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# THE STABILITY OF ELECTROMAGNETIC LEVITATION SYSTEMS FOR SOLID BODIES

by

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## Abstract

The stability of electromagnetic systems for levitating solid bodies is examined, with the aim of clarifying the principles governing the design of high speed transport electromagnetic suspension systems. Earnshaw's theorem is discussed, and its extension by Braunbek proved using energy theorems. Braunbek's result is that a stable system must incorporate material of less then free space permeability, which in practice would be superconductors, magneplanes or conducting material with A.C. eddy currents. Systems containing only material of greater permeability (e.g. iron) must be unstable. This leads to the analysis of the stability of mixed systems containing both types of material, particularly of those consisting of passive lumps of iron supported stably in a magnetic field provided by superconducting (fixed flux) surfaces. A theoretical method for assessing such systems, based on an extension of the $\nabla B^2$ force experienced by iron bodies in a free magnetic field, is complemented by looser qualitative arguments concerning the magnitude and direction of the forces experienced by the iron. Two dimensional systems with slowly varying gaps are analysed and a system with constant flux coils rather than fixed flux surfaces discussed.



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Contents

## 1. Introduction

With the increasing interest in high speed trains and surface transport systems, a number of different types of suspension systems have been suggested to replace the use of wheels and rails. Conventional railway systems using wheels begin to suffer excessive wear at speeds above perhaps 100 m.p.h., though this limit can be extended to perhaps 150 m.p.h. by careful design of the spring suspension, as in the advanced passenger train designed by British Rail. For speeds in the region of 300 m.p.h., however, suspension systems based on other principles have been proposed, including the use of air cushions or electromagnetic levitation.

This paper is a discussion of the stability of electromagnetic levitation systems, regarded mainly as an exercise in conventional electromagnetic theory, but oriented towards the problem of designing stable suspension systems with no essential electromagnetic losses and a predetermined stiffness. Schemes in which the traction and the levitation are provided by the same system are not discussed explicitly, nor are any detailed designs produced, but an attempt is made to clarify some of the principles involved in designing a levitation system and to illustrate them with simple examples.

The difficulties in designing a levitation system arise not so much in arranging an equilibrium position where the electromagnetic forces balance the other forces in the system as in ensuring that the system is stable to small departures from equilibrium. A stable equilibrium, more precisely defined below, is one in which the change in force as a body moves slightly restores the body to its original position. Whereas the force on a body in a given position is dependent only on the magnetic field around the body and not dependent on the arrangements for providing the field the change in field and the consequent change in force does depend on these arrangements. It is not surprising therefore that the way in which the field is provided is crucial to the stability of a levitation system.

Earnshaw's paper[(3)] on the stability of particles in various force fields seems to be the first reference historically. There he showed that particles in an inverse square law force field could not be in stable equilibrium; some other force field was necessary, whose nature depended on whether the forces were attractive or repulsive. This result can be applied to distributions of charge or current which are kept constant in size and direction as they move in a free space magnetic field (which might be provided by similar distributions). However it does not apply to systems containing

material whose permittivity (in the electrostatic case) or permeability (in the magnetic case) differs from that of free space. Braunbek[1] considered the effect of the presence of such material and concluded "Static, stable free suspension of a system I in the electric, magnetic and gravitational field of another system II is impossible unless diamagnetic material (or its equivalent, such as superconductors) is present in at least one of the two systems". Systems with paramagnetic material only he found must be unstable.

It seems clear from his experimental paper[2] that he thought in terms of passive diamagnetic bodies suspended stably in essentially free magnetic fields; in that paper he rules out the suspension of active systems under gravity above passive diamagnetic material as being impractical, and does not consider the possibility of passive systems above active diamagnetic material. In fact such systems can be constructed - once diamagnetic material is included in part of the system it is possible to have para (ferromagnetic) material in other parts and still make the system stable. The most interesting from the practical viewpoint is a superconducting system ("the train") suspended above a piece of iron ("the track"), which could in principle be the present railway lines, though whether this would be practicable is another matter. It is worth emphasising in passing that such systems have no essential electromagnetic losses: their performance depends on the magneto-static field and the "iron", a material of high permeability and is independent of any motion. A practical realisation of such a scheme might however involve parasitic electromagnetic losses, such as eddy currents in the "iron" if motion or alternating fields were involved and the iron was electrically conducting, or any losses associated with cooling the superconductors. These losses could be reduced by conventional methods like laminating or field tapering, according to economic considerations.

The basis of the theoretical work in this paper is the standard theorem (in a suitable form) concerning the change of mechanical energy available in systems when material of specified permeability is introduced. It is used not only to prove Braunbek's theorem, but also to provide a convenient method for calculating the mechanical energy potential and to discuss the behaviour of passive material (usually iron) supported in a magnetic field provided by the other part of the system.

Later in the paper the behaviour of such systems is examined in a more qualitative way in terms of the forces on the iron and the use of coils run at constant flux as the effectively diamagnetic material is considered. Finally some computational results are mentioned briefly and some general conclusions drawn.

## 2. Stability and Stiffness

Before proceeding further we define the sense in which stability and stiffness are used to discuss electromagnetic levitation systems in this paper. Only systems which can in principle be constructed from the following types of electromagnetic elements will be considered: coils or magnetic pole or dipole distributions (maintained if necessary) at constant intensity in free space, coils maintained at constant flux, and inert magnetic material of permeability different from that of free space. With these elements most boundary conditions can be represented: for example a body with the magnetic potential maintained constant on it as it moved around in the suspension system can be achieved by putting a current sheet of fixed strength on a body of infinite permeability. Again, a body with prescribed normal field component is equivalent to a body of low permeability $\mu$ surrounded by a fixed current sheet with strength proportional to $1/\mu$ in the limit $\mu \to 0$.

Let us consider a suspension system with two parts (the "track" and the "vehicle") each consisting of some combination of these elements rigidly fixed in each part. One part is fixed and the other movable and the position of a specified point in the movable part of the system is designated the point $\underline{r} = (x, y, z)$ in a Cartesian system in the fixed part. In addition the orientation of the movable part is specified by three other coordinates $\underline{\theta} = (\alpha, \beta, \gamma)$ which we may take to be the angles between a set of Cartesian axes fixed in the body and a set fixed in space. We shall only consider systems in which there is a mechanical energy potential $W(\underline{r}, \underline{\theta})$ such that the force $\underline{F}$ is given by

$$\underline{F} = - \nabla W$$

where the forces corresponding to the $\underline{\theta}$ coordinates are to be interpreted as torques. This implies that in any motion in which the system returns to the same configuration from which it started, no net work is done on the movable part. Of course there might be systems for which $\nabla \wedge \underline{F} \neq 0$, but this would be an embarrassment in levitation systems since they would correspond in some sense to devices for converting mechanical into electrical energy or vice versa, i.e. motors or generators. The equation for small displacements $(\underline{x}, \underline{\alpha})$ from an equilibrium position $(\underline{r}, \underline{\theta})$ may now be written:

$$||M|| \begin{bmatrix} \ddot{\underline{x}} \\ \ddot{\underline{\alpha}} \end{bmatrix} + ||W_{,ij}|| \begin{bmatrix} \underline{x} \\ \underline{\alpha} \end{bmatrix} = 0$$

where $||M||$ is a mass matrix (symmetric and positive definite) representing

the mass and moments of inertia. A necessary and sufficient condition for stability is that $\|W_{,ij}\|$, which is symmetric by definition, is positive definite. At this stage it is useful to distinguish between purely translatory motions (in which $\underline{\theta}$ is held constant), purely rotational motions (in which $\underline{r}$ is held constant) and combined motions. It may be that one of the constrained motions is unstable, but even if both are stable there is no guarantee that a combined motion is stable - the x and $\alpha$ parts of $\|W_{ij}\|$ can both be positive definite without the whole matrix so being.

Two comments might be made about rotational motions at a fixed position. Firstly, since W will be a periodic function of the angular displacements $\underline{\theta}$ it will always have a minimum at some point (and a maximum somewhere else). Thus a body constrained in translation will always find a stable angular position somewhere, and can never be stable at all angular positions - at best there would simple be no torque at any position, as in the case of an iron sphere. Secondly if a system were stable in translation, but unstable in rotation, it could be stabilised by putting another similar system far enough away and joining them mechanically. Thus a system could not be stable rotationally in all positions, but could always be stabilised if necessary at any given rotation by the addition of a similar system far enough away. Whatever the system is rotationally, a prerequisite for a completely stable system is stability to translational motions alone. We now consider the conditions required for stability in translation on the philosophy that if this is possible rotational stability can probably be arranged as well.

Stability in translation without rotation is assured if the $\underline{x}$ part of $\|W_{,ij}\|$ is positive definite, i.e. $W_{,ij}\, dr_i\, dr_j$ is positive for all $dr_i$. Whether this is so will depend on the geometric arrangement of the system. However a necessary, but not sufficient condition, for positive definiteness is that $W_{ii} (= -\nabla . \underline{F}) > 0$. In most of the following discussion of stability it turns out that it is possible to calculate the change in W as the system moves from position $\underline{r}$ to $\underline{r} + d\underline{r}$ in two stages as Braunbek[1] did. In the first the energy changes from $W(\underline{r})$ to $W'(\underline{r} + d\underline{r})$ and in the second from $W'(\underline{r} + d\underline{r})$ to $W(\underline{r} + d\underline{r})$. The first change, which corresponds to a translation holding all the current (or polarisation) distributions constant in the absence of any magnetic (or dielectric material), is a situation covered by Earnshaw's theorem for which $\nabla^2 W' = 0$. In the second change it is often possible to prove $W(\underline{r} + d\underline{r}) - W'(\underline{r} + d\underline{r})$ has a definite sign which depends only on the permeability of the material in the system. Thus it may turn out that the value of $\nabla^2 W = - \nabla . \underline{F}$, found by adding the two contributions

to the change in W, has a definite sign. It is in fact the trace of $\|W_{ij}\|$ and equal to the sum of the eigenvalues. It may then be argued that if $\nabla.\underline{F} > 0$, $\nabla^2 W < 0$ not all the eigenvalues can be positive; one must be negative and the system unstable. Such systems need not then be considered further. However even if $\nabla.\underline{F} < 0$ and $\nabla^2 W < 0$ not all the eigenvalues need be positive so stability is not assured. We then argue that by geometric design of the system, it is plausible that all the eigenvalues might be made positive. If this is achieved, rotational stability can then be considered.

Although this condition is necessary, but not sufficient, it has proved useful in eliminating certain types of system. However in many systems containing a mixture of materials of permeability both greater and less than that of free space $\nabla.\underline{F}$ can be of either sign.

## 3. Energy Theorems

Most standard texts on electromagnetism contain a proof of the theorem that increasing the permittivity in an electrostatic system for a fixed charge distribution decreases the electrostatic energy stored or vice versa. In another version of the theorem conducting bodies (of infinite permittivity) are introduced. For this case there is an expression for the energy change though the corresponding result for bodies of zero permittivity is not given.

These energy theorems can be used as the basis for proving Braunbek's theorem and related theorems about the stability of levitation systems. In the electrostatic case the stored electrostatic energy is the mechanical potential energy, since no other source of energy need enter the discussion. However versions of these theorems are needed that will apply to the analogous case of magnetic fields and magneto static forces. It is possible to replace fixed current distributions by equivalent polarisations (and vice versa) which always produce the same field externally and then appeal to the electrostatic analog. There may be some difficulties with multiply connected electromagnetic systems, since the external fields, i.e. those in between the fixed and movable parts of the system, do not have single valued potentials, which is implicit in the electrostatic calculation. These difficulties can probably be overcome by introducing extra polarisations and appealing to arbitrarily small holes in the two parts to allow relative motion of these extra polarisations. Although this might be acceptable, there is also a further difficulty: there does not seem to be any way of using the electrostatic analog to describe the behaviour of coils held at constant flux. In practice such coils may be used in levitation systems, so it is necessary to include them in the theory.

For this reason a proof of the energy theorem in a form suitable for magnetic materials with coils is set out below. When the change in mechanical energy available during a change in configuration is calculated, any energy changes at the coils that might be implied by the conditions imposed there must be included as well as the change in stored magnetic energy. The problem has to be closed in some way by specifying what happens at the coils otherwise the change in mechanical energy is not defined for finite changes. The natural conditions to impose at the coils are that the current through them, or alternatively the flux through them, is held constant during any displacement. The first corresponds to constant magnetisation (or highly ohmic coils driven by a constant voltage source) and the second to no energy alteration at the coils, or coils of zero resistance (superconducting perhaps).

Let $W$ be the energy available for doing mechanical work. In any change

in the relative position of parts of the system the change in W is given by

$$\delta W = \delta(\text{magnetic energy}) + (\text{change in coil energy}) \ . \qquad (3.1)$$

If $\underline{A}$ is the vector potential for the magnetic field in a material of variable permeability $\mu$ and $\underline{j}$ is the current distribution, the relevant Maxwell's equations take the form

$$\underline{j} = \nabla \wedge \left( \frac{\underline{B}}{\mu} \right)$$

$$\nabla . \underline{B} = 0 \qquad (3.2)$$

$$\text{or} \qquad \underline{B} = \nabla \wedge \underline{A} \ .$$

The flux through a coil may then be written

$$\Phi = \frac{1}{I} \int_{\text{coil}} \underline{j}.\underline{A} \, d\tau \qquad (3.3)$$

where I is the current through the coil and $\underline{j}$ the corresponding current distribution. This formulation applies to a distributed coil and $\underline{j}/I$ is a function only of the coil geometry and remains unchanged during any change of configuration.

The change in mechanical energy may now be written

$$\delta W = \frac{1}{2} \sum_{\text{coils}} \delta(\Phi I) - \sum_{\text{coils}} I \, \delta\Phi$$

$$= \frac{1}{2} \sum_{\text{coils}} (\Phi \, \delta I - I \, \delta \Phi) \qquad (3.4)$$

or in the corresponding form

$$= \frac{1}{2} \int_{\text{coils}} (\underline{A}.\delta\underline{j} - \underline{j}.\delta\underline{A}) \, d\tau \qquad (3.5)$$

Any ohmic dissipation that occurs in the coils does not appear since it is balanced by the extra voltage required at the coils. The change in mechanical energy between two states 1 and 2 in a system where the coils are held either

at constant current or constant flux becomes

$$W_2 - W_1 = \frac{1}{2} \sum_{\substack{\text{constant}\\ \text{current}}} I(\Phi_1 - \Phi_2) + \frac{1}{2} \sum_{\substack{\text{constant}\\ \text{flux}}} \Phi(I_2 - I_1)$$

which may be written

$$W_2 - W_1 = \frac{1}{2} \sum_{\text{all coils}} (I_2 \Phi_1 - I_1 \Phi_2) \ . \tag{3.6}$$

The corresponding distributed result is

$$W_2 - W_1 = \frac{1}{2} \int_\tau (\underline{j}_2 . \underline{A}_1 - \underline{j}_1 . \underline{A}_2) \, d\tau \tag{3.7}$$

where the constant flux condition has been used in the form

$$\int \underline{j}_1 . (\underline{A}_1 - \underline{A}_2) = \int j_2 . (\underline{A}_1 - \underline{A}_2) = 0 \tag{3.8}$$

(3.6) and (3.7) apply to different configurations 1 and 2 of the system. If the configurations are the same, then the 1 and 2 variables are the same. This can be contrasted with the reciprocal theorem for different variables in the same configuration, for which the right hand side of (3.6) and (3.7) would be zero.

Before proceeding we record a number of relations that will be useful later in this section. If $\underline{A}$, $\underline{B}$, $\underline{j}$ and $\underline{A}'$, $B'$, $\underline{j}'$ are two sets of variables satisfying Maxwell's equation (3.2) then

$$\int \underline{A} . \underline{j}' \, d\tau = \int \frac{\underline{B} . \underline{B}'}{\mu} \, d\tau - \int \frac{1}{\mu} (\underline{A} \wedge \underline{B}') . d\underline{S} \tag{3.9}$$

Again, if locally on a closed surface S $\frac{\underline{B}}{\mu} = \nabla \varphi$, at least in the surface, then

$$\int \frac{1}{\mu} \underline{A}' \wedge \underline{B} . \, d\underline{S} = \int \varphi \, \nabla \wedge \underline{A}' - \nabla \wedge (\varphi \underline{A}') . d\underline{S}$$

$$= \int \varphi \, \underline{B}' . d\underline{S} \tag{3.10}$$

Thus if $\underline{A}'$ etc. apply in the presence of a body of zero permeability

$$\int_{\mu=0} \frac{1}{\mu} \underline{A}' \wedge \underline{B} . dS = 0 \tag{3.11}$$

and if the body has infinite permeability

$$\int \frac{1}{\mu} \underline{A} \wedge \underline{B}' . d\underline{S} = 0 \tag{3.12}$$

either from (3.10) with $\varphi = 0$ or because $\underline{B}' \wedge d\underline{S} = 0$ .

