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THE EFFECT OF FLUCTUATIONS ON THE RESPONSE FUNCTION OF PLASMAS AND LIQUIDS

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Abstract

Turbulent, high density and low temperature plasmas are alike in having a non-negligible level of density fluctuations. A dielectric function of such plasmas is derived which incorporates both the self-correlation effects introduced by Dupree and the distinct particle correlation effects introduced by Ichimaru. Comparisons are made with theories of the response function of liquids.

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1. Introduction

In the conventional (R.P.A.) derivation of the plasma dielectric function particle correlations (microscopic density fluctuations) are ignored. This is correct for a plasma in or close to thermodynamic equilibrium provided the plasma parameter, $\gamma \equiv (n\lambda_D^3)^{-1}$, which measures the ratio of potential to kinetic energy, is small, which is so for most plasmas of laboratory and astrophysical interest. This approach will fail in at least two circumstances. Firstly, at low temperatures and/or high densities γ may not be small. This is certainly so for electrons in metals and may perhaps be so for plasmas produced by laser compression. (The need to treat the electrons quantum mechanically does not affect the substance of this argument. The main complicating feature is the existence of correlations due to "statistics" as well as those due to dynamical effects). Secondly, if the growth of microinstabilities results in plasma turbulence the magnitude of γ is not a guide to the importance of fluctuations. In these cases the effects of correlations must be included in the calculation of the dielectric function in a non-perturbative way.

Essentially the same problem occurs in the theory of the dynamical structure of liquids and it has attracted similar solutions. It may seem odd that there should be any connection between the theory of plasma turbulence and the theory of liquids. However, the term "turbulence" is normally used by plasma theorists to indicate little more than a state in which random fluctuations in microscopic quantities have to be taken seriously. On this definition a liquid at rest in thermodynamic equilibrium is turbulent!

Two quadratic correlation functions may figure in the calculation. The self-correlation function $G_s(\underline{r},\underline{v},t\,;\underline{r}',\underline{v}',t')$ is the probability that a particle is at position $(\underline{r},\underline{v})$ in phase space at time t , given that <u>the same</u> particle was at $(\underline{r}',\underline{v}')$ at an earlier time t' . The distinct particle correlation function $g_d(\underline{r},\underline{r}')$ is the probability that there is a particle at $\underline{r}$, given that there is another particle at $\underline{r}'$ (at the same time). Dielectric functions incorporating one or other (but not both) of these correlation functions have appeared in the plasma physics literature, derived in very dissimilar fashions. The first object of this paper is to derive a dielectric function incorporating both correlation functions and compare it with the expressions arising in existing theories. The second object is to compare this dielectric function with analogues which appear in theories of liquids, which are similar in principle but significantly different in detail.

The basic ideas may be explained rather simply by a brief account of existing theories. The R.P.A. dielectric function contains an expression

$$k^{-2} \int d\underline{v} \, \frac{\underline{k}.\frac{\partial F}{\partial \underline{v}}}{(\omega + \underline{k}.\underline{v})} \, , \tag{1}$$

and the elements of interest to us are $(\omega + \underline{k}.\underline{v})^{-1}$ and k^{-2} . The expression $\delta(\underline{v} - \underline{v}')(\omega + \underline{k}.\underline{v})^{-1}$ is the Fourier-Laplace transform of the self-correlation function, G_s , of a freely moving particle. The theories of Dupree (1966,1967), Weinstock (1969), and others (in which $(\omega + \underline{k}.\underline{v})^{-1}$ is replaced by some kind of diffusive propagator) may be regarded as attempts to incorporate G_s more realistically. These theories are too well known to require description here.

The distinct particle correlation function may be introduced by considering the term k^{-2} in (1). This is the Fourier transform of the unshielded Coulomb potential. It appears plausible that when correlations are strong k^{-2} should be replaced by a shielded potential, involving g_d . Singwi et al (1968) (discussing electrons in metals) described a way in which this can be done.

