $n\varphi - m\theta$ :

$$\overline{H} = H_o(\theta) + \eta\, H_1(\theta, n\varphi - m\theta, a(t)) . \tag{8}$$

The $\theta$ dependence in $\overline{H}$ is a consequence of the toroidal geometry only and is then of order $\epsilon \equiv r/R$ , where $R$ is the major radius. $\overline{H}$ is a periodic function of $\theta$ and $n\varphi - m\theta$ . It must be noted that in order to be consistent with the approximations implied by our physical model the order $\eta$ of the dissipative helical perturbation must be much smaller than the order $\epsilon$ characterising the toroidicity of the unperturbed equilibrium:

$$\eta \ll \epsilon < 1 .$$

We find it convenient to separate in the Hamiltonian the dependence of order $\epsilon$ and the dependence of order $\eta$ by introducing new angle variables, instead of the $\theta, \varphi$ , through the following transformation

$$P = n\,p_2 + m\,p_3\,, \qquad W_o = \frac{\theta}{n}$$
$$p = m\,q\,p_3\,, \qquad w_o = \frac{n\varphi - m\theta}{qmn} \tag{9}$$

where m,n are integers different from zero.

Starting from the canonical system (7) (where $x^i \equiv (r,\theta,\varphi)$ ) and expressing the Hamiltonian in terms of the new variables, one obtains

$$\dot{P} = -\frac{\partial\overline{H}}{\partial W_o}\,, \qquad \dot{W}_o = \frac{\partial\overline{H}}{\partial P}$$
$$\dot{p} = -\frac{\partial\overline{H}}{\partial w_o}\,, \qquad \dot{w}_o = \frac{\partial\overline{H}}{\partial p} \tag{10}$$

showing that the transformation (9) is canonical.

In the new variables the Hamiltonian $\overline{H}$ and the invariant J take the form

$$\overline{H} = H_o(P,W_o) + \eta H_1(P,p,W_o,w_o,a(t))$$
$$J = \oint P dW_o + \oint p dw_o \quad . \tag{11}$$

The $W_o$ dependence in $\overline{H}$ is of order $\epsilon$ while the $w_o$ dependence is of order $\eta$ . In the expression for J the effect of the perpendicular drifts on the guiding centre orbit was neglected by putting $dx^1 \equiv dr = 0$ .

## 3. THE HELICAL FLUX

The contribution of the inertial term $M\vec{v}$ of the canonical momentum to the time derivative of J is given by the expression

$$\frac{d}{dt} M \oint v\, ds = \frac{M}{2} \oint \frac{d}{dt}(v^2)\, \frac{ds}{v} \tag{12}$$

where the deformation of the orbit during one cycle was neglected, so that d/dt commutes with the line integral for J . Eq (12) is the change of the kinetic energy during one cycle and will be assumed as negligible in the following. The only contribution to $\frac{dJ}{dt}$ then comes from the $A_i$ . In the absence of the helical perturbation $\eta H_1$ one has $w_o$ = const along the unperturbed trajectory represented by the line of force with $q = m/n$. In this case one obtains $J_{w_o} \equiv \oint p \, dw_o = 0$ and $J$ reduces to the first part only

$$J = J_{W_o} \equiv \oint P \, dW_o = \frac{e}{c} \oint (n A_2 + m A_3) \, dW_o = \frac{e}{cn} \int_0^{2\pi n} (n A_2 + m A_3) \, d\theta$$

$$= \frac{e}{c} \oint_{w_o = \text{const}} A_i \, dx^i \equiv - \frac{e}{c} \chi \tag{13}$$

where $-\chi$ is the magnetic flux across a helical ribbon enclosed by the axis located at $r = 0$ and a closed curve defined by $w_o$ = const , $r$ = const .

In the presence of the dissipative magnetic perturbation, $J_{W_o}$ is no longer conserved and the same holds for the helical flux $-\chi$ . In order to calculate the order of magnitude of the time dependent terms of $J_{W_o}$ we shall use standard results of the canonical perturbation theory (see e.g. Ref [8]).