We now use (3.7) to obtain energy theorems about the change in mechanical energy available when the configuration is changed, not by displacement of different parts but by the alteration of the permeability - in particular for small changes in permeability and for cases where finite passive bodies of either zero or infinite permeability are introduced. For small changes $\delta\mu$ in $\mu$ the conditions at the coil do not affect the calculation. Using (3.5), Maxwell's equation (3.2) and the identity (3.9) we find the usual expression

$$\delta W = -\frac{1}{2} \int \frac{B^2}{\mu^2} \delta\mu \, d\tau \tag{3.13}$$

Increasing the permeability always decreases W and vice versa.

When finite bodies of zero or infinite permeability are introduced, then from (3.7)

$$W'_{\text{after}} - W_{\text{before}} = \frac{1}{2} \int_{\substack{\text{between} \\ \text{bodies}}} (\underline{A}.\underline{j}' - \underline{A}'.\underline{j}) \, d\tau \tag{3.14}$$

or using (3.9)

$$= \frac{1}{2} \int_{\text{bodies}} \frac{1}{\mu} (\underline{A}' \wedge \underline{B} - \underline{A} \wedge \underline{B}') . d\underline{S} \tag{3.15}$$

which may be written

$$= \frac{1}{2} \int_{\substack{\mu=0 \\ \text{bodies}}} - \int_{\substack{\mu=\infty \\ \text{bodies}}} \frac{1}{\mu} [ (\underline{A} - \underline{A}') \wedge (\underline{B} - \underline{B}') - \underline{A} \wedge \underline{B} ] . d\underline{S} \tag{3.16}$$

(3.16) is the same as (3.15) as shown by an inspection of the missing terms which are $\int \frac{1}{\mu} \underline{A}' \wedge (\underline{B}' - 2\underline{B}) . d\underline{S}$ for the $\mu = 0$ bodies and $\int \frac{1}{\mu} (2\underline{A} - \underline{A}') \wedge \underline{B}' . d\underline{S}$

for $\mu = \infty$ bodies. Both of these are zero from the relations (3.11) and (3.12). If only bodies of one type or the other are present, then integrals always have the same sign. Using (3.9) the $\int \frac{1}{\mu} \underline{A} \wedge \underline{B}$ is done over the bodies, inside which $\underline{j} = 0$ so that the $\underline{A}.\underline{j}$ contribution disappears. The other integral $\frac{1}{\mu}(\underline{A} - \underline{A}') \wedge (\underline{B} - \underline{B}')$ is done over the space between the bodies and $(\underline{A} - \underline{A}').(\underline{j} - \underline{j}')$ contribution is zero because of the conditions of constant current or constant flux (3.8). We are left with

$$W' - W = \pm \frac{1}{2}\left[ \int_{\substack{\text{outside}\\ \text{bodies}}} \frac{(B - B')^2}{\mu} \, d\tau + \int_{\substack{\text{inside}\\ \text{bodies}}} \frac{B^2}{\mu} \right] \qquad (3.17)$$

where the positive sign applies to bodies of zero permeability and the negative to bodies of infinite permeability. $\underline{B}$ is the original field in the absence of the bodies, and $\underline{B}'$ the field in their presence. On the bodies $\underline{B}'.d\underline{S}$ or $\underline{B}' \wedge d\underline{S} = 0$ so that $(\underline{B}' - \underline{B})$ satisfies the same boundary conditions as $-\underline{B}$ (the original field). At all other places $\underline{B}'$ satisfies the same conditions as $\underline{B}$ so that $\underline{B}' - \underline{B}$ satisfies the corresponding homogeneous condition - for example if $\underline{B}.d\underline{S}$ specified at a surface then $(\underline{B}' - \underline{B}).d\underline{S}$ is zero there and so on. Calculating $W' - W$ using continuous distributions has somewhat lost sight of the fact that the fields might have been prescribed on surfaces by, say, their normal component or potential.

Expressions for $W$ and $W' - W$ for the special case of surfaces on which the magnetic field potential $\varphi(\underline{B} = \nabla\varphi)$ or its normal component $\frac{\partial\varphi}{\partial n}$ as the surfaces are moved can be obtained from (3.1) from first principles or by using the present results. Inside the surfaces there are current distributions and permeable material which are so arranged to keep $\varphi$ or $\frac{\partial\varphi}{\partial n}$ constant as the case may be. Outside $\mu = \mu_o$ Then the magnetic energy change between two states is as in (3.14) except that now the integral is over the internal volumes of the surfaces. The identity (3.10) may be substituted in (3.15) to give

$$W' - W = \frac{1}{2\mu_o} \int_{\substack{\text{inside}\\ \text{surfaces}}} \left( \varphi \frac{\partial\varphi'}{\partial n} - \varphi' \frac{\partial\varphi}{\partial n} \right) dS$$

Taking account of the sign change by referring dS to the region external to the bodies this can be written

$$W' - W = \frac{1}{2}\int_{\frac{\partial\varphi}{\partial n}} (\varphi' - \varphi)\frac{\partial\varphi}{\partial n}\, dS - \int_{\varphi} \varphi\left(\frac{\partial\varphi'}{\partial n} - \frac{\partial\varphi}{\partial n}\right) dS$$

$$\text{i.e.}\quad \delta W = \frac{1}{2}\int_{\frac{\partial\varphi}{\partial n}} \delta\varphi\frac{\partial\varphi}{\partial n}\, dS - \int_{\varphi} \varphi\frac{\partial}{\partial n}(\delta\varphi)$$

$$\text{or}\quad W = \frac{1}{2}\int_{\frac{\partial\varphi}{\partial n}} - \int_{\varphi} \varphi\frac{\partial\varphi}{\partial n}\, dS \tag{3.18}$$

As might be expected W is independent of the details of what actually goes on inside the surfaces. Once $\varphi$ or $\frac{\partial\varphi}{\partial n}$ is given on the surfaces the external field is fixed by the geometrical arrangement of the surfaces with respect to each other.

By using the identity (3.10) as before an expression corresponding to (3.16) for the change in energy on insertion of bodies with $\mu = \infty$ or $\mu = 0$, can be found

$$W' - W = \frac{1}{2}\int_{\mu=0} - \int_{\mu=\infty}\left[(\varphi' - \varphi)\frac{\partial(\varphi' - \varphi)}{\partial n} - \varphi\frac{\partial\varphi}{\partial n}\right] dS \tag{3.19}$$

where $\varphi'$ obeys the same conditions as $\varphi$ except on the bodies, where $\frac{\partial\varphi'}{\partial n} = 0$ or $\varphi' = 0$ respectively.

Although the original system has non-uniform permeability, the change in mechanical potential energy has been calculated only for two types of permeability variation, namely small arbitrary variations (3.13) and piecewise discrete changes to $\mu = 0$ or $\mu = \infty$ (3.17). It may be possible to find expressions for the energy change for arbitrary finite changes in permeability, though it does not appear easy. However in much of what follows we shall only require to know whether $W' - W$ is positive or negative. In such cases we may appeal to the idea that any distribution of permeability greater than free space (even anisotropic) can be achieved by a set of bodies of infinite permeability arranged on a suitably fine scale. A similar remark would hold for material of less than free space permeability and bodies of zero permeability. If a knowledge of the sign was all we needed it might appear we could appeal to (3.13) and a succession of increments $\delta\mu$. In fact such arguments can be

misleading, just in those cases for example which represent artificial conditions at the coils in systems that are designed to provide stable suspension of iron bodies.

Finally, for use in later sections, we obtain a result concerning the change in energy for small displacements of a system of non-uniform permeability in which some of the coils are maintained at constant current and others at constant flux. In systems of free space permeability, where the result is probably of most use, we could proceed by calculating $\nabla . \underline{F} = -\nabla^2 W$ directly following the extensions indicated by Thornton[9] using formulae for $\underline{F}$ like

$$\underline{F} = \frac{1}{2} \sum_{ik} I_i I_k \nabla L_{ik}$$

where $L_{ik}$ is the inductance between coils i and k. $\nabla^2 L_{ik} = 0$ and $L_{ik}$ is a positive definite matrix.

Alternatively we can consider the difference between the change in energy in two different types of displacement between two configurations of the system. In one of these displacements all the coil currents are held constant, and in the other coil currents or the coil fluxes are held constant. Denoting the initial configuration variables by 1, and the final configuration variables by 2 and 2′ we use (3.6) to give $W_2 - W_1$ and $W_2' - W_1$. We also use the reciprocal theorem for different states of the same configuration 2 and 2′ in the form

$$\sum_{\text{all coils}} (I_2' \Phi_2 - I_2' \Phi_2) = 0 \qquad (3.20)$$

Then straight addition, subtraction and rearrangement gives

$$W_2' - W_2 = \frac{1}{2} \sum_{\text{all coils}} (I_2' - I_2)(\Phi_2' - \Phi_2) + (I_2 - I_1)(\Phi_2' - \Phi_2) + (I_2' - I_2)(\Phi_1 - \Phi_2')$$

Now $I_1 = I_2$ for all the coils since 1 to 2 is a constant current displacement, and from 1 to 2′ either $I_2' = I_1 = I_2$ because the current is constant, or $\Phi_1 = \Phi_2'$ since the flux is constant. Thus only the first term remains giving

$$W_2' - W_2 = \frac{1}{2} \sum_{\text{all coils}} (I_2' - I_2)(\Phi_2' - \Phi_2) \qquad (3.21)$$

Thus the change in W in a (partially) constant flux displacement is always greater than the corresponding change in W when the coils are held at constant current, by an amount equal to the magnetic energy of the field caused by the difference in currents between the two states. This result is true even if material is of non-uniform permeability, though it may be more useful for the special case of coils in free space.

4. Earnshaw's theorem

Earnshaw[(2)] considered the stability of a particle in the inverse square law (attractive) field produced by other particles. He assumed the force $\underline{F}$ on the particle to be given by

$$\underline{F} = \nabla V$$

where

$$V = \Sigma \frac{m}{r}$$

where m is the "size" of the other particles and r the distance to them. Using $\nabla^2 V = 0$ he showed that the surfaces of constant V near a point of neutral attraction ($\nabla V = 0$) are hyperboloids of one or two sheets and that for small displacements from this point the motion of the particle must be unstable in at least one of the principal directions, although it may be stable in the other two. He concluded that to get stability, $\nabla^2 V$ must be negative and went on to discuss the different force laws that would achieve this.

In the context of electromagnetic levitation a more appropriate form of Earnshaw's theorem is obtained by examining the force on a distributed charge distribution $\rho$ or current distribution $\underline{j}$ due to an (electric) field or (magnetic) field provided by another set of charge or current distributions in free space. Then

$$\underline{F}(\underline{r}) = \int \rho(\underline{r}')\,[\nabla_{\underline{r}} V(\underline{r}+\underline{r}') + \nabla_{\underline{r}'} V'(r')]\,d\tau'$$

or

$$\underline{F}(\underline{r}) = \int \underline{j}(\underline{r}') \wedge [\nabla_{\underline{r}} V(\underline{r}+\underline{r}') + \underline{B}'(\underline{r}')]\,d\tau'$$

Here $\underline{r}$ denotes a fixed point of the body and $\underline{r}'$ a variable point within the body referred to $\underline{r}$ as origin. V is the potential due to the fixed charge/current distribution and $\nabla V'$ and $\underline{B}'$ the extra induced potential or field due to $\rho$ or $\underline{j}$. Now where the charges or currents supplying the original fields are kept constant and there are no other boundary conditions the fields $\nabla V'$ and $\underline{B}'$ are the free space fields generated by $\rho(r')$ and $\underline{j}(r')$ and do not depend on $\underline{r}$. The contribution to $\underline{F}$ from these is zero (as explicit calculation shows, putting $\nabla^2 V' = \rho'$, or curl $\underline{B}' = \underline{j}'$. These are really Maxwell stress calculations, but we could say physically that a charged body or set of currents in free space cannot provide a net force on itself).

It is clear from the first equation that $\nabla \wedge \underline{F}$ and $\nabla . \underline{F} = 0$ because $\nabla^2 V = 0$. Since $\nabla_{\underline{r}} = \nabla_{\underline{r}'}$ when operating on V, this is also true for the second equation, but not immediately obvious. Briefly $\nabla_{\underline{r}} \wedge \underline{F} = \int(\underline{j}(\underline{r}').\nabla_{\underline{r}\equiv\underline{r}'}) \nabla_r V \, d\tau'$ and from $\nabla . \underline{j} = 0$, $j_n = 0$ and Gauss's theorem $\nabla \wedge \underline{F} = 0$. $\nabla . \underline{F} = 0$ immediately.

The point thus established, perhaps at unnecessary length, is that in constant charge or current systems in free space there is a force potential $\underline{F} = - \nabla W$ between the parts of the system that satisfies $\nabla . \underline{F} = 0$ for translations of one part with respect to the other without rotation. This might be called Earnshaw's theorem in the context of electromagnetic levitation. However as soon as rotations are permitted $\nabla . \underline{F}$ need not be zero, and may be of either sign. The ones that would occur naturally in a body permitted to rotate would be those that decreased W, now interpreted as a mechanical energy potential that depends on the angular as well as the spatial position of the body. $\nabla . \underline{F}$ would be $> 0$ for such a body, i.e. stability impossible.

5. Braunbek's theorem

Braunbek[1] considered electrostatic and electromagnetic levitation systems which contained constant polarisation or current distributions and material whose permittivity or permeability differed from the free space value. He concluded

"Static, stable free suspension of a system I in the electric, magnetic and gravitational field of another system II is impossible unless diamagnetic material (or its equivalent such as superconductor) is present in at least one of the two systems".

His proof for the electrostatic case he applied to the electromagnetic case by the use of fixed magnetic polarisations which are equivalent to fixed current distributions. A relative change of position between the two parts of a system (two systems in his notation) is split up into two changes of configuration. During the first of these the polarisations induced in places where the permittivity differs from that of free space are frozen at their initial values so that $\nabla.\underline{F} = -\nabla^2 W = 0$ from Earnshaw's theorem. In the second the system in its new position is allowed to relax to its electromagnetic equilibrium state and a second contribution to $\nabla^2 W$ calculated. However for reasons not yet fully understood he adopts an artificial definition of energy in the intermediate stage. This definition is designed to give the correct energy in an electromagnetic equilibrium. It yields an expression for $\nabla^2 W$ in the second stage which is positive definite if the permittivity (permeability) is everywhere greater than free space, but it is not obviously negative definite in the cases where the permittivity is everywhere less, unless it is only very slightly less.

A proof of Braunbek's theorem is given below in which a displacement is dividedin a similar manner into two separate changes in configuration. An examination of the actual energy charges allows a somewhat stronger statement to be obtained, which can be summarised as follows:

(a) Systems consisting only of constant current or of charge distributions in free space are marginally unstable, i.e. $\nabla.\underline{F} = 0$ (Earnshaw's theorem).

(b) In systems where the magnetic permeability $\mu$ is always greater than or equal to its free space value $\mu_o$ are $\nabla.\underline{F} > 0$ and the system must be unstable, whereas in systems where $\mu \leqslant \mu_o$ $\nabla.\underline{F} < 0$ and the system may be stable.

(c) In mixed systems, where $\mu$ is both < and > $\mu_o$ in different parts of the

system, $\nabla.\underline{F}$ may have either sign, and can have different values for different relative positions of the two parts of the system.

In the notation of the last part of (b) Braunbek[1] showed that $\nabla.\underline{F}$ may be $< 0$ and hence the system may be stable. The proof below shows that if $\mu < \mu_o$ $\nabla.\underline{F}$ must be $< 0$. However this still does not guarantee stability.

