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COULOMB LOGARITHMS FOR D.C. AND R.F. CONDUCTIVITY OF PLASMAS

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by

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A B S T R A C T

This note gives a brief discussion of the dependence of the Coulomb logarithm ($\ell n \Lambda$) and Gaunt factor on electron temperature, density, and wave frequency in the Rayleigh–Jeans limit. The physical bases for the cut-offs appropriate in various limiting cases are clarified, and the independence of quantum and shielding corrections is pointed out. Formulas with refined numerical coefficients are displayed for cases of experimental interest. In particular, a numerical correction is deduced to Spitzer's $\ell n \Lambda$, applicable to the D.C. conductivity. The results permit improved precision in the calculation of conductivity, collision frequency, and bremsstrahlung emission over a wide range of plasma parameters.

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It is well known that calculation of the electrical conductivity of a highly ionized plasma gives rise to a logarithmically divergent factor of the form (SPITZER, 1962)

$$\ell n \, (b_{max}/b_{min}),$$

where $b_{max}/b_{min} \equiv \Lambda$ is the ratio of maximum to minimum impact parameter for the electron-ion "collision" process. A similar factor arises in the closely related theory of radio-frequency bremsstrahlung emission by a plasma (BRUSSARD and VAN DE HULST, 1962). The ratio Λ is then prevented from diverging by the invocation of physical arguments, usually in the form of an ad hoc "cut-off", which in turn is usually uncertain by a numerical factor of order unity. The purpose of this Note is to clarify the physical parameters involved and display refined numerical coefficients for important limiting cases of experimental interest.

Among the quantities containing the logarithmic term in question are: SPITZER'S (1962) D.C. conductivity:-

$$\sigma_{dc} = \frac{4\sqrt{2}}{\pi^{3/2}} \frac{(4\pi\varepsilon_0)^2}{e^2 \, m^{1/2}} \frac{(kT)^{3/2} \, \gamma_E}{Z \, \ell n \, \Lambda} ; \qquad (1)$$

"high-frequency" ($\omega \gg \omega_p$) effective electron-ion collision frequency for momentum transfer (SHKAROFSKY, 1961):-

$$\langle \nu_{ei} \rangle = \frac{4(2\pi)^{1/2}}{3} \left(\frac{e^2}{4\pi\varepsilon_0} \right)^2 \frac{Z \, n \, \ell n \, \Lambda}{m^{1/2} \, (kT)^{3/2}} ; \qquad (2)$$

and bremsstrahlung emission power density (energy-time^{-1}-volume^{-1}- radian frequency interval^{-1}) (GREENE, 1959):-

$$\rho_\omega = \frac{16}{3} \left(\frac{2\pi}{3} \right)^{1/2} \left(\frac{e^2}{4\pi\varepsilon_0} \right)^3 \frac{Z \, n^2 \, \mu \, G}{(mc^2)^{3/2} \, (kT)^{1/2}} \, \exp \left(-\frac{\hbar\omega}{kT} \right). \qquad (3)$$

The connection between (1) and (2) has been neatly systematized by SHKAROFSKY (1961). The relation between the conductivity point of view, which yields (1) and (2), and the bremsstrahlung point of view, yielding (3), has been discussed by OSTER (1961), THEIMER (1963), and others; in particular, the quantity G,

traditionally known as the <u>Gaunt factor</u>, is found to be related to the logarithmic term by the equivalence:-

$$G = \frac{\sqrt{3}}{\pi} \ln \Lambda. \tag{4}$$

The term γ_E in (1) is a correction factor to account for electron-electron collisions. In (3), the refractive index μ and the exponential term are usually written explicitly, although some authors incorporate them in the Gaunt factor.

In the theoretical treatment of these processes it is necessary as a first step to consider electrons of a given velocity; then, one may suitably average over the velocity distribution. For brevity, we here consider only the final, velocity-averaged quantities, assuming specifically a Maxwellian distribution. Our concern is, then, with the dependence of Λ (or G) upon wave frequency ω, electron temperature T, electron density n, and ionic charge number Z.