To find the dielectric function, one applies a small external electric field, E^{ext} , and calculates the density response. Singwi et al point out that the evolution of the one particle distribution function is described by

$$\left(\frac{\partial}{\partial t}+\underline{v}\cdot\frac{\partial}{\partial \underline{r}}\right) f_1(\underline{r},\underline{v},t) - \frac{e^2}{m}\iint d\underline{r}'d\underline{v}' \,\nabla_{\underline{r}}\, |\underline{r}-\underline{r}'|^{-1}\cdot\frac{\partial}{\partial \underline{v}} f_2(\underline{r},\underline{v},\underline{r}',\underline{v}',\underline{t})$$

$$= \frac{e}{m} E^{ext}(\underline{r},t)\cdot\frac{\partial f_1}{\partial \underline{v}}(\underline{r},\underline{v},t) \;, \tag{2}$$

(i.e. the exact first member of the BBGKY hierarchy). f_2 is the two-particle distribution function.

In the conventional theory one puts

$$f_2 = f_1(\underline{r},\underline{v},t)\, f_1(\underline{r}',\underline{v}',t) \tag{3}$$

and recovers the Vlasov equation, but Singwi et al suggest that in the presence of strong correlations it might be more reasonable to use

$$f_2 = f_1(\underline{r},\underline{v},t)\, f_1(\underline{r}',\underline{v}',t)\, g_d(\underline{r}-\underline{r}') \tag{4}$$

where g_d is held fixed throughout the calculation. (Of course equation (4) is exact in the absence of E^{ext}, assuming there are no velocity correlations (as is the case in thermodynamic equilibrium).) The effect is to replace the Coulomb force by a shielded force:

$$\underline{\nabla}\frac{1}{|r-r'|} \rightarrow g_d(\underline{r}-\underline{r}')\,\underline{\nabla}\cdot\frac{1}{|\underline{r}-\underline{r}'|} \tag{5}$$

This is equivalent to the replacement

$$k^{-2} \rightarrow k^{-2}\int \frac{d^3 j}{(2\pi)^3}\,\frac{\underline{k}\cdot\underline{j}}{j^2}\, g_d(\underline{k}-\underline{j}) \tag{6}$$

The same result was also obtained by Ichimaru (1974). (A closer look at these theories is postponed to section 5.)

This analysis does not contain the slightest trace of the Dupree-style orbit diffusion. This is not remedied in more elaborate versions of the same idea such as a truncation similar to equation (4) at the level of three-particle correlations (Ichimaru 1974) or allowing for the change in g_d in the external field via its functional derivative with respect to the density perturbation (which is related to three-particle correlations, for systems in thermodynamic equilibrium) (Vashista and Singwi, 1972). The reason is that the functions appearing in these theories are single time or equal time functions, whereas G_s is a two-time function.

It is also the case that in the Dupree-type orbit diffusion theories, referred to above, distinct particle correlations do not appear. In fact the particles are treated as independently propagating in a given random electric field (whose covariance is put equal to the covariance of the true electric field).

In sections 2, 3 and 4 a dielectric function incorporating both correlation functions is derived. The method used is an extension of one due to Hubbard (1967). A more satisfying, but more difficult, method might be to set up a hierarchy, analogous to the BBGKY hierarchy, in which many-times self and distinct particle correlation functions appear explicitly and truncate it by a two-times version of equation (4).

2. The Response of a Single Particle

For simplicity we restrict ourselves to a single species calculation. We begin by singling out for special attention one particle. We refer to it as the "red" particle and refer to the others as the "blue" particles.