We consider a system with a fixed and movable part with a mutual force $\underline{F}$ between them. Each part contains material of variable permeability and coils whose current is held constant during any change in configuration. As an extra case coils held at constant flux will also be considered. An important preliminary step in the proof is to surround each part of the sytem by a current distribution (perhaps a sheet) in the free space region between the parts, and at the same time remove the currents in any of the coils inside the current distributions. These additional distributions are constructed so that the field inside them is reduced to zero, but the field outside them, between the two parts of the system, is unchanged. Provided the permeable material is present the addition of any fraction of such a current distribution and a pro rata change in the currents in the original distributions will not change the fields outside, and hence the force between the two parts of the system. In particular removal of the current distribution with the permeable material present inside will not change the external field. Coils which are held at constant flux will have zero flux after the addition of the current distribution.

We now use the energy arguments in section 3 to calculate the change in mechanical potential energy W for small displacements. With the extra current distributions in place and the system in its original position, we remove the permeable material from the system. In general this would cause a change in W, but because the field in the permeable material has been arranged to be zero there is no change. Then we make a set of small displacements between the two parts of the system. For these displacements $\nabla.\underline{F} = 0$ if all the coil currents are held constant, and $\nabla.\underline{F} < 0$ if any of the coils are held at constant flux (3.21). Then in each of the new positions we re-insert the permeable material. By the energy theorems of the previous section this gives a positive contribution to the change in W, i.e. a negative contribution to $\nabla.\underline{F}$ provided all the material has permeability $\mu < \mu_o$ and a positive contribution to $\nabla.\underline{F}$ if $\mu > \mu_o$. If $\mu$ is $< \mu_o$ in some places and $> \mu_o$ in others the contribution may be of either sign.

Thus for constant current coils we get the statements (a), (b) and (c)

given above. In addition in systems with some constant flux coils and material with $\mu < \mu_o$ $\nabla.\underline{F} < 0$. This is not unexpected since constant flux coils would be expected to behave like $\mu < \mu_o$ material. However with constant flux coils and material for which $\mu \gtrsim \mu_o$ or $\mu$ is unspecified the two contributions are of opposite sign and from these arguments we cannot say what might happen. A special case is discussed in section 6.

A similar difficulty arises when the movable part of the system is free to rotate as it is displaced. If a constant current system is slightly displaced from a position of stable rotational equilibrium without being allowed to rotate, $\nabla.\underline{F} = 0$ for such displacements, and the additional release of mechanical energy W as it is permitted to adopt its new slightly different orientation will make a negative contribution to the change in W. Hence the total $\nabla.\underline{F} = - \nabla^2 W > 0$. If only material with $\mu > \mu_o$ is present then $\nabla.\underline{F} > 0$ for the displacements without rotations and permitting rotations only increases $\nabla.\underline{F}$. If only material with $\mu < \mu_o$ is present the contribution due to $\nabla.\underline{F}$ from the constrained displacement is negative and from the rotation positive. The sum may then in general be of either sign. To summarise, if the movable part of the system is not constrained any extra rotation makes a positive contribution to $\nabla.\underline{F}$ which may in general be larger than the negative $\nabla.\underline{F}$ in $\mu < \mu_o$ systems.

However it would be surprising if passive bodies with $\mu < \mu_o$ for which $\nabla.\underline{F} < 0$ in a constrained displacement in a constant current system (or free field) became unstable with $\nabla.\underline{F} > 0$ when allowed to rotate. That $\nabla.\underline{F} < 0$ even if rotations are permitted may be seen by using the two stage argument in reverse. The inert body is surrounded by an appropriate current distribution which reduces the field inside to zero in the usual way. If we allowed this current sheet to be displaced without rotation as before, then re-inserted in the $\mu < \mu_o$ material, we would get the usual zero and negative contributions to $\nabla.\underline{F}$ but permitting the final rotation would make the usual positive contribution. Alternatively, knowing that the body does rotate, we could instead displace the current sheet without rotation, then rotate it, and then insert the body. The first stage makes no contribution to $\nabla.\underline{F}$, and the second a negative contribution if we appeal to the physical argument that the current sheet (without the permeable material inside) is in unstable equilibrium and work must be done to rotate it to its new equilibrium position. This is because the dipole moment of the $\mu < \mu_o$, and hence of the equivalent current sheet, is antiparallel to the magnetic field. The current sheet is therefore unstable, because its dipole

moment does not change in magnitude as it rotates, although the original $\mu < \mu_o$ material is stable because its dipole moment does so change. An example of this is given in appendix 4. The final insertion of the $\mu < \mu_o$ material makes another negative contribution to $\nabla.\underline{F}$ ; so that (the total) $\nabla.\underline{F} < 0$ for a passive $\mu < \mu_o$ body moving in free magnetic field, even if it rotates freely.

6.1 Behaviour of iron or superconducting flat plates

Provided that no part of the system is allowed to rotate, either by symmetry or by constraint, any active system opposite a passive flat plate is stable if the plate is superconducting and unstable if it is iron. A more general statement is that if in any system a superconducting plate can be laid along a naturally occurring field stream surface (through which no flux passes) it is stable and correspondingly an iron plate on a natural constant potential surface is unstable. This may be proved using the energy theorems of section 3. Moving the superconducting plate is equivalent to removing it (when there is no energy change since this is the one position in which it does not affect the fields) and replacing it in a slightly different position, when the mechanical energy W goes up. In the case of the iron plate it goes down. In general such plates could be curved, and the parts of the system supplying the field different on both sides. If the plate is flat, then the parts are images of each other with the same geometry and the same potentials for the superconducting plate, and opposite potentials for the iron plate. With a one sided system there will now be a net force on the plates - repulsive for the superconducting plate and attractive for the iron plate but their stability is the same as that of the combined system.

Note that these remarks imply that all null-flux systems consisting of a flux excluding plate between some provider of magnetic field are stable and cannot be made unstable (i.e. arbitrarily destiffened) by putting iron in the system providing the field. However they can be if iron is attached to the plate itself nor need $\nabla.\underline{F}$ be negative in general for curved flux excluding plates as explicit examples show (section 9).

If we anticipate some of the remarks of the next section, then these results may be extended to the behaviour of passive material inserted into an initially uniform field near a flat plate. The initial field may be in any direction though the natural directions would be parallel to a superconducting plate or perpendicular to an iron one. As remarked in section 7 the change in mechanical energy (3.18) or (7.2) may be used as the mechanical energy potential and provided the bodies do not rotate during any displacements the system behaves exactly like an active body with the magnetisation specified by the initial field.

Systems in which rotations of various parts may occur during translation pose a problem. If the rotations are constrained during a translation, the

extra energy released when the system is permitted to take up its new equilibrium configuration by rotations without translation will decrease the stability (if any) of the constrained translation. While permitting rotations may not make it unstable we cannot say that it will not.

Nevertheless one feels that a single active body opposite a flat superconducting plate which, because of its asymmetry, rotated when pushed nearer the plate would still be stable. If so there may be an argument to prove this like the one in section 5 for a rotating body of less than free space permeability. However a system with more than one rotating body might conceivably be unstable, because it seems possible that as it was pushed nearer the plate there might be circumstances in which some of the bodies were rotated into a position where they were unstable and flipped over. In such a case a lurch in the translatory behaviour of the system would be expected. Further thinking about the implications of such systems is required.

6.2 Constant flux coils and iron bodies

In section (5) the sign of $\nabla.\underline{F}$ for various systems has been established. For mixed systems containing material of any permeability, and coils held at constant current or constant flux the sign of $\nabla.\underline{F}$ is not determined. It is not even determined in the case of passive iron bodies supported by constant flux coils.

However in the special case of a passive body supported by a single coil held at constant flux in an equilibrium position with no force (i.e. a situation where the magnetic forces are much larger than any other forces, such as gravity) we can show that $\nabla.\underline{F} > 0$ and thus that the equilibrium must be unstable, no matter what the shape of the coil or the iron.

Let the flux through the coil be $\Phi$ when the current is $I$. Then the flux and current can be related by the inductance

$$\Phi = L\,I$$

where $L(\underline{r})$ is defined in the presence of the passive iron body (and any other static ones that might be attached to the coil) and varies as the position $\underline{r}$ of the iron body varies.

The magnetic energy stored in the system is $\frac{1}{2} LI^2$ and the energy removed at the coils if the flux $\Phi$ changes is $I\,d\Phi$. Using (3.1) or (3.4) in situations where L changes as the body moves the force $\underline{F}$ on the body is given by

$$\underline{F} = -\nabla\left(\frac{1}{2}LI^2\right)_\Phi = -\nabla\left(\frac{1}{2}\,\Phi^2/L\right)_\Phi$$

or

$$= \nabla\left(\frac{1}{2}\,LI^2\right)_I$$

$$= \frac{I^2}{2}\,\nabla L \quad \text{in general} \qquad (6.10)$$

(6.10) applies whatever the condition at the coil, as it should: the force can only depend on the instantaneous current and the geometry. By direct calculation

$$(\nabla.\underline{F})_I = \frac{I^2}{2}\,\nabla^2 L \qquad (6.11)$$

$$(\nabla.\underline{F})_\Phi = -\frac{\Phi^2}{2}\,\nabla^2\left(\frac{1}{L}\right)$$

$$= \frac{I^2}{2}\left[\nabla^2 L - \frac{2(\nabla L)^2}{L}\right] \qquad (6.12)$$

Now $(\nabla.\underline{F})_I > 0$ for this is a constant current situation with iron present. From (6.11) $\nabla^2 L > 0$. If $\underline{F} = 0$, $\nabla L = 0$ and from (6.12) $(\nabla.\underline{F})_\Phi > 0$ as well. This means that one constant flux coil cannot hold a piece of material of permeability greater than free space in stable equilibrium with $\underline{F} = 0$, even if rotations of the iron are not permitted. Any rotations would only increase $\nabla.\underline{F}$. At least two coils must be necessary, and that they are sufficient is shown by an example in appendix 5, and experimentally in the accompanying paper.

There would be some stiffening effect from the constant flux condition at the coil when the second term in (6.12) was non zero and the analysis above does not show whether $(\nabla.\underline{F})_\Phi$ can be negative. We may speculate that even if $\underline{F} \neq 0$, due to gravity say, that the stiffening effect is never enough and that $\nabla.\underline{F} >$ always. A proof of this speculation, or alternatively, a counter example, has yet to be found.

We may further speculate that, even if such a one coil (constant flux) system is always unstable with iron for which $\mu = \infty$, it might be possible to make it stable if the iron saturated.

The precise role of saturation in these situations is not at present clear. If we say that for small changes in the field a partially saturated piece of iron behaves like material with constant polarisation and a permeability given by the local value of $\frac{dB}{dh}$ then, as Braunbek[1] pointed out, this would not stabilise a constant current situation. But it is possible to hold a piece of magnetic material with constant polarisation in stable suspension in the field of a single coil held at constant flux provided the polarisation is big enough compared to the coil current. However with a saturated piece of iron only a limited amount of constant polarisation is available, proportional to the exciting current in the coil, and it might not be enough.

There are systems which depend on saturation in the iron for stability. An example is the one discussed by Guderjahn and Wipf[4] where an A.C. excited or travelling coil is placed below a flat eddy current repulsion sheet with an iron sheet attached underneath the eddy current sheet. When far below the sheet the coil is attracted by the iron, and would be unstable if the iron did not saturate. If the iron does saturate as the coil approaches the sheet then some flux penetrates to the eddy current sheet and eventually the attractive force becomes repulsive. So saturation can be beneficial in the sense that systems may be designed that would not be stable unless it occurred.

## 7. Calculation of the potential W and slowly varying fields

In a particular application the best method of obtaining the force on part of an electromagnetic system may well be to solve the field equations and calculate the Maxwell stresses over a suitable surface.

However conceptually and for the analysis of simple systems it is useful to consider calculating the force $\underline{F}$ as $-\nabla W$ where W is given by (3.18) in the form:

$$W = \frac{1}{2\mu_o}\int_{\frac{\partial\varphi}{\partial n}} - \int_{\varphi} \varphi \frac{\partial\varphi}{\partial n}\, dS + \frac{1}{2}\sum_{\Phi} - \sum_{I} \Phi I \qquad (7.1)$$

where the integrals correspond to bodies on which $\frac{\partial\varphi}{\partial n}$ or $\varphi$ is fixed and the sums to coils where the flux $\Phi$ or current I is fixed. (7.1) actually applies to systems with variable permeability $\mu$ provided $\varphi$ is redefined so that $\underline{B} = \mu\nabla\varphi$ and the integrals are written in the form $\int \varphi \underline{B}.d\underline{S}$, though it may only be useful for bodies with free space between them.

A disadvantage of calculating W from (7.1) is that in cases where the movable part of the system is small and possibly passive (that is containing no sources of current) its behaviour might be expected to depend only on the field and system elements nearby and not on the distant field. However if (7.1) is used the boundary conditions far away appear to be important, in spite of the fact that the field is only distorted locally, in the sense that if the far field was specified by constant flux coils or constant $\frac{\partial\varphi}{\partial n}$ surfaces the positive sums and integrals would be taken in (7.1), which gives $W = \frac{1}{2} \times$ magnetic energy, whereas constant current coils or $\varphi$ surfaces would give $W = -\frac{1}{2}$ magnetic energy. Of course $\nabla W$ would be the same in both cases.

A better idea of the effect of introducing a passive body is gained by using for W the expressions for the change in W as the body is introduced, in the forms (3.17) or (3.19). Firstly it is now clear that W really only depends on the local field and the local boundary conditions. Secondly, if the initial undisturbed field is exactly uniform then the passive body (provided its orientation is fixed or does not matter) in the original system behaves exactly like an equivalent system in which an active body with the appropriate boundary conditions ($\varphi$ or $\varphi_n$ fixed on its surface) interacts with fixed elements with a homogeneous boundary condition corresponding to the possibly non-homogeneous one in the original system. To take a specific

example, consider an iron sphere in an initially uniform field near a fixed superconducting flat surface. The field may be in any direction, in particular perpendicular or parallel to the surface. The sphere behaves in exactly the same way as a sphere magnetized with a cos $\theta$-like potential maintained on its surface, i.e. uniform magnetisation oriented in the original field direction, above an inert superconducting surface. In this situation, from section (6.1) or from experiment, we know the sphere experiences a stable repulsive force, and so would a superconducting sphere in the same situation. The stability depends on the flat surface - if it was iron both sorts of sphere would be unstable. One may speak of images in the wall: in the case of the superconducting wall both spheres set up an image of the same sign and experience a repulsive force which decreases with distance from the wall - hence stability, whereas with the iron wall the images are of opposite sign and bodies experience an attractive force which falls as the distance from the wall increases - hence instability.

This idea can be extended to take account of non-uniformity in the initial field. More precisely we can consider bodies placed in fields which vary gradually, in the sense that the scale length of a typical field variation is much greater than the size of the body. Then $W$ can be calculated using a suitable mean value of the field. If we again consider iron or superconducting bodies inserted into a system with (say) a superconducting flat surface, then far away from such a surface $W$ may be found from (3.17) or (3.19) and will take the form

$$W = S_{ij} B_i B_j \tag{7.2}$$

where $S_{ij}$ is a symmetric positive definite tensor for a superconducting body and negative for an iron body. $S_{ij}$ might be called the electromagnetic shape of the body and in a free field depends only on its actual shape.

If the body has a fixed orientation $S_{ij}$ is fixed, but $W$ may vary as $\underline{B}$ varies. However if the body is not fixed in orientation it will rotate at a given position until it reaches a minimum value of $W$, i.e. the principal axes of $S_{ij}$ will rotate until the direction with the least eigenvalue is aligned with the field. A long flat iron body, for which $W$ (7.2) is negative, will align itself to cause the maximum local field distortion, namely parallel to the field: a similar superconducting body for which $W$ is positive will cause the minimum field distortion and will also align itself parallel to the field.