Introducing the Hamilton-Jacobi function $S(W_o \, w_o \, J_W \, J_w)$ generating a transformation from the $(W_o \, w_o \, J_{W_o} \, J_{w_o})$ variables associated with $H_o$ , to the new $(W w \, J_W \, J_w)$ variables associated with $H_o + \eta H_1$ and expanding

$$S = S_o + \eta S_1 + \eta^2 S_2 + \ldots \tag{14}$$

one can calculate the perturbed $J_{W_o}$ in the form (compare e.g. Ref [8])

$$J_{W_o}(W,w) = J_W + \left[ \eta \frac{\partial S_1}{\partial W_o} + \eta^2 \left( \frac{\partial S_2}{\partial W_o} - \frac{\partial^2 S_1}{\partial W_o^2} \frac{\partial S_1}{\partial J_W} \right) \right]_{W_o = W, w_o = w} \tag{15}$$

where $J_W$ is invariant.

Since $H_1$ is periodic in $W_0$, $w_0$, it can be expanded in Fourier series

$$H_1 = \sum_{n_1 n_2} H^{(1)}_{n_1 n_2} (J_{W_o}, J_{w_o}) \exp i(n_1 W_o + n_2 w_o) \tag{16}$$

where we have neglected, for the moment, the time dependence described by the adiabatic parameters.

Expressing $S_1$ also in Fourier series

$$S_1 = \sum_{n_1 n_2} S^{(1)}_{n_1 n_2} (J_W, J_w) \exp i(n_1 W_o + n_2 w_o) \tag{17}$$

one finds in the canonical perturbation theory [8] that

$$S^{(1)}_{n_1 n_2} = - \frac{H^{(1)}_{n_1 n_2}}{2\pi i(n_1 \nu_1 + n_2 \nu_2)} \tag{18}$$

where $n_1 \neq 0$ , $n_2 \neq 0$ and

$$\nu_1 = \frac{\partial H_o(J_{W_o})}{\partial J_{W_o}} , \nu_2 = \frac{\partial H_o(J_{W_o})}{\partial J_{w_o}} = 0 \quad . \tag{19}$$

Using these results one can now discuss the order of magnitude of the terms in the r.h.s. of the Eq (15) for $J_{W_o}(W,w)$. The terms $\partial S_1/\partial W_o$ and $\partial S_2/\partial W_o$ do not give any contribution after averaging over one cycle. The only contribution then results from the last term, quadratic in $S_1$ . Now the order of magnitude of $S_1$ is the same as the order of magnitude of $H^{(1)}_{n_1 n_2}$ for $n_1 \neq 0$ , $n_2 \neq 0$ . But the terms with $n_1 \neq 0$ describe just the $W_o$ dependence of $H_1$ , which is of order $\epsilon$ in the toroidicity. The time dependence of $J_{W_o}(W,w)$ is then $O(\epsilon^2 \eta^2)$ . It follows that the helical flux $-\chi = \frac{c}{e} J_{W_o}(W,w)$ is conserved up to the first order in $\epsilon$ .

## 4. FINAL REMARKS

As is known from the theory of the adiabatic invariants [8], any action variable $J$ which is invariant for a time-independent Hamiltonian, remains an adiabatic invariant when the parameters $a(t)$ of the Hamiltonian change slowly. We can then state, from the result of the foregoing section, that the helical flux is adiabatically conserved with respect to the dissipative helical perturbation $\eta H_1(a(t))$, apart from terms of order $\eta^2 \epsilon^2$. The same conclusion cannot be drawn for the toroidal flux $\Phi$ and the poloidal flux $\Psi$, for the reason that the invariants from which their conservation could be derived, cease to exist already in the order $\eta$. Indeed $\Phi$ and $\Psi$ are defined by the relations

$$\Phi \equiv \int_0^{2\pi} A_2 \, d\theta \quad , \quad \Psi \equiv -\int_0^{2\pi} A_3 \, d\varphi \quad . \tag{20}$$

In the absence of the dissipative perturbation $H_1$, the magnetic configuration is axisymmetric and the variables $\theta$ and $\varphi$ can be separated in the canonical formalism. The invariance of $\Phi$ and $\Psi$ then follows (provided that the inertial terms can be neglected) from the existence of the following two invariants for the unperturbed motion of the circulating particles

$$J_{02} = \int_0^{2\pi} p_2 \, d\theta_o \ , \ J_{03} = \int_0^{2\pi} p_3 \, d\varphi_o \tag{21}$$