The problem may be specified in terms of three dimensionless parameters:-

$$P_1 = \frac{\hbar\omega}{kT} \tag{5}$$

$$P_2 = \frac{kT}{Z^2 R_y} = \frac{2}{m} \left(\frac{4\pi\varepsilon_0 \hbar}{e^2} \right)^2 \frac{kT}{Z^2} \tag{6}$$

$$P_3 = \frac{\omega_p^2}{\omega^2} = \frac{n}{n_c} = \frac{e^2}{\varepsilon_0 m} \frac{n}{\omega^2} \tag{7}$$

Here R_y = 13.6 eV is the Rydberg energy (ionisation potential of hydrogen), ω_p is the usual plasma frequency, and n_c is the cut-off density for the wave frequency ω. The first two parameters measure the importance of quantum effects in the radiation and collision processes, respectively; the third measures collective processes, such as the Debye shielding of heavy-ion scatterers by the plasma electrons. In the general case, the effective Λ, to be used in relations such as (1)-(3), depends in a complicated way upon the various parameters, and must be found by numerical computation (BRUSSARD and VAN DE HULST, 1962; GREENE, 1959). However, the <u>Rayleigh-Jeans limit</u>, $P_1 = \hbar\omega/kT \ll 1$, is well satisfied

for most laboratory plasmas throughout the radio-frequency and infra-red domains, and will be assumed in the remainder of this Note. In this limit, we wish to point out that the roles of the other two parameters become independent of each other, a considerable simplification (DE WITT, 1958). In fact, recalling the fundamental definition of Λ as the ratio of impact parameters, b_{max}/b_{min}, we deduce from the following physical arguments that, in effect, b_{min} is a function of P_2 alone, and b_{max} of P_3 alone.

The impact parameter b_{min} emerges naturally from an impact point of view as a close impact radius, which for classical mechanics is the impact parameter for a 90° deflection, $b_0 = \langle Ze^2/4\pi\varepsilon_0 mv^2 \rangle \approx Ze^2/4\pi\varepsilon_0 kT$, with numerical uncertainties arising from details of the velocity averaging (SPITZER, 1962). For quantum mechanics, however, the angular deflection may be regarded as the spreading of the electron wave passing through an aperture of radius b (MARSHAK, 1940). Thus for "90° spreading", the close impact radius b_{min} is of the order of the deBroglie wavelength $\langle \hbar/mv \rangle \approx \hbar/(mkT)^{1/2}$. The ratio of these alternative evaluations is essentially $P_2^{1/2}$. As is well known, the classical description is dominant at low temperatures, the quantum at high. The transition region is discussed in more detail below.

Meanwhile, for a very dilute plasma such that $P_3 = \omega_p^2/\omega^2 \ll 1$, collective electron interactions are negligible and the appropriate cut-off, b_{max}, is the impact parameter for which the wave field goes through a complete period during the "duration" of the collision, $\sim b/v$; that is, $b_{max} \rightarrow \langle v/\omega \rangle \approx (kT/m)^{1/2}/\omega$. For larger impact parameters the collision is rendered ineffective by the reversal of the wave field (SILIN, 1960; YOSHIKAWA, 1961; CHANG, 1962). On the other hand, for a relatively dense plasma ($\omega_p^2 \gg \omega^2$), the field of a scattering ion is shielded by the surrounding cloud of plasma electrons beyond a distance of the order of the Debye length $\lambda_D = (kT/m)^{1/2}\omega_p$, which is then the appropriate cut-off (SPITZER, 1962). The ratio of these alternatives is simply $P_3^{1/2}$.

Now to put these physical arguments on a more quantitative basis, we review present knowledge of the limiting cases and the respective transition regions. A well known special case arises in bremsstrahlung emission in the classical ($P_2 \ll 1$), unshielded ($P_3 \ll 1$) limit; that is, the variables of the problem are ordered $\hbar\omega_p \ll \hbar\omega \ll kT \ll Z^2 R_y$. This completely classical analysis, without many-electron complications, can be dealt with precisely (KRAMERS, 1923). OSTER (1961) has reviewed this limit in detail, pointing out that no ad hoc cut-off arises and giving careful attention to the velocity averaging. The result for this case (and $P_2^{\frac{1}{2}}/P_1 \gg 1$) is:-

$$\Lambda \rightarrow \frac{2}{\gamma}^{5/2} \frac{4\pi\varepsilon_0}{e^2 m^{\frac{1}{2}}} \frac{(kT)^{3/2}}{Z\omega} = \frac{4}{\gamma^{5/2}} \frac{P_2^{\frac{1}{2}}}{P_1} , \qquad (8)$$

where $\gamma = 1.781 = \exp(0.577)$ is Euler's constant and the numerical coefficient is presumed exact.