As such a body moves under gravity (say) close to a horizontal (superconducting) wall the $S_{ij}$ in (7.2) become complicated functions of $h$, the distance from the wall, and the orientation of the body with respect to the wall. In general the final orientation of the body will be neither parallel to the wall nor in the same direction with respect to the field as it is far from the wall, but at some orientation and distance from the wall that minimises $W$. In the special case of spherical bodies in three dimensions or circular cylinders in two the orientation of the body itself when near the wall will be irrelevant by symmetry, but there should still be a distinction between the effect of fields perpendicular to the wall, $B_n$, and parallel, $B_s$. One would expect

$$W = \frac{1}{2\mu_o}[k_s(h)\,B_s^2 + k_n(h)\,B_n^2] \tag{7.3}$$

It turns out (appendix 1) that in the two dimensional case of a circular cylinder near a flat wall the field orientation does not matter: $k_s(h) = k_n(h) = k(h)$ say. Then

$$W = \frac{k(h)}{2\mu_o}B^2$$

or in a slightly different non dimensional form

$$W = \frac{k\left(\frac{h}{a}\right)B^2V}{2\mu_o} \tag{7.4}$$

where $a$ is the radius of the body and $V$ its volume. For large $h/a$ $k$ assumes the value appropriate to a body in a free field (rather like the calculation of virtual mass in fluid mechanics). For a superconducting sphere $k = 1\frac{1}{2}$, for an iron sphere $k = -3$ : for cylinders $k = \pm 2$.

Hence we get the well known result that iron bodies tend to move to regions of high $B^2$ to minimise $W$ and superconducting bodies to regions of low $B^2$, and they finally reside (in the absence of other forces) in regions of maximum or minimum $B^2$. It is not difficult to construct fields with interior points at which $B^2$ is a minimum - indeed any position where $\underline{B} = 0$ must qualify. However $B^2$ can never have a maximum internally (since $\nabla^2 B^2 = 2\,\varphi_{ij}\,\varphi_{ij} > 0$) so an iron body would find no internal equilibrium position (if it could it would violate Braunbek's theorem which asserts instability for systems containing only iron and free fields).

Now consider the case of an iron body drifting in a slowly varying

field, seeking a maximum $B^2$, which must lie on the boundary of the region where the field is specified. Suppose matters have been so arranged that at the edge it encounters a locally flat wall on which $\frac{\partial\varphi}{\partial n}$ is specified. In the equivalent system it will behave as an insulating wall and produce a repulsive stabilising force which will predominate when the body is close to the wall compared to a typical distance over which the field varies. For large h k will take the value appropriate to the body when immersed in a uniform field of infinite extent and as h decreases k will rise or fall in such a way that $k'' > 0$ for the stable situations like an iron body approaching an insulating flat plate and $k'' < 0$ for the unstable ($\nabla.\underline{F} = -\nabla^2 W$). In appendix 1 the calculation of k(h) for circular cylinders in an initially uniform field in two dimensions is done analytically (giving an infinite series) for the case where the surrounding wall is also circular. An odd feature of the result is that even in this case W does not depend on the field orientation but only on the distance between the centres of the two circles. The flat wall is then a special case. In a general three dimensional situation we might distinguish eight cases as representing the range of practical interest, formed by permuting the situations of an iron or an insulating body approaching an iron or insulating wall in an initially uniform field which is either parallel or perpendicular to the wall (appendix 3). However in the two dimensional case described above there are only essentially two calculations to do because firstly the field orientation does not matter and secondly by the inverse property of flux and potential lines in two dimensions k for iron walls and insulating bodies is $-k$ for the same shaped insulating wall and iron body. The curves of k versus h are sketched in figure 1, actually with four curves just to clarify the situation, but the curves of negative k are just the images of the curves with positive k for the reason given above. Note that the force becomes infinite as the body touches the wall for the "like on like" cases, in two dimensions. In the analogous situation of a spherical ball in three dimensions the curves will have differing shapes (though with the required monotonic curvature) and will also depend on the initial field orientation (see appendix 3). The case of most practical interest of an iron ball approaching an insulating wall with an initially uniform field parallel to the wall, although in principle easy to solve, unfortunately requires a three dimensional calculation.

Figure 1 shows explicitly that as an iron cylinder approaches an insulating wall the force ($\underline{F} = -\nabla W = -\frac{dW}{dH}$) is repulsive and falls as h

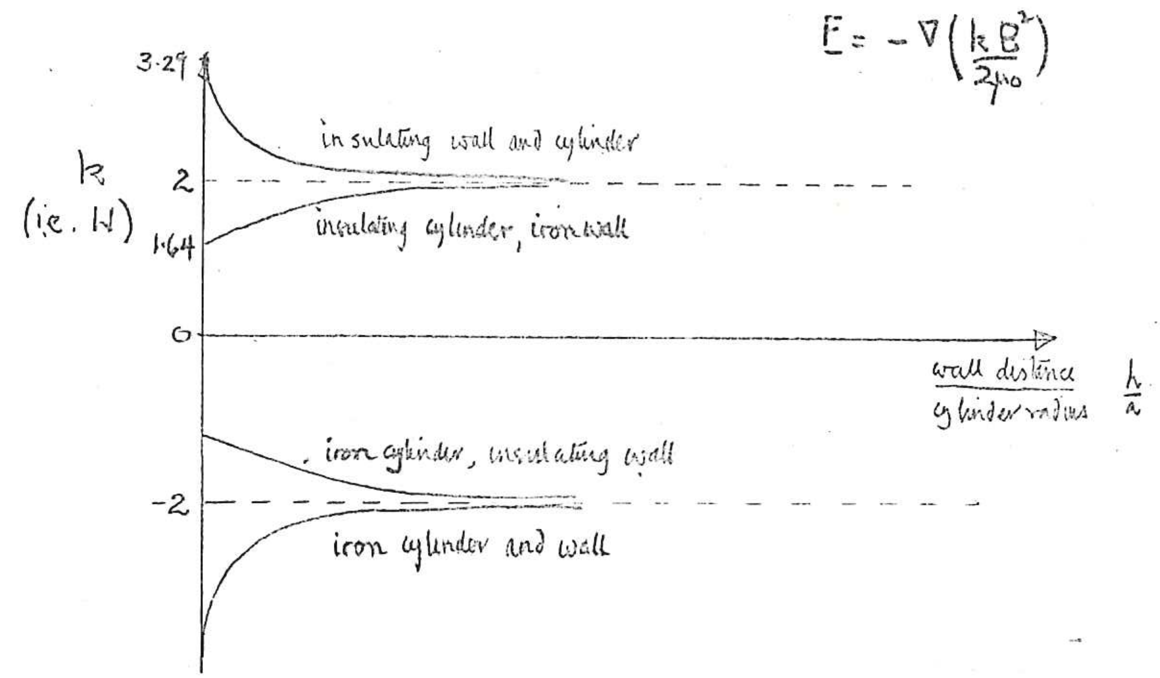
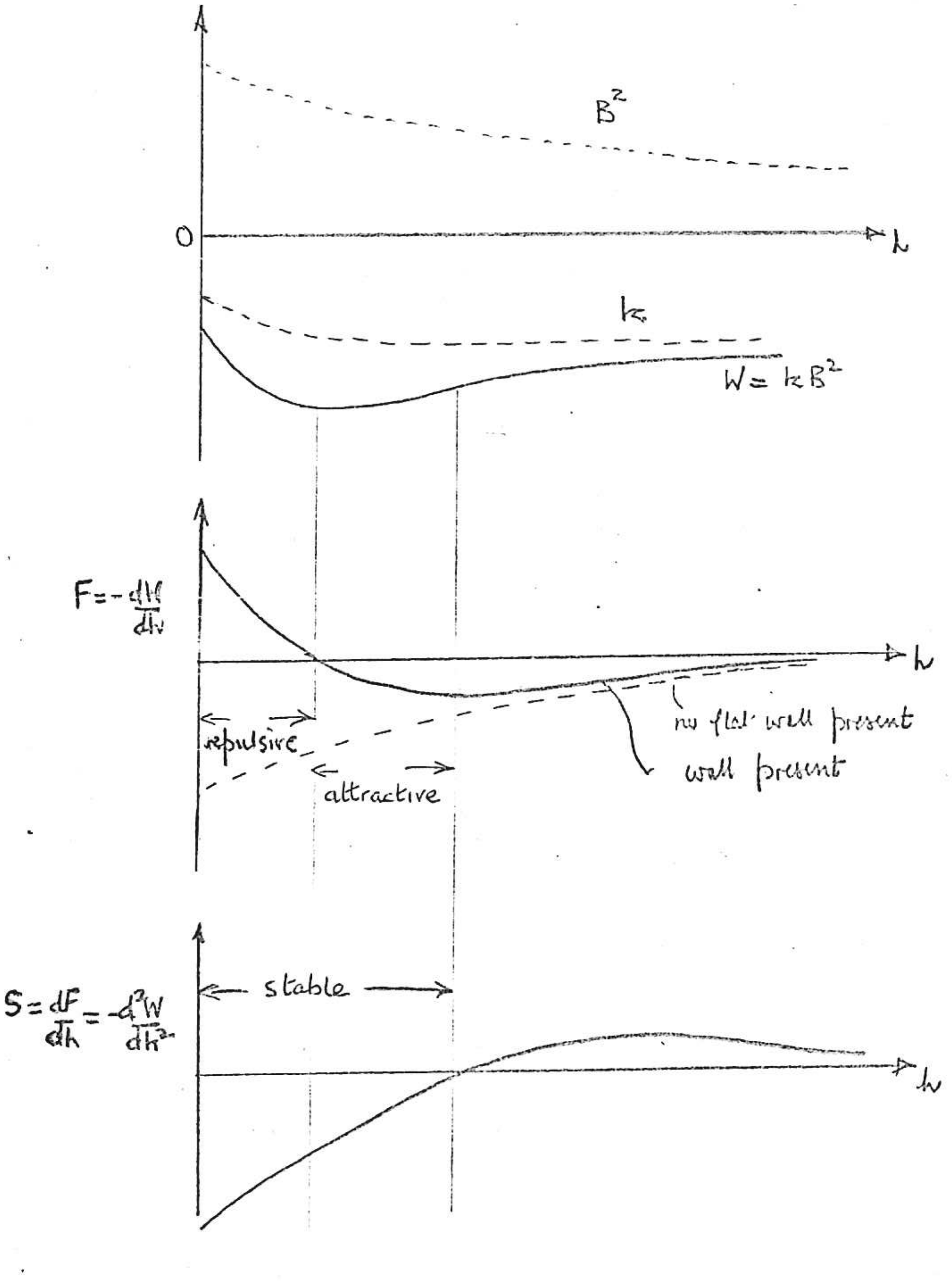



Figure 1. Initially uniform field of arbitrary orientation; 2-D circular cylinder inserted

increases, a stable situation. Now suppose we superimpose on this a slow variation in $B^2$ so that far from the wall the iron performs the $\nabla B^2$ drift described previously. Then the force potential W is now the product of the variable k and $B^2$ and to ensure stability we require $\frac{d^2W}{dh^2} > 0$ and also the relevant components of $\nabla^2 W$ in the directions transverse to the wall also $> 0$. The iron still performs the usual $\nabla B^2$ drift transverse to the wall which only affects the drift in the perpendicular direction. If we want to ensure stability in the transverse direction we must arrange that $B^2$ has a maximum on an axis perpendicular to the wall when the axis is, roughly speaking, approached transversely. If stability in the transverse direction is assured, the iron cannot possibly be stable as it approaches the wall until it interacts with the wall, so $\frac{d^2W}{dh^2} < 0$ (i.e. unstable) until this point.

Figure 2 shows the curves of W against the wall distance h for this case. The dotted curves represent the situation without the wall present, and for the wall stabilisation without any variation in $B^2$. The full curve




Slightly non-uniform field with a flat stabilising wall

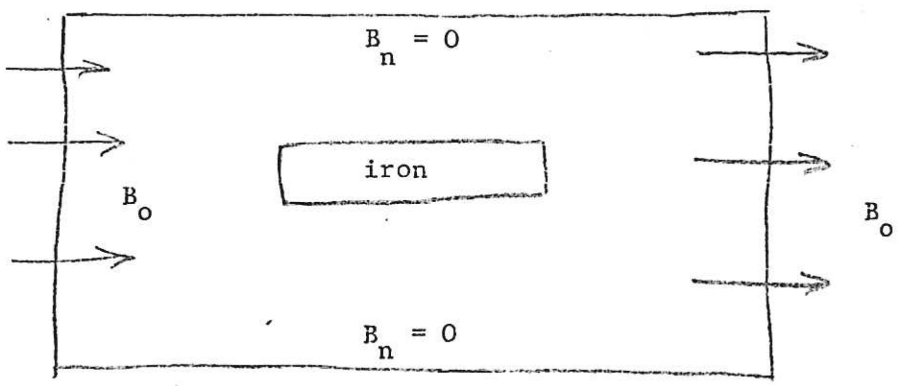
Figure 2

represents the effect of combining both according to (7.10) and the second part of the figure the corresponding force curve. Where the stiffness $S = -\frac{d^2W}{dh^2} < 0$ the body is stable though the force may be positive or negative. For this simple case it appears that the stiffness is only small where the force is attractive, though there may be some possibility of craftily arranging the original distribution of $B^2(h)$ (which must always have $\frac{d^2}{dh^2} B^2(h) > 0$ ) to flatten the force curve, i.e. reduce the stiffness, in the repulsive force region. To what extent this can be done is not yet clear - it may be that it is possible to anything in principle, but difficult to concoct the required field in practice.

8. Qualitative arguments for assessing the stability of mixed systems

The suggestion of the previous section 7 was that stability of an iron body near an insulating wall could be achieved by suitably balancing the stabilising effect of the wall and the unstable $\nabla B^2$ drift of the iron body in a free field. The essential assumption was a gradual variation in the field and we now try to extend this to more severe field variations. We examine in a qualitative way the stabilising and destabilising features of a levitation system, and hope to establish some rule of thumb principles that will help in later computer calculations. It seems likely that ultimately computation will be necessary to establish the optimum arrangements for a levitation system, and so we consider a few simple systems in order to see how they work. The arguments offered are suggestive rather than definitive.

We shall only consider systems made out a surface on which the flux is prescribed, and another surface of inert iron on which the potential is zero. Consider first the system shown in figure 3, a magnetic pipe with magnetic insulators



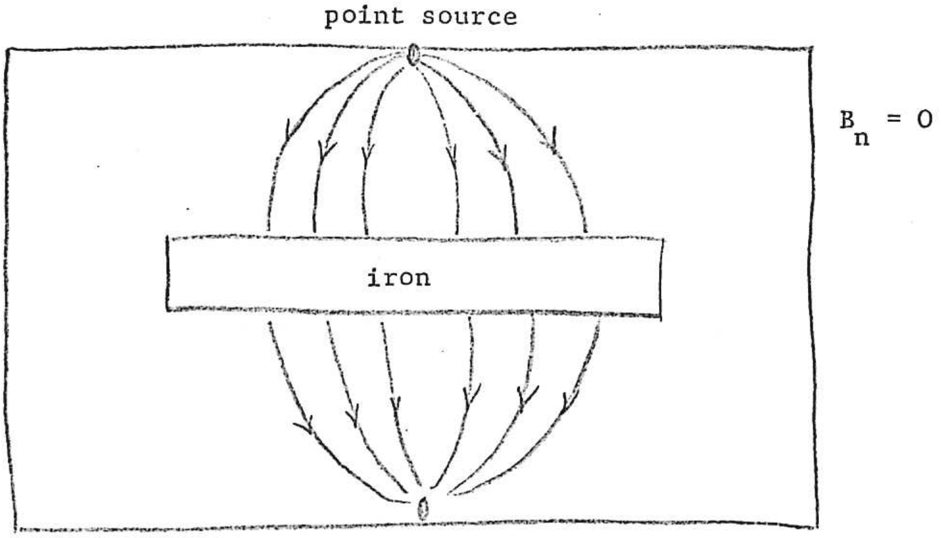

a magnetic pipe

Figure 3

top and bottom and a uniform magnetic flux prescribed at either end. The system is two dimensional, though the argument would apply equally to a circular pipe. If the pipe is long it does not matter much what the boundary conditions at either end are - they could be uniform flux or a fixed magnetic potential across the pipe, as long as the magnetically insulating side walls are present. Before any iron is introduced the field inside is uniform. Now introduce a thin strip of iron in the middle. Then in a long pipe the total force on the iron will be in the vertical direction since by symmetry the horizontal force is zero provided the iron is far from the ends.