$J_{o3} = 2\pi \, p_3$ being even an exact invariant. When the dissipative helical perturbation $H_1$ exists, $J_{02}$ and $J_{03}$ are no longer constants. Their time behaviour cannot be calculated with the perturbation formalism of the foregoing section because the case is degenerate (see Ref [8]). Indeed the unperturbed motion occurs along a closed line of force, so that one has the relation $m \dot{\theta}_o - n \dot{\varphi}_o = 0$. Since the angle variables are linear functions of time with frequencies $\omega_1$ and $\omega_2$ respectively, one has the relation

$$m \, \omega_1 - n \, \omega_2 = 0$$

which is characteristic of the degeneracy. However the time behaviour of $J_{02}, J_{03}$ can be obtained directly by solving the canonical system:

$$\dot{J}_{02} = - \eta \frac{\partial H_1}{\partial \theta_o}$$
$$\dot{J}_{03} = - \eta \frac{\partial H_1}{\partial \varphi_o} \quad . \tag{22}$$

We once again expand $H_1$ in a Fourier series with respect to $\theta_o$ and $m\,\theta_o - n\,\varphi_o$ :

$$H_1 = \sum_{n_1 n_2} H^{(1)}_{n_1 n_2}(J_{01}) \exp(i\, n_1\, \theta_o) \exp[i\, n_2(m\,\theta_o - n\varphi_o)] \quad . \tag{23}$$

We write $H_1$ in the form

$$H_1 = F(m\,\theta_o - n\,\varphi_o) + G(\theta_o\, , m\,\theta_o - n\,\varphi_o) \tag{24}$$

where

$$\begin{aligned} F &= \Sigma_{n_2}\, H^{(1)}_{on_2}(J_{oi}) \exp[i\, n_2(m\,\theta_o - n\,\varphi_o)] \\ G &= \sum_{n_1 \neq 0, n_2} H^{(1)}_{n_1 n_2}(J_{oi}) \exp(i\, n_1 \theta_o) \exp[i n_2(m\,\theta_o - n\,\varphi_o)] \quad . \end{aligned} \tag{25}$$

While integrating the canonical system, one substitutes for $\theta_o(t)$ and $\varphi_o(t)$ the expressions of the unperturbed motion, which imply $m\,\theta_o(t) - n\,\varphi_o(t) = \text{const}$ . One then obtains the following result for $J_{02}(t)$ :

$$J_{02}(t) = J_{02}(o) - \eta \frac{\partial F}{\partial \theta_o} t - \eta \sum_{n_1 \neq 0, n_2} \frac{H^{(1)}_{n_1 n_2}(n_1 + n_2 m)}{n_1 \omega_1} \exp(i\, n_1 \theta_0) \exp[i\, n_2(m\theta_o - n\varphi_o)] \, .$$

We see that the contribution of order $\eta$ coming from $F$ is a secular term increasing linearly with time. A comparable result can be obtained for $J_{03}(t)$ .

## APPENDIX

In our physical model the basic Hamiltonian (1) describes, at zero order in $\lambda$ , the particle motion in the magnetic configuration including the collective magnetic effects resulting from the particle interaction, while the collisional coulomb interaction is considered at the order $\lambda$ . As we know from section 3, the motion with $\lambda = 0$ can be described in the lowest order of the small Larmor radius expansion introducing angle and action variables $(W_j\, w_j\, J_{W_j}\, J_{w_j})$ for each particle labelled by $j$ . In the presence of the coulomb interaction, the particle coordinates can no longer be separated. However the action variables remain constant up to the order $\lambda$ . For instance, applying the canonical perturbation theory to the Hamiltonian (1) and transforming to new angle variables $\tilde{W}_j , \tilde{w}_j$ one has that at first order in $\lambda$ , $J_{W_j}$ is modified as follows (compare Eq (15) ) :

$$J_{W_j}\,(\tilde{W}_j,\tilde{w}_j) = \tilde{J}_{W_j} + \lambda \left( \frac{\partial S_1}{\partial W_j} \right)_{W_j = \tilde{W}_j\,,\; w_j = \tilde{w}_j} \qquad \text{(A1)}$$

where $\tilde{J}_{w_j}$ is invariant. The first order part $S_1$ of the Hamilton-Jacobi function can be expanded, as usual, in a Fourier series:

$$S_1 = \sum_{1j}^{N} \sum_{n_{1j}\,n_{2j}} S_1(\tilde{J}_{W_j},\tilde{J}_{w_j}) \exp i(n_{1j}\, W_j + n_{2j}\, w_j) \qquad \text{(A2)}$$

where $N$ is the number of particles in the system.

It is then seen that the average of $\partial S_1/\partial W_j$ over one period of the uncorrelated motion of the particles is equal to zero, so that $J_{W_j}$ is invariant in the average up to the first order in $\lambda$ .

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