Another limiting case is the quantum mechanical ($P_2 \gg 1$) analog of the preceding, which may for our purposes be found using the Born approximation (SAUTER, 1933). This limit has also been systematically reviewed by OSTER (1961) with the result:-

$$\Lambda \rightarrow \frac{4}{\gamma} \frac{kT}{\hbar\omega} = \frac{4}{\gamma} \frac{1}{P_1} , \qquad (9)$$

where again the numerical coefficient is presumed exact. The cross-over, where (8) and (9) are equal, occurs at $P_2 = \gamma^3$, that is:-

$$kT = \gamma^3 Z^2 R_y = (77eV)Z^2 = (890,000°K)Z^2. \qquad (10)$$

The transition, as a function of P_2, between the classical limit (8) and the quantum limit (9) has been evaluated numerically by GREENE (1959) and OSTER (1963). The exact value of Λ changes smoothly between the two asymptotes, and at the cross-over is approximately two-thirds ($\ell n \frac{2}{3} \approx -0.4$) of the value given by either asymptote. To good approximation then one may use the asymptotic form only, following the well-known rule stated by SPITZER (1962) but with the revised cross-over temperature (10).

In their pioneering work on D.C. conductivity, Spitzer and colleagues were concerned with the classical ($P_2 \ll 1$), d.c. ($P_3 \gg 1$) limit. In addition to pointing out the significance of the Debye length, rather than the mean inter-ionic spacing $n^{-\frac{1}{3}}$, as the proper cut-off, they defined the well known ratio:-

$$\Lambda_{Sp} \equiv \frac{\lambda_D}{b_0} = \frac{3}{2\pi^{\frac{1}{2}}} \left(\frac{4\pi\varepsilon_0}{e^2} \right)^{3/2} \frac{(kT)^{3/2}}{Z\, n^{\frac{1}{2}}} = \frac{3}{\sqrt{2}} \frac{1}{P_1} \left(\frac{P_2}{P_3} \right)^{\frac{1}{2}} , \tag{11}$$

which has since been widely used in the interpretation of plasma experiments. The number of electrons in a "Debye sphere" is $\frac{4}{3}\pi n\, \lambda_D^3 = \frac{1}{9} Z\, \Lambda_{Sp}$, which must be large if the statistical arguments used in the theory are to be valid. Because Λ_{Sp} is large and occurs inside a logarithm, no great care was used in performing the velocity averaging; accordingly it is to be expected that (11) is in error by a numerical factor of order unity. In passing, we note that Spitzer's Λ_{Sp} may be used to rewrite (2) in the form:-

$$\frac{\langle \nu_{ei} \rangle}{\omega_p} = \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \frac{\ell n\, \Lambda}{\Lambda_{Sp}} \tag{12}$$

which emphasizes the fact that $\langle \nu_{ei} \rangle \ll \omega_p$ in most plasmas in which electron-ion collisions are the dominant relaxation process.

Spitzer's limit ($P_2 \ll 1 \ll P_3$) may be approached from two different points of view; an impact theory (SPITZER, 1962) in which b_{min} is well defined while b_{max} requires an <u>ad hoc</u> cut-off, and a wave theory (DAWSON and OBERMAN, 1962) in which b_{max} is well defined but b_{min} requires the cut-off. KIHARA and AONA (1963) have recently shown how to merge the two complementary theories to obtain a precise numerical coefficient. Their result, properly averaged over a Maxwellian distribution, is:-

$$\Lambda \to \frac{4}{3\gamma^2 e^{\frac{1}{2}}} \Lambda_{Sp} = 0.25\, \Lambda_{Sp} . \tag{13a}$$

A further refinement to this limiting case arises from ion correlations. The calculations of DAWSON and OBERMAN (1963) and ITIKAWA (1963) yield, for the Spitzer limit including equilibrium ion correlations:-

$$\Lambda \to \frac{4}{3\gamma^2} \frac{1}{(1+Z)^{\frac{1}{2}}(1+1/Z)} \Lambda_{Sp}. \tag{13b}$$

For instance, for hydrogen and helium:-

$$\Lambda \rightarrow \begin{cases} \dfrac{2}{3\gamma^2} \; \Lambda_{Sp} = 0.21 \; \Lambda_{Sp} & (Z = 1) \qquad\qquad (13c) \\[2em] \dfrac{4}{3^{7/4}\gamma^2} \; \Lambda_{Sp} = 0.18 \; \Lambda_{Sp}. & (Z = 2) \qquad\qquad (13d) \end{cases}$$

Easily remembered approximations to (13c)-(13d) are to replace Λ_{Sp} by $\frac{1}{5} \Lambda_{Sp}$ or to subtract 1.6 from $\ell n \Lambda_{Sp}$. To the author's knowledge, (13a)-(13d) have not previously been given explicitly and are not generally known by experimentalists. Although the numerical correction is small (typically ~ 15% in $\ell n\Lambda$), it is not negligible in many plasma experiments. Indeed, but perhaps coincidentally, a number of workers have recently reported experimental results in which the electron temperature inferred from the d.c. conductivity, using (1), has been somewhat higher than that inferred from spectroscopic measurements, as would result from using (11) in place of (13b) (RYNN, 1964; HIRSCHBERG, 1964; BUTT et al., 1964).