Again by symmetry there is no net vertical force on the iron when it is midway between the top and bottom of the walls. The question is: in what direction is the force as the iron is displaced vertically, say upwards. It is clear by inspection that when the iron is displaced as far as it can be so that the top face is on the magnetically insulating wall no flux can get into the top of the iron - the flux on the sides can only produce (equal and opposite) horizontal forces and the flux that goes in and out of the bottom must produce vertical forces all of which are in the negative vertical direction. Thus when it is on the wall there is an attractive force pulling it back towards the midway position. There seems no reason for the force to change sign as we approach the midway position though it will become smaller, and if we assume the force is monotonically changing then this is a stable system with a zero net force at the midway position. It is more difficult to say whether its stiffness increases, i.e. what the exact shape of the graph of $F_y$ against y is, but one might guess that $-\frac{\partial F}{\partial y}$ was an increasing function of y since in the special case of a infinitely thin plate $\frac{\partial F}{\partial y}$ presumably has some finite value at the midway position, but on the wall $F_y$ is infinite (due to the singular behaviour in the magnetic field which is $\sim \frac{1}{d^{\frac{1}{2}}}$ where d is the small distance from the ends of the plate). The system is also probably stable to rotations by the same argument, since if the plate was rotated till it blocked the pipe, the return couple would be stabilising.

Another version of the same system is shown in figure 4.




unstable magnetic box

Figure 4

Here the field is provided by a source and sink on the top and bottom walls, all the rest of the walls being magnetically insulating again. The ends are far away and the iron plate in the middle is long compared to the distance between the top and bottom walls, so thatonce again it only experiences a vertical force due to the flux entering its top surface and leaving its bottom. There is actually a simple formula for the flux distribution provided by this system on the plate surfaces, if we assume it is infinitely long, but the important point is that if the distance from a magnetically insulating wall to the plate surface is h, then most of the flux enters in a characteristic distance of order h along the plate and falls off quickly either side. Thus $B_y$ on the plate is given by a formula of the form $B_y(x) = \frac{1}{h}\,\text{fn}\,(\frac{x}{h})$ where the function is given by the exact calculation. The factor $\frac{1}{h}$ implies that the total flux $\int B_y dx$ is independent of h, but the total attractive force $\int B_y{}^2\,dx \propto \frac{1}{h}$. Thus the attractive force is larger the smaller h, so that as the plate is displaced towards one of the sources it experiences a net attractive force in that direction. In the context of section 7 this might be regarded as an extreme case of $\nabla B^2$ destabilisation overcoming the stabilising effect of the wall. This system is probably unstable to rotations as well.

Clearly one could consider a system that consisted of the two systems together, the field being the sum of that provided by specifying the flux at the ends, and by specifying it as source and sink on the top and bottom walls, in a variable proportion. Although the problem is not linear one would imagaine that as the proportion of the second field increased the system would move from stability to instability.

Another way of looking at the problem qualitatively is to examine the flux distribution around an iron body set in a magnetic field, as in figure 5.

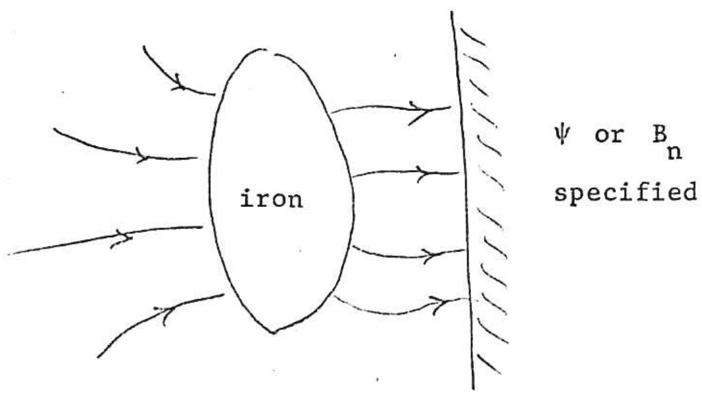
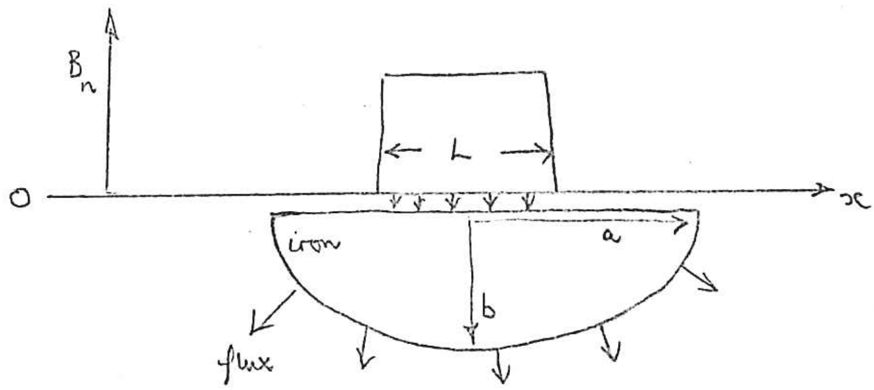



Figure 5

Flux enters on one side and leaves on the other. What constitutes one side or the other is clear in the case of a long thin body whose ends extend beyond the field, but vague in the general case. However pursuing this line of thought, the flux in and out is constant, i.e. $\int B_n\, dS$ is the same on both sides, but the force is attractive and $\int B_n^2\, dS$. We imagine the flux to be prescribed on the wall with an appropriate sink or source at a large distance in the negative x-direction. When the body is far from the wall the net force on it is small and we assume effectively zero. We also assume that as the body is moved towards the wall the force increases monotonically. This assumption is probably true for bodies with smoothly varying contours and smoothly varying flux distributions on the wall, though one can also conceive it might be possible to construct bodies and flux distributions for which the force did not vary montonically with distance from the wall. With the assumption of monotonicity it is again a question of finding the direction of the force when the body is close to the wall. Thinking in terms of a body with two sides, one opposite the wall and one facing the sink/source at infinity the attractive force $\int B_n^2\, dS$, when the flux $\int B_n\, dS$ is prescribed, is larger when $B_n$ is more bunched or more non uniform. Now provided the body effectively extends beyond the source of the flux at the wall the non-uniformity on the negative x side is determined by the shape of the body on that side, whereas the nonuniformity on the wall side is determined by the body shape and by the prescribed flux distribution. If the wall flux distribution had no effect then the body would be effectively in a free magnetic field and iron bodies in free fields are unstable. However when the wall flux distribution has a significant effect and constrains the flux to be more uniform than it would otherwise be, the attractive force on this side will be lessened and in the final situation the net force might be repulsive. With the assumption of a montonically varying force in the wall direction, and of a much smaller sideways force, this would imply stability. On the other hand a flux distribution that caused a greater concentration of flux than would naturally occur would be likely to increase the instability.

The direction of the force when the body is close to the wall would have to be found by calculation. In most cases this would probably be numerical but in simple cases analysis is possible. As an example, we treat the case of a semi-elliptical body shown in figure 6. The rear flat surface covers the prescribed flux distribution completely when the body is close to the wall, and under these conditions there will be a negligible sideways force when the body is displaced a small distance from the wall.

On the outward-looking face the flux distribution is normal to the surface




Elliptical iron body near a flat wall with specified normal field

Figure 6

and is the same as for an elliptical body held at constant potential in free space. It is shown in standard texts on electromagnetic fields that in this case

$$B_n \propto \left( \frac{x^2}{a^4} + \frac{y^2}{b^4} \right)^{-\frac{1}{2}}$$

where (x, y) lies on the surface of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The flux through the surface is $\int_{x=-a}^{a} B_n \, ds$

and since the only force on the surface is normal to it with magnitude $\frac{B_n^2}{2\mu}$, the force component in the y direction is

$$\frac{1}{2\mu} \int B_n^2 \, ds \cos\theta$$

or

$$\frac{1}{2\mu} \int_{x=-a}^{a} B_n^2 \, dx$$

The integrals are elementary. If we prescribe the amount of flux to be $\pi$ for convenience, we find a normalising factor of $\frac{1}{ab}$ in front of the above equation for $B_n$, which varies from $\frac{1}{b}$, at $x = \pm a$, $y = 0$ on the side walls to $\frac{1}{a}$ at $x = 0$, $y = -b$ in the centre. The integral for the force in the (negative) y direction on the curved surface becomes :

$$F_y = -\frac{1}{2\mu} \cdot \frac{1}{ea} \log \frac{1+e}{1-e}$$

where $e^2 = 1 - \frac{b^2}{a^2}$ . In practice we are only interested in cases where $b < a$ .

The attractive force on the flat rear surface of the body will depend on the flux distribution, but its minimum value will be when the flux is uniform and spread over the whole of the back surface. If it is uniform and spread over a distance L then to achieve a total flux $\pi$ $B_n = \frac{\pi}{L}$ and

$$F_y = \frac{1}{2\mu} \frac{\pi^2}{L} \quad \text{where} \quad L \leqslant 2a \, .$$

There will be a net force away from the wall provided

$$\frac{1}{ea} \log \frac{1+e}{1-e} > \frac{\pi^2}{L} > \frac{\pi^2}{2a} \, .$$

When $e = 0$ , i.e the ellipse is a circle, the left hand side is $\frac{2}{a}$ and the equality is not satisfied. For larger b ($e^2 < 0$) the left hand side is even smaller. It increases as b is decreased and is just satisfied when

$$e \simeq 1 - \frac{1}{75} \, , \quad \frac{b}{a} \simeq \frac{1}{6}$$

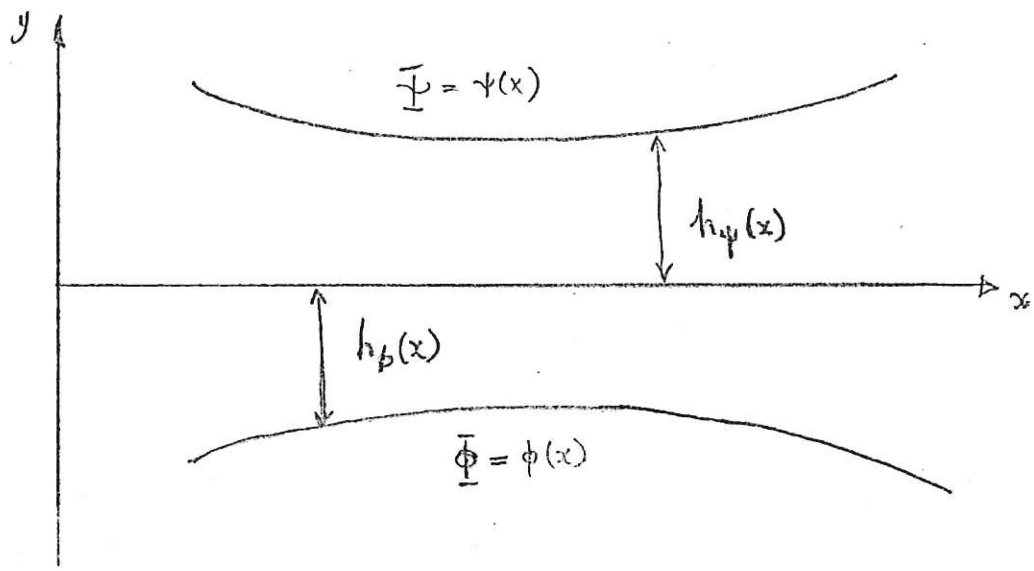
Thus elliptical bodies thinner than $\frac{b}{a} \simeq \frac{1}{6}$ or the corresponding value for a smaller L would experience a repulsive force when close to the wall, and the fatter cnes an attractive force. Ultimately for bodies thin enough to be a flat plate the repulsive force tends to infinity logarithmically because of the singularity in the field distribution at the ends. The effect of this singularity would technically disappear when the flat plate was a small but finite distance from the wall, but the trend - a large repulsive force - would still be there.

The behaviour of this body to rotations is more difficult to guess, but although this might be of practical interest, in principle we are only concerned with establishing the behaviour of bodies undergoing small translations, for the reasons set out in the earlier sections.

## 9. Analysis of mixed systems with slowly varying gaps

With the aim of providing examples or counter examples we analyse a particular class of mixed systems, namely those consisting of surfaces on which the magnetic potential $\varphi$ or its normal component $\frac{\partial\varphi}{\partial n}$ are specified. We consider a two dimensional system in terms of $\varphi$ and the corresponding stream function $\psi$. There are only two surfaces (figure 7), one movable and one fixed, both infinitely long in the x direction and separated by a small gap in the y direction - small in this context meaning small compared to the characteristic length of significant variations of the field in the x direction. On one surface $\varphi$ is specified, on the other $\psi$. In a given position the force $\underline{F}$ between the surfaces can be calculated by evaluating the Maxwell stress tensor along the x axis (see appendix 1) which is taken to lie between the surfaces. As figure 7 shows the shape of the surfaces are defined by their positive distances $h_\varphi(x)$ and $h_\psi(x)$ from the x axis. Dashes denote differentiation with respect to x.

The force $\underline{F}$ is found by evaluating the field components to the second order in quantities like $h\frac{d}{dx}$ and calculating the Maxwell stresses. It is also possible to produce an explicit formula for the potential function from which $\underline{F}$ is derived according to (3.18) but the function and hence the field has to be evaluated to the third order to give $\underline{F}$ to the second order.




Notation for a slowly varying gap

Figure 7

The results from appendix 1 are :

$$F_x = \int_{-\infty}^{\infty} dx \left\{ \varphi'\psi' - (h_\varphi\psi')'\psi' - (h_\psi\varphi')'\varphi' - \varphi''\psi'' \frac{(h_\varphi + h_\psi)^2}{2} + h'_\varphi h'_\psi\varphi'\psi' \right\} \quad (9.1)$$

$$F_y = \int_{-\infty}^{\infty} dx \left\{ \frac{\varphi'^2 - \psi'^2}{2} + (h_\psi\varphi')'\psi' - (h_\varphi\psi')'\varphi' + \frac{1}{2}[h_\psi\psi'' + (h_\varphi\psi')']^2 - \frac{1}{2}[h_\varphi\varphi'' + (h_\psi\varphi')']^2 \right\}$$

(9.2)

where $\underline{F}$ is the force on the upper ($\psi$) surface. These formulae assume that $\varphi$ and $\psi$ tend to constant values at the ends, so that integrals of any quantities that are total differentials (for example $\varphi'\psi'' + \varphi''\psi'$) are ignored. They have the comforting feature of being unaffected by changing $h_\varphi$ by $h_o$ and $h_\psi$ by $-h_o$ where $h_o$ is constant (i.e. moving the x axis). Again if we write $-h_\varphi$, $-h_\psi$ for $h_\varphi$, $h_\psi$ and $\psi$ for $\varphi$ and $-\varphi$ for $\psi$, then the magnetic field is rotated through 90° and the Maxwell stresses, $F_x$ and $F_y$ change sign.

To calculate $\nabla.\underline{F}$ we envisage making small displacements of the upper surface in the x and y directions. If we designate the position of the upper surface by $(x_o, y_o)$ and change $(x_o, y_o)$ then the operator $\frac{\partial}{\partial x}$ is equivalent to repeating the integrals for $\underline{F}$ with $\psi(x)$ and $h_\psi(x)$ replaced by $\psi(x-\epsilon)$ and taking the limit $\epsilon \to 0$. Thus $\frac{\partial}{\partial x_o} \int \theta(\psi, h_\psi, \varphi, h_\varphi)\, dx = -\int(\theta_\psi\psi' + \theta_{h_\psi} h'_\psi)dx$ or alternatively $\int(\theta_\varphi\varphi' + \theta_{h_\varphi} h'_\varphi)\, dx$ assuming all quantities disappear at the integral limits. The operator $\frac{\partial}{\partial y_o}$ is equivalent to $\frac{\partial}{\partial h_\varphi}$ or $\frac{\partial}{\partial h_\psi}$, both giving the same result since the formulae contain only $(h_\varphi + h_\psi)$, apart from derivatives of $h_\varphi$ and $h_\psi$ which do not affect the answer.