Imagine that the external field couples only to the red particle. Let $f_R(\underline{r},\underline{v},t)$ be the density in phase space of the red particle,

$$f_R(\underline{r},\underline{v},t) \equiv \delta(\underline{r} - \underline{r}_R(t))\ \delta(\underline{v} - \underline{v}_R(t)) \ . \qquad (7)$$

The evolution of f_R is described by

$$\left(\frac{\partial}{\partial t} + \underline{v}\cdot\frac{\partial}{\partial \underline{r}} + \frac{e}{m}\ \underline{E}_B\cdot\frac{\partial}{\partial \underline{v}} \right) f_R = -\frac{e}{m}\underline{E}^{ext}\ \frac{\partial f_R}{\partial \underline{v}} \ , \qquad (8)$$

where $\underline{E}_B$ is the electric field produced by the blue particles.

The external field will not affect $\underline{E}_B$ directly, but it will do so indirectly via the coupling of the blue particles to the (perturbed) red particle. This effect must be very small, so we ignore it. Then Δf_R, the change in f_R linear in $\underline{E}^{ext}$, satisfies

$$\left(\frac{\partial}{\partial t} + \underline{v}.\frac{\partial}{\partial \underline{r}} + \frac{e}{m} \underline{E}_B.\frac{\partial}{\partial \underline{v}} \right) \Delta f_R = -\frac{e}{m} \underline{E}^{ext} . \frac{\partial f_R}{\partial \underline{v}} \tag{9}$$

Equation (9) may be formally solved, thus

$$\Delta f_R(\underline{r},\underline{v},t) = -\frac{e}{m} \int_0^t dt' \int d\underline{r}' \int d\underline{v}' \; \Gamma_s(\underline{r},\underline{r}',\underline{v},\underline{v}',t,t') \; \underline{E}^{ext}(\underline{r}',t').\frac{\partial}{\partial \underline{v}'} f_R(\underline{r}',\underline{v}',t') , \tag{10}$$

where the Green function Γ_s satisfies

$$\left(\frac{\partial}{\partial t} + \underline{v}.\frac{\partial}{\partial \underline{r}} + \frac{e}{m} \underline{E}_B.\frac{\partial}{\partial \underline{v}} \right) \Gamma_s(\underline{r},\underline{r}',\underline{v},\underline{v}'t,t') = \delta(\underline{r}-\underline{r}') \; \delta(\underline{v}-\underline{v}') \; \delta(t-t') \tag{11}$$

Since the red particle is identical in properties to the blue particles we have

$$G_s \equiv < \Gamma_s > , \tag{12}$$

$$n F(\underline{v}) \equiv N < f_R(\underline{r},\underline{v},t) > , \tag{13}$$

where n is the number density of particles, N is the total number of particles, $F(\underline{v})$ is the one-particle velocity distribution function (assumed independent of $\underline{r}$ and t, for simplicity) and the angular brackets indicate an ensemble average.

Using equations (12) and (13) we can average equation (10), with the result

$$< \Delta f_R(\underline{r},\underline{v},t) > = -\frac{ne}{Nm} \int_0^t dt' \int d\underline{r}' \int d\underline{v}' \; G_s(\underline{r},\underline{r}'\underline{v},\underline{v}'t,t')\underline{E}^{ext}(\underline{r}',t').\frac{\partial F(\underline{v}')}{\partial \underline{v}'} \tag{14}$$

In performing the average we have ignored the correlation between Γ_s and f_R. This is equivalent to supposing that the history of $\underline{E}_B(\underline{r},\tau)$ for $t > \tau > t'$ is almost independent of the fact that the red particle was at $(\underline{r}',\underline{v}')$ at time t'. That this is so is very plausible but difficult to demonstrate.

We now obtain the density response of the red particle by integrating both sides of equation (14) over $\underline{v}$. If a Fourier-Laplace transformation is also performed the result is

$$< \Delta \rho_R(\underline{k},\omega) > = - \frac{ne}{Nm} \iint d\underline{v}\ d\underline{v}'\ G_s(\underline{k},\underline{v},\underline{v}'\omega)\ \underline{E}^{ext}(\underline{k},\omega) . \frac{\partial F(\underline{v}')}{\partial \underline{v}'} \tag{15}$$

(In obtaining equation (15) we assumed that $G_s = G_s(\underline{r} - \underline{r}', \underline{v}, \underline{v}', t - t')$, i.e. that the system is statistically homogeneous and stationary.)