The transition between the unshielded and the shielded limits (i.e., variation with P_3) requires careful consideration of the rather subtle, collective aspects of the electron dynamics (OBERMAN, RON and DAWSON, 1962; RON and TZOAR, 1963). Numerical results, calculated from different points of view, have been given by DAWSON and OBERMAN (1962) and by OSTER (1964) in the classical limit neglecting ion correlations. These calculations did not provide an exact numerical coefficient for Λ but may be normalized by either respective asymptotic value, (8) or (13a), the same coefficient being obtained in either case. The numerical curve departs insignificantly from the asymptotic forms except just above the plasma frequency, where DAWSON and OBERMAN (1962) find a slight "bump" due to coupling to longitudinal plasma oscillations. To very good approximation, one may use the asymptotic forms, only, on the respective sides of the cross-over where (8) and the appropriate version of (13a)-(13d) are equal, namely:-

$$P_3 = \frac{\omega_p^2}{\omega^2} = \begin{cases} \dfrac{\gamma}{2e} = 0.33 & \text{(14a)} \\[2em] \dfrac{\gamma}{2(1+Z)^{1+1/Z}} & \text{(14b)} \\[2em] \dfrac{\gamma}{8} = 0.22 \quad (Z = 1) & \text{(14c)} \\[2em] \dfrac{\gamma}{6\sqrt{3}} = 0.17 \quad (Z = 2) & \text{(14d)} \end{cases}$$

We have so far discussed three of the four possible limiting cases, yielding (8), (9) and (13). The remaining quantum ($P_2 \gg 1$), shielded ($P_3 \gg 1$) case is implicit in the work of RON and TZOAR (1963) and has been mentioned in a brief paper by KONSTANTINOV and PEREL' (1962). It may be found by applying the P_2-variation ratio of (9) to (8) to the $P_3 \gg 1$ limit of (13b), with the result, not previously published:-

$$\Lambda \rightarrow \left[\frac{8}{\gamma(1+Z)^{1+1/Z}} \right]^{\frac{1}{2}} \frac{kT}{\hbar\omega_p} = \left[\frac{8}{\gamma(1+Z)^{1+1/Z}} \right]^{\frac{1}{2}} \frac{1}{P_1 P_3^{\frac{1}{2}}} \quad . \tag{15}$$

Our physical picture implying the independence of $b_{min}(P_2)$ and $b_{max}(P_3)$ is entirely consistent with detailed calculations which show that, in the limit $P_3 \ll 1$ (GREENE, 1959; OSTER, 1963), $P_1\Lambda(P_2)$ is independent of frequency and electron density and that, in the limit $P_2 \ll 1$ (DAWSON and OBERMAN, 1962; OSTER, 1964), $P_2^{-\frac{1}{2}} \Lambda(P_3)$ is independent of electron temperature. Although numerical interpolation data are available from the cited calculations, for most practical purposes it suffices to use for $\Lambda(P_2,P_3)$ the asymptotic forms $[(8), (9), (13),$ or $(15)]$ appropriate to the four domains defined by the cross-over conditions (10) and (14).

Electron-electron collisions become important for low frequencies, below the collision frequency (2) (SPITZER, 1962; SHKAROFSKY, 1961). It is customary to express this correction to the low-frequency conductivity (1) by the factor $\gamma_E(Z)$.

A method for computing "exact" coefficients in this context has been given by ITIKAWA (1963).

We have dealt here only with the case of an isotropic plasma. The anisotropy introduced by a static magnetic field severely complicates the situation, making it more difficult to ascribe simple physical meaning to the various collision terms (ELEONSKII, ZYRANOV and SILIN, 1962; OBERMAN and RON, 1963; OBERMAN and SHURE, 1963). The magnetic field is important when the gyroradius is shorter than the Debye length, i.e., when the cyclotron frequency, $\omega_b = eB/m$, is comparable to or above the plasma frequency ω_p.

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