Applying these rules we find

$$\left.\begin{aligned}
\frac{\partial F_x}{\partial x_o} &= \int dx \{\varphi''\psi' - (h'_\varphi\psi')'\psi' + (h'_\psi\varphi')'\varphi' + O(h^2)\} \\
\frac{\partial F_x}{\partial y_o} &= \int dx \{-(h_\varphi + h_\psi)\varphi''\psi'' + O(h^2)\} \\
\frac{\partial F_y}{\partial x_o} &= \int dx \{(h_\psi\varphi'')'\psi' + (h_\varphi\psi'')'\varphi' + O(h^2)\} \\
\frac{\partial F_y}{\partial y_o} &= \int dx \{\varphi''\psi'(\text{or } -\psi''\varphi') + \psi''[h_\psi\psi'' + (h_\varphi\psi')'] - \varphi''[h_\varphi\varphi'' + (h_\psi\varphi')'] + O(h^2)\}
\end{aligned}\right\}$$

(9.3)

Thus we can check that, to the first order in $h$, $\frac{\partial F_x}{\partial y_o} = \frac{\partial F_y}{\partial x_o}$ and calculate $\nabla.\underline{F}$:

$$\nabla.\underline{F} = \frac{\partial F_x}{\partial x_o} + \frac{\partial F_y}{\partial y_o} .$$

Under the transformation $(\varphi, \psi) \to (\psi, -\varphi)$ referred to above, $\nabla.\underline{F}$, indeed $\frac{\partial F_x}{\partial x_o}$ and $\frac{\partial F_y}{\partial y_o}$, appear not to change sign as they should, but this is because under the change $(h_\varphi, h_\psi)$ to $(-h_\varphi, -h_\psi)$ the operators $\frac{\partial}{\partial x_o}$ and $\frac{\partial}{\partial y_o}$ change sign as well.

To zero order in $h$, we see that

$$\frac{\partial F_x}{\partial x_o} = \frac{\partial F_y}{\partial y_o} = \frac{1}{2} \quad \nabla.\underline{F} = \int dx\ \varphi''\psi' ,$$

so that where both surfaces are "active", i.e. $\varphi \not\equiv 0$, $\psi \not\equiv 0$ the system may be stable or unstable, depending on the sign of the integral and if stable has to order $h$ the same stiffness in all directions. This is a result of $h$ being small: the principal directions are actually displaced $O(h)$ from the $x_o$ and $y_o$ axes.

The cases where one of the surfaces is passive, i.e. $\varphi$ or $\psi \equiv 0$, is probably of more practical interest, particularly if $\varphi = 0$ since this represents the case of the inert lump of iron opposite a "superconducting" surface on which the flux distribution is specified.

In this case the zero order terms in the force gradients disappear, leaving only first order terms in the h's, which may be of either sign. Explicitly

$$F_x = \int -\ (h_\varphi\psi')'\psi' + O(h^3) \tag{9.4}$$

$$F_y = \int -\ \frac{\psi'^2}{2} + \frac{1}{2}[h_\psi\psi'' + (h_\varphi\psi')']^2 + O(h^3) \tag{9.5}$$

$$\frac{\partial F_x}{\partial x_o} = \int -\ (h'_\varphi\psi')'\psi' + O(h^2) \tag{9.6}$$

$$\frac{\partial F_y}{\partial y_o} = \int \psi''[h_\psi\psi'' + (h_\varphi\psi')'] + O(h^2) . \tag{9.7}$$

Thus there is an attractive force between the two parts of the system. If $h_\varphi$ is

constant, i.e. the iron is flat, then $F_x = 0$ (explicitly and also from first principles, since the force is always normal to the iron surface) and

$$\frac{\partial F_y}{\partial y_o} = \psi''^2 (h_\psi + h_\varphi) > 0 ,$$

i.e. the system is unstable, weakly to first order in h, a special case of the general result. Bending the superconducting surface, i.e. choosing $h_\psi$, makes no difference. However bending the iron surface does, since a little examination of (9.6) and (9.7) shows it is possible to choose $h_\varphi$, $\psi$ and $h_\psi$ so that $\nabla . \underline{F} < 0$ and indeed $\frac{\partial F_x}{\partial x_o}$ and $\frac{\partial F_y}{\partial y_o}$ individually $< 0$.

These results can be regarded as showing that it is possible, in principle, to construct a system, one part of which is an inert iron surface and the other a flux specified surface, which is either stable or unstable. Conversely, by using the $(\varphi, \psi) \rightarrow (\psi, -\varphi)$ transformation we can similarly show that systems with a specified $\varphi$ surface (constant current coils backed by iron) and an insulating flux surface (a magneplane) may be either stable or unstable; that if the magneplane is flat they are (only proved in the small gap approximation here) naturally stable, but that by suitably bending the magneplane surface and adjusting the flux distribution on the iron they can be made unstable (or with less bending their stiffness decreased). The formulae show that in this approximation $\frac{\partial F_y}{\partial y_o} > \frac{\partial F_x}{\partial x_o}$ so that if both are negative the system will be stiffer in the x direction than in the y.

## 10. Computational results and conclusions

With the use of a suite of programs[8] available at Culham for computing two dimensional electromagnetic fields, some simple systems have been analysed. The results are very limited since only a very small amount of computing has been done with the aim of justifying the rather qualitative ideas set out in the previous sections. The most interesting system consists of a circular iron cylinder inside a square box rather like that in figure 3, but with walls curving inwards top and bottom with zero flux prescribed on them and flat walls at either side with uniform flux. The diameter of the cylinder is one quarter the length of the side of the box, and the top and bottom are curved in order to increase the field in the centre, and attract the cylinder there.

The forces on the cylinder (no torque by symmetry) are calculated in the following way. The standard input procedures of the program are used to specify the shape of the box and its associated flux distribution, and the position and shape of the iron cylinder. With the appropriate boundary conditions the finite difference equations determining the magnetic field distribution are solved on a suitably fine rectangular mesh. The forces (and couple in general) are then found by integrating the formulae for the Maxwell stresses around a closed rectangular contour around the cylinder. For convenience the contour is made up of appropriate mesh lines and the accuracy of the calculation can be estimated by calculating the stresses around a number of different contours for which the results should be the same. The cylinder is then moved a small distance and the whole calculation repeated to give the forces at the new position. By doing this at a large enough number of different positions the force gradients and $\nabla.\underline{F}$ throughout the region can be determined.

The results, given in greater detail in an earlier version of this paper, show that as the iron cylinder is moved from the centre towards the top curved wall, the force in this direction, although it may be initially towards the wall (unstable), eventually becomes repulsive (stable) provided the wall curvature is not too large, while the sideways force is always stable. When the iron is moved towards a side wall, a repulsive force is again experienced, and the force parallel to the side wall is now weakly unstable. This is presumably because the field strength increases towards the corners of the box and the side wall provides no stabilization in this direction. At the centre in the examples tried, the cylinder was sometimes unstable in one direction, but $\nabla.\underline{F}$ was never positive - perhaps because the iron cylinder was never sufficiently small to be far enough away from the wall.

These results, as far as they go, support the theoretical ideas of the previous sections. It is likely that in the design of a practical system

mathematical analysis will only suggest fruitful lines of approach and that in the end considerable computation will be required. For systems in which a passive piece of iron is levitated by an active system we can make the following progressively less definitive statements:

(a) There must be some material like superconductors or A.C. flux surfaces which behave essentially like diamagnetic material (Braunbek).

(b) Such material in the form of a flux wall will stabilise the equilibrium of a passive piece of iron provided the original undisturbed magnetic field is not too non-uniform. The behaviour of the system will probably involve judicious arrangement of the original field non-uniformity which can be arranged to drive the iron to the wall and the stabilising influence of the wall.

(c) Constant flux coils can be used instead of fixed flux surfaces for stabilisation, but their effect is weaker. Careful design of the slight non-uniformity in original field is likely to be even more important.

## 11. Acknowledgements

I am grateful to Dr P A Davenport for introducing me to these problems and to C Ll Thomas for writing the two-dimensional Maxwell stress routine.

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Appendix 1: Analysis of mixed systems with slowly varying gaps

To find the mutual force $\underline{F}$ in the situation described in section 9 a solution of the field problem with the boundary conditions specified in figure 7 is required. If $\Phi + i\Psi$ is the solution in terms of the potential and stream-function then the force potential W can be calculated from (3.18). In a two dimensional situation (3.18) can be written

$$W = \int_{\Psi} - \int_{\Phi} \Phi \, d\Psi$$

Given the particular contours of figure 7, and assuming that the fields are such that $\Phi = \Psi = 0$ at $x = \pm\infty$, $\int_{-\infty}^{\infty} \frac{d}{dx}(\Phi\Psi)\,dx = 0$ and taking account of the change in direction of integration along the $\Phi$ surface W may be rewritten in the anti-symmetric form

$$W = \int_{\Psi} \Phi\Psi' dx - \int_{\Phi} \Psi\Phi' dx \qquad (A.1)$$

where a prime denotes differentiation with respect to x .

W can actually be evaluated from (A.1), but to find $\Phi + i\Psi$ as an expansion in powers of $h_{\varphi}$, $h_{\psi}$ and to calculate $\underline{F} = -\nabla W$ to second order in h requires knowledge of W to third order. It is actually simpler to calculate $\underline{F}$ directly from the Maxwell stresses, since $\Phi + i\Psi$ is now only required to second order.

In a two dimensional situation the force on a body surrounded by a contour may be written

$$F_x - iF_y = -\frac{i}{2\mu_o} \int (B_x - iB_y)^2 \, d(x+iy)$$

For the particular case when the contour is the x axis the forces on the upper half of the plane (i.e. the top half of the system) become

$$F_x = -\frac{1}{\mu_o} \int_{-\infty}^{\infty} B_x B_y \, dx$$
$$F_y = \frac{1}{2\mu_o} \int_{-\infty}^{\infty} (B_x^2 - B_y^2) \, dx \, . \qquad (A.2)$$

Now $\Phi + i\Psi$ may be written as some function of (x + iy) as follows, where R and I are <u>real</u> functions

$$\Phi + i\Psi = R(x+iy) + iI(x+iy) \tag{A.3}$$

from which we have

$$\begin{aligned} (B_x)_{y=0} &= R'(x) \\ (B_y)_{y=0} &= -I'(x) \end{aligned} \tag{A.4}$$

and the forces from (A.2) become

$$\begin{aligned} F_x &= \frac{1}{\mu_o}\int_{-\infty}^{\infty} R'I'\,dx \\ F_y &= \frac{1}{2\mu_o}\int_{-\infty}^{\infty} (R'^2 - I'^2)\,dx \,. \end{aligned} \tag{A.6}$$

It remains only to evaluate R and I in terms of the functions $\varphi(x)$, $\psi(x)$, $h_\varphi(x)$ and $h_\psi(x)$. Expanding (A.3) to second order in $y^2$ we find

$$\begin{aligned} \Phi &= R - \frac{y^2}{2} R'' - yI' \\ \Psi &= I - \frac{y}{2} I'' + yR' \,. \end{aligned} \tag{A.7}$$

Substituting into (A.7) the boundary conditions

$$\begin{aligned} \Phi(y = -h_\varphi) &= \varphi(x) \\ \Psi(y = h_\psi) &= \psi(x) \end{aligned} \tag{A.8}$$

we find

$$\begin{aligned} \varphi &= R - \frac{h_\varphi^2}{2} R'' + h_\varphi I' \\ \psi &= I - \frac{h_\psi^2}{2} I'' + h_\psi R' \end{aligned} \tag{A.9}$$

and these can be inverted to second order to give R and I in terms of $\varphi$ and $\psi$, as follows

$$\begin{aligned} R &= \varphi - h_\varphi(\psi - h_\psi\varphi')' + \frac{h_\varphi^2}{2}\varphi'' \\ I &= \psi - h_\psi(\varphi - h_\varphi\psi')' + \frac{h_\psi^2}{2}\psi''. \end{aligned} \tag{A.10}$$

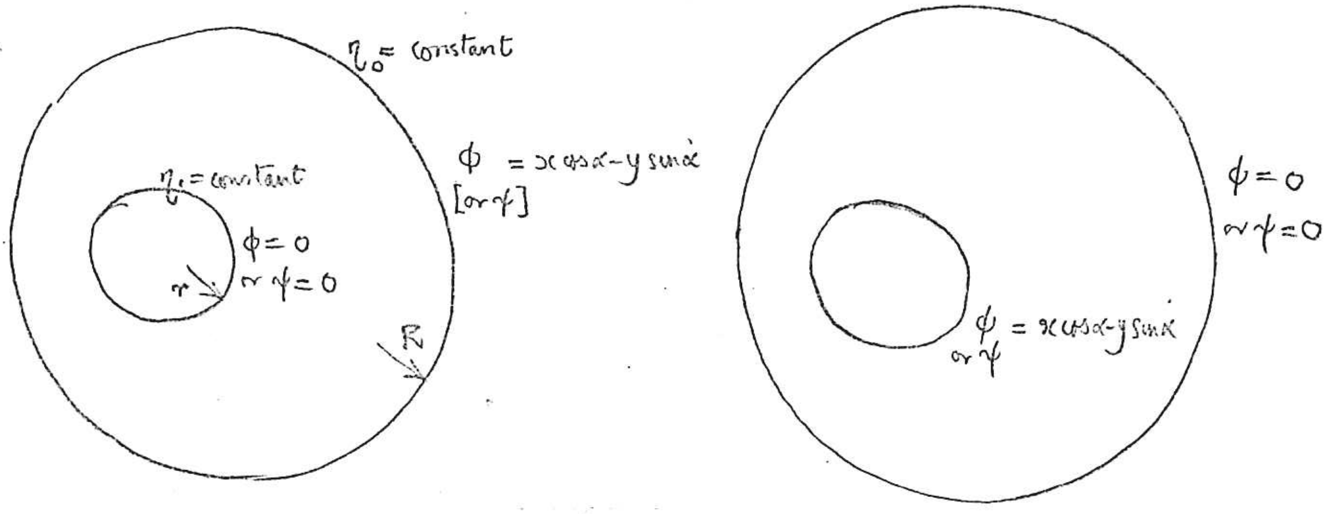
Substituting these values of R and I from (A.10) into (A.6), and retaining only terms up to second order in $h_\varphi$, $h_\psi$ then gives the formula (9.1) and (9.2) for $F_x$ and $F_y$.

## Appendix 2: The force potential for circular cylinders

The formula 3.18 for the force potential W may be written in two dimensions in terms of the potential $\varphi$ and the stream function $\psi$ as

$$W = \left(\frac{1}{2\mu_o}\right)\int_{\psi} - \int_{\varphi} \varphi \, d\psi \qquad (A.11)$$

with due regard to the direction of integration around different bodies ($\partial\varphi/\partial n$ in (3.18) always points out of the free space region). Consider the case of a cylindrical circular hole on which either the potential $\varphi$ or the stream-function $\psi$ is maintained at such a value that without any body inside the magnetic field is uniform (of unit value). Another circular body of infinite or zero permeability, i.e. on which $\varphi$ or $\psi$ is held zero, is inserted into the first circular hole. Then A.11 can be used to calculate the force potential W once the magnetic field functions $\varphi$ and $\psi$ have been found. The force potential should also be given by the equivalent system in which $\varphi$ or $\psi = 0$ on the original surfaces and $\varphi$ or $\psi$ is specified on the inserted surface. Because of the interchangeability of $\varphi$ and $\psi$ in two dimensions we need only consider the two cases where $\varphi$ is specified on the outer surface, and either $\varphi$ or $\psi = 0$ on the inner surface. The original situation and the equivalent situation are summarised in figure A1.




original system | equivalent system

Circular system inside a circular hole

Figure A1

The original uniform magnetic field has an arbitrary orientation $\alpha$, which turns out not to affect W. $\varphi$ and $\psi$ could be found by conformal mapping, no doubt, but it is also convenient to work in terms of the bi-cylinder coordinate system given, for example, in Moon and Spencer, page 89. Variables $\eta$ and $\xi$ are defined by

$$x = \frac{a \sinh \eta}{\cosh \eta - \cos \xi}$$

$$y = \frac{a \sin \xi}{\cosh \eta - \cos \xi} \tag{A.12}$$

The curves of constant $\eta$ are circles with centre (a coth $\eta$, 0) and radius a cosech $\eta$. Given the radius of the two circles R and r and the distance h between their centres the two values of $\eta$ $\eta_0$ and $\eta_1$ describing the circles and a can be found explicitly :

$$\cosh \eta_1 = \frac{R^2 - r^2 - h^2}{2hr}$$

$$\cosh \eta_0 = \frac{R^2 - r^2 + h^2}{2hR}$$

$$a^2 = \Pi \frac{(R \pm h \pm r)}{4h^2} \tag{A.13}$$

also $$\cosh(\eta_1 - \eta_0) = \frac{R^2 + r^2 - h^2}{2Rr} .$$

We also have the identity:

$$x \cos\alpha + y \sin\alpha = 2a \sum_{n=0}^{\infty} e^{-n\eta} \cos(n\xi - \alpha) - a \cos\alpha \qquad \eta > 0 \tag{A.14}$$

Laplace's equation becomes:

$$\frac{\partial^2\varphi}{\partial\eta^2} + \frac{\partial^2\varphi}{\partial\xi^2} = 0$$

admitting solutions of the type $e^{\pm n\eta} e^{\pm ni\xi}$. Matching solutions for $\varphi$ and the associated stream function $\psi$ can thus be written

$$\varphi = \Sigma \cos(n\xi - \alpha_n)\,[A_n \sinh n(\eta - \eta_1) + B_n \cosh n(\eta - \eta_1)]$$

$$\psi = \Sigma \sin(n\xi - \alpha_n)\,[A_n \cosh n(\eta - \eta_1) + B_n \sinh n(\eta - \eta_1)] \tag{A.15}$$

Thus, with the use of (A.14), ignoring the constant $a \cos \alpha$, the solution to the potential and stream function on the original system may be written:

$$\left.\begin{array}{l} \varphi \text{ given on } \eta = \eta_o \\ \\ \varphi = 0 \text{ on } \eta = \eta_1 \end{array}\right\} \quad \begin{array}{l} \varphi = 2a \displaystyle\sum_{n=0}^{\infty} \frac{e^{-n\eta_o}}{\sinh n(\eta_o - \eta_1)} \sinh n(\eta - \eta_1)\cos(n\xi - \alpha) \\ \psi = 2a \displaystyle\sum_{n=0}^{\infty} \frac{e^{-n\eta_o}}{\sinh n(\eta_o - \eta_1)} \cosh n(\eta - \eta_1)\sin(n\xi - \alpha) \end{array} \tag{A.16}$$

and

$$\left.\begin{array}{l} \varphi \text{ given on } \eta = \eta_o \\ \\ \psi = 0 \text{ on } \eta = \eta_1 \end{array}\right\} \quad \begin{array}{l} \varphi = 2a \displaystyle\sum_{n=0}^{\infty} \frac{e^{-n\eta_o}}{\cosh n(\eta_o - \eta_1)} \cosh n(\eta - \eta_1)\cos(n\xi - \alpha) \\ \psi = 2a \displaystyle\sum_{n=0}^{\infty} \frac{e^{-n\eta_o}}{\cosh n(\eta_o - \eta_1)} \sinh n(\eta - \eta_1)\sin(n\xi - \alpha) \end{array} \tag{A.17}$$

Then the force potential W can be calculated from (3.18) as $-\frac{1}{2\mu_o}\int_{\eta=\eta_o} \varphi\, d\psi$ giving for the two cases

$$W = -\frac{4\pi a^2}{2\mu_o} \sum_{n=0}^{\infty} n\, e^{-2n\eta_o} \left.\begin{array}{l}\coth \\ \tanh\end{array}\right\} n(\eta_1 - \eta_o) \tag{A.18}$$

The force potential may also be calculated for the equivalent system from (7.8); denoting this by $W'$ we have

$$W' = \mp \frac{1}{2\mu_o}\left[\int_{\eta=\eta_1} \varphi\, d\psi + \pi r^2\right] \tag{A.19}$$

The negative sign in (A.19) applies to the case where $\varphi$ is specified on $\eta = \eta_1$, the positive sign when $\psi$ is. The term $\pi r^2$ (constant in this case of an originally uniform field) comes from the integration of the term $\int \varphi \frac{\partial \varphi}{\partial n}$ in (3.19). Since the calculation of $W'$ is the same as the calculation of W with the roles of the parameters $\eta_o$ and $\eta_1$ reversed $W'$ can be written down immediately

from (A.18), namely:

$$W' = \mp \frac{1}{2\mu_o}\left[4\pi a^2 \sum_{n=0}^{\infty} n\, e^{-2n\eta_1} \begin{matrix}\coth\\ \tanh\end{matrix}\, n(\eta_1 - \eta_o) + \pi r^2\right] \qquad \text{(A.20)}$$

In all cases the signs can be checked since $\int \varphi\, d\psi$ should be positive as it corresponds to $\int (\nabla\varphi)^2\, d\tau$ in the general case. Using the relation (A.13) we can also check explicitly that:

$$W - W' = -\frac{1}{2\mu_o}\left[4\pi a^2 \sum_{n=0}^{\infty} n\left(e^{-2n\eta_o} \mp e^{-2n\eta_1}\right) \begin{matrix}\coth\\ \tanh\end{matrix}\, n(\eta_1 - \eta_o) \mp \pi r^2\right]$$

$$= -\frac{1}{2\mu_o}\left[4\pi a^2 \sum_{n=0}^{\infty} n\left(e^{-2n\eta_o} \pm e^{-2n\eta_1}\right) \mp \pi r^2\right]$$

$$= -\frac{1}{2\mu_o}\left[\pi a^2 (\operatorname{cosech}^2 \eta_o \pm \operatorname{cosech}^2 \eta_1) \mp \pi r^2\right]$$

$$= -\frac{1}{2\mu_o}\pi R^2$$

Thus $W - W'$ is constant, confirming that either may be used as the force potential. However $W'$ is more convenient since $W \to \infty$ as $R \to \infty$. An important feature of $W$ or $W'$ is that they are independent of $\alpha$, the orientation of the originally uniform field. For a given $R$ and $r$, $W$ and $W'$ depend only on $h$, the distance between the centres of the two circles, and thus the inner cylinder is in an axisymmetric potential well (or hump) although the field configuration itself is not axisymmetric.

In the special case $R - h \to \infty$, $r$ finite which represents a cylinder inserted into a uniform field far from any wall the condition $\frac{R-h}{r} \gg 1$ implies from (A.13) $\eta_1 - \eta_o \to \infty$, ie $\eta_1 \to \infty$ so that $a = r \operatorname{cosech} \eta_1 \to re^{\eta_1}/2$. Then from (A.18) we can check

$$W' \to \mp \frac{1}{2\mu_o}\left[\pi r^2 e^{2\eta_1} \sum_{n=0}^{\infty} ne^{-2n\eta_1} + \pi r^2\right]$$

$$\to \mp \frac{1}{2\mu_o}[\pi r^2 + \pi r^2]$$

which is consistent with the value of $k$ in (7.4).

An important special case is that of a cylinder near a flat wall. Then setting the distance from the wall $R - h - r = d$ and taking the limit $R, h \to \infty$, $d$ and $r$ being given we have:

$$\eta_o = 0$$

$$\cosh \eta_1 = \frac{d}{r} + 1$$

and

$$W' = \mp \frac{\pi r^2}{2\mu_o}\left[ 4 \sinh^2\eta_1 \sum_{n=0}^{\infty} n e^{-2n\eta_1} \begin{matrix}\coth \\ \tanh\end{matrix} (n\eta_1) + 1 \right]$$

$$= \frac{\pi r^2}{2\mu_o} \cdot k_p \quad \text{say} \qquad \text{(A.21)}$$

Again, as $\frac{d}{r} \to \infty$, $\eta_1 \to \infty$ and $k_p \to \mp 2$, $-2$ for the case of an iron cylinder and iron wall, and $+2$ for an insulating cylinder and iron wall.

After some manipulation of standard formulae the series giving $k_p$ can be summed explicitly in terms of elliptic functions. Using standard notation we define the argument $k$ of the elliptic functions $K$, $K'$, $E$ etc. by

$$\cosh \eta_1 = 1 + \frac{d}{r} = \cosh\left(\frac{\pi K'}{K}\right).$$

Then it may be shown that for the two representative cases,

$$k_p = -\sinh^2\eta_1\left[\frac{4K^2}{\pi^2}\left\{\left(1 - \frac{E}{K}\right) - \frac{1 + k^2}{3}\right\} + \frac{1}{3}\right]$$

and

$$k_p = \sinh^2\eta_1 \left[\frac{4K^2}{\pi^2}\left(\frac{1}{3} - \frac{k^2}{6}\right) - \frac{1}{3}\right] \qquad \text{(A.22)}$$

In particular, as $\frac{k}{r}$ and $\eta_1 \to \infty$, $k \to 0$ and $k_p \to \mp 2$ as before. As $\frac{d}{r} \to 0$, $\eta_1 \to 0$, $k \to 1$, and by expanding in terms of $\eta$(small) we find respectively

$$k_p = -\left[\frac{\pi^2}{3} - 2\sqrt{2}\sqrt{\frac{d}{r}} + 0\left(\frac{d}{r}\right)\right] \quad \left(\frac{\pi^2}{3} = 3.29\right)$$

$$\text{(A.23)}$$

and

$$k_p = \frac{\pi^2}{6} + \frac{2}{3}\left(\frac{\pi^2}{6} - 1\right)\left(\frac{d}{r}\right) + \frac{1}{9}\left(\frac{\pi^2}{10} - 3\right)\left(\frac{d}{r}\right)^2 + 0\left(\frac{d}{r}\right)^3$$

The first of these expressions, for the like on like case (iron cylinder on an iron wall) implies that the force $(-\nabla k)$ tends to infinity like $\left(\frac{d}{r}\right)^{-\frac{1}{2}}$ near

the wall, whereas for the other case (actually an insulating cylinder near an iron wall, but applying to an iron cylinder near an insulating wall with a change of sign) the force is finite.

Expansions in small k may also be done when $\frac{d}{r}$ is large and yield

$$k_p = \mp 2 \left[ 1 \pm \frac{1}{4\left(1+\frac{d}{r}\right)^2} + 0\left(\frac{r}{d}\right)^3 \right] . \qquad (A.24)$$

Thus the force $\sim \left(\frac{r}{d}\right)^3$ for large d with the same sign, representing the force between a dipole and its image in an iron wall. Whether the dipole is made of iron with $\varphi$ specified on its surface or of an insulator with $\frac{\partial\varphi}{\partial n}$ does not matter far from the wall.

These expansions give the shape of the curve approximately for the mixed (second) case, even at intermediate points, as explicit comparison with the exact result at $\frac{d}{r} = 1$ shows.

Appendix 3: Force estimates for bodies close to walls

Consider the force on an inert magnetic body as it approaches a flat wall. If the magnetic field was originally uniform then by considering the equivalent situation of a magnetised body near an inert wall as described in section 7 we see that the force is always attractive if the flat wall is iron, repulsive if the wall is magnetically insulating and its magnitude always increases as the body approaches the wall. The question we now examine is whether the force is finite or infinite when the body touches the wall. This depends on the shape of the body, and we restrict ourselves to a discussion of smoothly curved bodies near a flat wall of which the archetypes might be circular cylinders in two dimensions or spheres in three. The eight permutations of boundary conditions and initial field orientation are sketched below in figure A2.

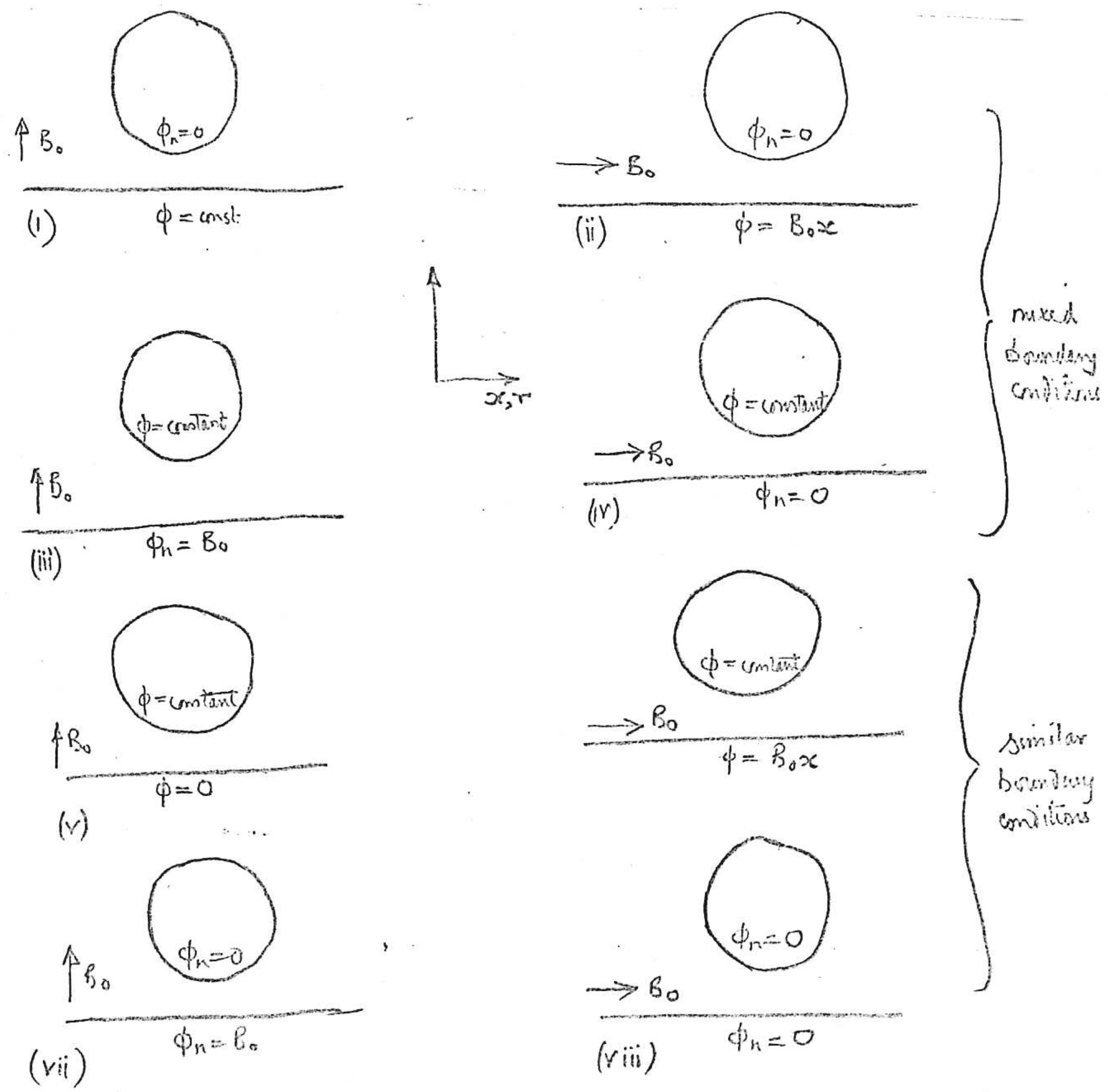



Figure A2 Inert bodies near flat walls

For the first four cases with mixed boundary conditions the field is fixed by the local boundary conditions on either side of the gap. $\varphi_n$ specifies the field across the gap and $\varphi$ (or its gradient) the field along it. In all these mixed cases the fields, and hence the Maxwell stresses calculated from them, are finite†. Thus the force on a body with these mixed boundary conditions when it touches the wall is finite, and its value depends on the shape of the body and the initial field orientation (in general).

However where the boundary conditions are of the same type, as in cases (v) to (viii) the fields close to the point of contact become indefinitely large - because of squeezing between the magnetically insulating surfaces as in cases (vii) and (viii) or because of a small mismatch in potential between the body and the wall as in cases (v) and (vi). The question is whether these infinite fields are sufficiently extensive or large to cause infinite forces.

We give the calculation of the force estimate for one of the cases, say case (v). If $\varphi_o$ is the (yet unknown) constant value of $\varphi$ on the body (assumed symmetrical in $x$), and if the gap is given by $g = h + \alpha x^2$ where $h$ is the distance of the bottom of the body from the wall and $\alpha^{-1}$ twice the radius of curvature here then the field $B$ is perpendicular to the gap and given by:

$$B = \frac{\varphi_o}{g} .$$

Then the flux emitted by the bottom surface passing through the body and being emitted into space is given by:

$$\text{flux} \sim \int_o^R (x) \frac{\varphi_o}{g} \, dx \qquad (A.30)$$

The force is found by integrating the Maxwell stresses $\frac{1}{2\mu} \int B^2$ over the bottom surface, on the assumption that the integral over the rest of the body will only have a finite contribution, so that

$$\text{force} \sim \int_o^R (x) \frac{\varphi_o^2}{g^2} \, dx \qquad (A.31)$$

where the extra $x$ in the integrals applies to the three dimensional case and the integration is taken up to a typical distance $R$, representing the size of the body.

If we make the substitutions $t = x\sqrt{\frac{\alpha}{h}}$ in these integrals then the upper

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† If the body is flat and has sharp edges it may be that field is locally infinite and sufficiently lop-sided to give rise to an infinite force when the body touches the wall; for example a flat iron disc on an insulating wall with an initially transverse field.

limit of integration becomes $T = R\sqrt{\frac{\alpha}{h}} \sim \sqrt{\frac{R}{h}}$, which is large. The flux integrals become:

$$\text{flux}_{2D} \sim \frac{\varphi_o}{\sqrt{\alpha h}}\int_o^T \frac{dt}{1+t^2} \sim \frac{\varphi_o}{\sqrt{\alpha h}}$$

$$\text{flux}_{3D} \sim \frac{\varphi_o}{\alpha}\int_o^T \frac{tdt}{1+t^2} \sim \frac{\varphi_o}{\alpha}\log\sqrt{\frac{R}{h}}$$

since the flux integral in three dimensions diverges for large T.

The force integrals become :

$$\text{force}_{2D} \sim \frac{\varphi_o^2}{\sqrt{\alpha h^3}}\int_o^T \frac{dt}{(1+t^2)^2} \sim \frac{\varphi_o^2}{\sqrt{\alpha h^3}}$$

$$\text{force}_{3D} \sim \frac{\varphi_o^2}{\alpha h}\int_o^T \frac{tdt}{(1+t^2)^2} \sim \frac{\varphi_o^2}{\alpha h} .$$

Now the argument is that as the body touches the wall the flux approaches a constant value determined by the flux emitted from the top of the body into the free space. The flux integrals then determine $\varphi_o$ and the force integrals become :

$$\text{force}_{2D} \sim \frac{1}{\sqrt{h}}$$
$$\text{force}_{3D} \sim \frac{1}{h\log^2\left(\frac{R}{h}\right)} . \qquad (A.32)$$

Similar arguments may be applied to the other cases, (vi), (vii) and (viii), though there are some differences in cases (vi) and (viii) since these are not axisymmetric in the three dimensional case. The conclusion is that all the two dimensional cases behave the same way as given by (A.32), as might have been expected given the interchangeability of potential and stream function in two dimensions and the general result that the force potential for circular cylinders is independent of the initial field orientation. In three dimensions cases (vi) and (vii) turn out to have the same behaviour, namely

$$\text{force}_{3D} \sim \log\left(\frac{R}{h}\right) \qquad (A.33)$$

and in case (viii) the force is finite, although the field does become infinite at the point of contact. For case (viii) it is strictly necessary to solve the two dimensional problem of magnetic field flow between the insulating surfaces with a "small" gap $g$ given by $g = h + \alpha r^2$ as above and a potential $\varphi \to B_o r \cos\theta$ at large $r$. The analysis is simplified when $h = 0$ (i.e. the body touches the wall), giving

$$\varphi \sim r^n \cos\theta$$

for small $r$ where $n = \sqrt{2} - \frac{1}{2} = 0.92$. The corresponding force integral $\int B^2 dS$ over the bottom surface of the sphere converges for small $r$ and thus even when the body touches the wall the (repulsive) force is finite.

Thus we conclude that for smoothly curved bodies approaching flat walls the force between the wall and body approaches a value which, in terms of distance $h$ from the wall, is

1) finite in all the mixed cases.

2) $\sim \frac{1}{\sqrt{h}}$ in all the two dimensional like on like cases.

3) is rather strongly infinite (A.32) for the case of an iron body on an iron wall with the field originally normal to the wall, in three dimensions.

4) is rather weakly infinite (A.33) for the cases of iron on iron with an originally parallel field, and insulator on insulator with an originally normal field, in three dimensions.

5) is finite for the case of insulators with an originally transverse field, in three dimensions.

Appendix 4 Behaviour of bodies in slowly varying fields

The table below sets out the behaviour of two types of bodies in magnetic fields which vary slowly in the sense of section (7). The bodies consist either of material with a fixed polarisation, or with permeability $\mu <$ or $> \mu_o$ (0 and $\infty$ say), and may either have a fixed orientation or be free to rotate so that the couple on them is zero. The effective dipole moment $\underline{M}$ is consistent with the formula for the force $\underline{F}$ and couple $\underline{G}$ on the body given by

$$\underline{F} = -\nabla W = (\underline{M}.\nabla)\underline{B}$$

$$\underline{G} = \underline{M} \wedge \underline{B}$$

where $\underline{B}$ is a free space magnetic field ($\nabla \wedge \underline{B} = 0$). $S_{ij}$ is the energy tensor (7.2) with eigenvalues $\lambda_i$ and is positive definite for a body of material $\mu < \mu_o$ and negative when $\mu > \mu_o$.

| Type of body | Potential W | Dipole moment $\underline{M}$ | $\nabla.\underline{F} = -\nabla^2 W$ |
|---|---|---|---|
| Polarisation fixed in strength and direction | $-\underline{M}.\underline{B}$ | $\underline{M}$ | 0 |
| Polarisation fixed in strength but free to rotate | $-MB$ | $M = M\frac{B}{B}$ | $\frac{M}{B^3}[B^2 B_{i,k} B_{i,k} - B_i B_j B_{i,k} B_{j,k}]$ |
| Magnetised body with fixed orientation | $\frac{1}{2} S_{ij} B_i B_j$ | $M_i = -S_{ij} B_j$ | $-S_{ij} B_{i,k} B_{j,k}$ |
| Magnetised body free to rotate | $\frac{1}{2}\lambda_m B^2$ | $M = -\lambda_m B$ | $-\lambda_m B_{i,k} B_{i,k}$ |

A uniformly polarised body free to rotate might have M of either sign, but would be unstable in rotation if M was negative (since for a given $|M|$, W is a minimum when $M > 0$ and the body would flip round). $\nabla.\underline{F}$ is positive when $M > 0$ as the expression in brackets is positive definite so that a freely rotating polarised body is not stable in translation.

A magnetised body when permitted to rotate also adopts a position in which $\underline{W}$ is a minimum, i.e. it aligns itself with the principal direction of $S_{ij}$ in which the eigenvalue $\lambda_m$ is least. This implies that $\nabla.\underline{F}$ in translation of a freely rotating body is always as large as it can be, but still negative for body with $\mu < \mu_o$. When the two types of bodies are free to rotate and have the same instantaneous dipole moment $\underline{M} = -\lambda_m \underline{B}$, $\nabla.\underline{F}$, has the same sign but is numerically bigger for the magnetised body.

## Appendix 5 A constant flux two coil system

A brief analysis of a two coil system designed to suspend a piece of iron is given below. The arrangement is like a Helmholz pair, with the flux through the coils kept constant and the iron suspended in between them. The iron is treated as a small spherical or circular (2-D) body in the manner of section 7.

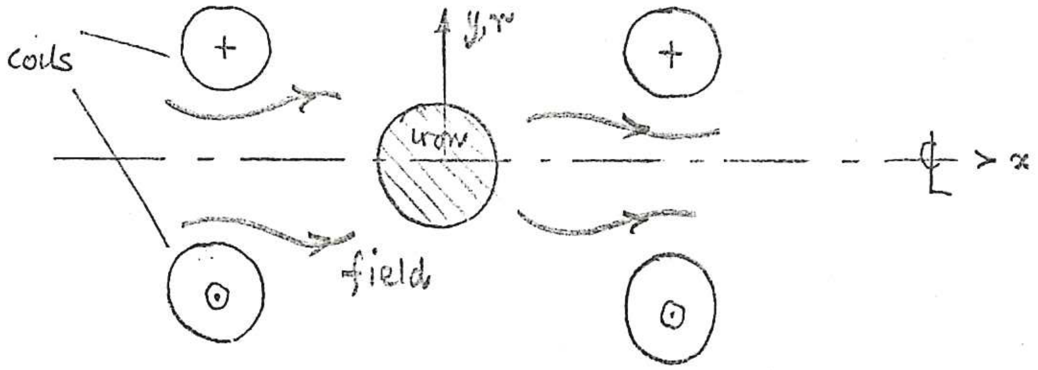


figure A3

First we consider the behaviour of the system when the coils are run at constant current. Suppose the magnetic field on the axis is $B_x = B(x)$ . This, in the symmetric Cartesian (x, y) or axisymmetric (r, x) cases, specifies the field everywhere (apart from singularities). To calculate the equivalent mechanical energy potential $W = \frac{k}{2\mu_0} \underline{B}^2 V$ (section 7) a knowledge of $\underline{B}^2$ is required, where $\underline{B}$ is the free space magnetic field generated by the coils.

By expanding about the axis for small y or r we can get the following formulae for $B_x$ , $B_r$ and related quantities, all in terms of the axial field B(x) and its derivatives in the axial (x) direction, which are denoted by dashes :

$$\left.\begin{aligned} B_y &= - y B' + O(y^3) \\ B_r &= - \frac{r}{2} B' + O(r^3) \\ \underline{B}^2 &= B^2 - y^2(B''B - B'^2) + O(y^4) \\ &= B^2 - \frac{r^2}{2} (B''B - \frac{B'^2}{2}) + O(r^4) \\ \nabla B^2 &= [2BB' , - 2y(B''B - B'^2)] \\ &= [2BB' , - r(B''B - \frac{B'^2}{2})] \\ \nabla^2 B^2 &= 4B'^2 \text{ or } 3B'^2 \quad \text{respectively} \end{aligned}\right\} \qquad (A\ 50)$$

Thus $\nabla.\underline{F} \sim \nabla^2 B^2$ is always $> 0$ since $k < 0$ for an iron body, except where $B' = 0$ ($B' = 0$ at the equilibrium position at the centre. This is a consequence of the small body theory being in error by higher order terms in the body size to magnetic field scale length ratio. $\nabla.\underline{F}$ would be strictly positive to higher order.) By symmetry (x, y) or (r, x) are the principal directions and the stiffnesses are given by

$$F_{x,x} \sim VB^2_{xx} = 2V(BB'' + B'^2)$$

$$F_{y,y} \sim VB^2_{yy} = -2V(B''B - B'^2) \qquad \text{(A 51)}$$

or

$$F_{r,r} = -V\left(B''B - \frac{B'^2}{2}\right).$$

Thus where $B' = 0$ the stiffnesses in the axial and transverse directions $\sim BB''$ and are of opposite sign. At the Helmholz spacing $B'' = 0$ and all the forces disappear to this order; at separations rather greater $B'' > 0$ and the system is transversely stable but axially unstable. At lesser spacings the situation is reversed.

It would be possible to treat the constant flux case by considering the equivalent system of a variably magnetised body between two coils with no flux through them. In fact we adopt a different approach and analyse the behaviour of the system, assigning self-inductances $L_1$, $L_2$ to the coils, and mutual inductance $M$, where all are defined in the presence of the iron. Then the force on the body $\underline{F}$ can be written

$$\underline{F} = -\nabla\,(\text{magnetic energy})_{\text{constant flux}}$$

$$= +\nabla\,(\text{magnetic energy})_{\text{constant current}}$$

and in general

$$\underline{F} = \frac{1}{2}\left(I_1^2\,\nabla L_1 + I_2^2\,\nabla L_2 + 2I_1I_2\nabla M\right). \qquad \text{(A 53)}$$

Using the following symmetry relations at the equilibrium point

$$L_1 = L_2 = L\,,\quad \frac{\partial L_1}{\partial x} = -\frac{\partial L_2}{\partial x}\,,\quad \frac{\partial^2 L_1}{\partial x^2} = \frac{\partial^2 L_2}{\partial x^2}\,,\quad \frac{\partial L_1}{\partial y} = \frac{\partial L_2}{\partial y} = 0 \quad \frac{\partial^2 L_1}{\partial y^2} = \frac{\partial^2 L_2}{\partial y^2}\,,$$

$\frac{\partial M}{\partial x} = 0$, $\frac{\partial M}{\partial y} = 0$ and putting $I_1 = I_2 = I$ we find for a constant current situation

$$(\nabla . \underline{F})_I = I^2 \nabla^2 (L + M) \qquad \text{(A 54)}$$

If the fluxes in the coils are held constant, then $I_1$ and $I_2$ will vary, and extra terms involving $\nabla I_1$ and $\nabla I_2$ will appear in $\nabla . \underline{F}$. Using the above symmetry relations and the conditions $\nabla \Phi_1 = \nabla \Phi_2 = 0$ where

$$\Phi_1 = L_1 I_1 + M I_2$$
$$\Phi_2 = MI + L_2 I_2 \qquad \text{(A 55)}$$

we find

$$(\nabla . \underline{F})_\Phi = (\nabla . \underline{F})_I - \frac{2I^2 (\nabla L)^2}{L - M} \qquad \text{(A 56)}$$

(A 54) and (A 56) apply to the individual stiffness $F_{x,x}$ and $F_{y,y}$ or $F_{r,r} + \frac{F_r}{r} \sim 2F_{r,r}$. It can be seen that the extra term in (A 56) stabilises the system in the axial directions only since $L_y = L_r = 0$.

We now identify the constant current formula (A 54) with the previous calculation. Using the fact that in a single coil calculation the force $\underline{F}$ on the iron body can either be written

$$\underline{F} = \frac{1}{2} I^2 \nabla L$$

or

$$\underline{F} = - \nabla \frac{kV}{2\mu_o} B^2$$

we find

$$L = L_o - \frac{kV}{2\mu_o} \frac{B^2}{I^2} \qquad \text{(A 57)}$$

where $L_o$ is the inductance in the absence of the iron and $B^2$ the field caused by a current I in the coil at the position of the iron. $\frac{B^2}{I^2}$ is a function of the coil goemetry, and is not sensitive to the coil spacing in the present arrangement (unlike $B''$). For the stiffness in the constant flux system we now find from (A 56), (A 51) and $(\nabla L)^2$ from (A 57) that

$$\frac{F_{x,x}}{I^2} \sim 2VBB'' - \frac{cV^2}{L_o - M_o}$$

$$\frac{F_{y,y}}{I^2} \sim 2VBB'' , \quad \frac{F_{r,r}}{I^2} \sim - VBB'' \qquad \text{(A 58)}$$

where the values of B have been normalised to unit current in the coils, and

c is a constant from (A 57) incorporating the value of $\nabla(\frac{B^2}{I^2})$ for the single coil. Since the body has been assumed small $L - M \simeq L_o - M_o$.

Equation (A 58) suggests that to achieve stable suspension $B''$ must be positive, i.e. the separation must be slightly greater than for Helmholz coils, and so arranged that, say, the stiffnesses in the two directions are the same. The bigger V and the smaller $L_o$ the greater the range of coil spacings for which stabilisation can be achieved. Actually calculations for a large circular iron cyclinder indicate that there is an optimum V, since a large piece of iron makes the positive contribution of the higher order terms to $(\nabla.\underline{F})_I$ mentioned above significant. The destabilising effect of the iron and the distribution of the stiffness depends on the uniformity of the original field, whereas the stabilising effect of the coils depends more on their global arrangement. The stiffness distribution effect is represented in (A 58) by the way $B''$ changes sign at the Helmholz coil spacing; the positive value of the sum of the stiffnesses $(\nabla.\underline{F} > 0)$ has been lost in higher order terms, and the stabilising effect of the coils is represented by the second term in (A 58) which, although small, only varies slowly as the coil spacing is changed. The accompanying paper[5] describes an experiment which appears to confirm these ideas, but it must be admitted that it works in régime where the iron body can no longer be regarded as small.