3. The Effective Field

At this point we restore the coupling between the external field and the blue particles and recognise that the "external" field experienced by the red particle is the sum of the true external field and the field due to the perturbed density of blue particles, $\Delta \rho_B(\underline{r},t)$. This latter has a mean and a fluctuating part and we propose to include only the first of these $< \Delta \rho_B(\underline{r},t) >$, ignoring a correlation whose significance is difficult to assess.

If the red particle had not been singled out we would have had

$$< \Delta \rho_B(\underline{r}', t) > = \frac{(N-1)}{N} < \Delta \rho(\underline{r}',t) > \tag{16}$$

$$\approx < \Delta \rho(\underline{r}', t) > , \tag{17}$$

(where $\Delta \rho$ is the perturbation in the density of all the particles) in which case the effective field $\hat{E}(\underline{r},t)$ (that is, the mean field experienced by the red particle at $(\underline{r},t)$) would be

$$\hat{\underline{E}}(\underline{r},t) = \underline{E}^{ext}(\underline{r},t) - e\int d\underline{r}' \frac{\partial}{\partial \underline{r}} \frac{<\Delta \rho(\underline{r}',t)>}{|\underline{r}-\underline{r}'|} \tag{18}$$

Hubbard suggested that the existence of distinct particle correlations can be taken into account by replacing equation (18) by

$$\hat{\underline{E}}(\underline{r},t) = \underline{E}^{ext}(\underline{r},t) - e\int d\underline{r}'\, g_d(\underline{r}-\underline{r}') \frac{\partial}{\partial \underline{r}} \frac{<\Delta \rho(\underline{r}',t)>}{|\underline{r}-\underline{r}'|} \tag{19}$$

Clearly, the only information about the red particle taken into account in equation (19) is its position. Velocity correlations and time-delayed effects are ignored.

We now take the Fourier-Laplace transform of equation (19), assuming for simplicity that the system is isotropic so that g_d is a function of $|\underline{r}-\underline{r}'|$ alone, to obtain

$$\hat{\underline{E}}(\underline{k},\omega) = \underline{E}^{ext}(\underline{k},\omega) + 4\pi e <\Delta \rho(\underline{k},\omega)> i\frac{\underline{k}}{k^2}\int d^3j\, \frac{\underline{k}.\underline{j}}{j^2}\, g_d(\underline{k}-\underline{j}) \tag{20}$$

4. The Dielectric Function

We now make the fundamental approximation: that the red particle responds to the effective field in the same way as it did to the fictitious external field of section 2. So we replace $\underline{E}^{ext}$ in equation (15) by $\hat{\underline{E}}$. If, in addition, we use the fact that

$$<\Delta \rho_R> = N^{-1} <\Delta \rho>, \tag{21}$$

we obtain, after a slight rearrangement

$$<\Delta \rho(\underline{k},\omega)> \left\{ 1 + i\frac{\omega_p^2}{k^2}\left[\int \frac{d^3j}{(2\pi)^3}\frac{\underline{k}.\underline{j}}{j^2}\, g_d(\underline{k}-\underline{j})\right] \iint d\underline{v}\, d\underline{v}'\, G_s(\underline{k},\underline{v},\underline{v}',\omega)\, \underline{k}.\frac{\partial F}{\partial \underline{v}'}(\underline{v}') \right\}$$

$$= -\frac{ne}{m}\int d\underline{v}'\, G_s(\underline{k},\underline{v},\underline{v}',\omega)\, \underline{E}^{ext}(\underline{k},\omega).\frac{\partial F}{\partial \underline{v}'}(\underline{v}') \tag{22}$$

We now introduce a potential $e\varphi^{ext}$ for the external field and define the susceptibility $\chi(\underline{k},\omega)$ by

$$\chi(\underline{k},\omega) \equiv \frac{<\Delta \rho(\underline{k},\omega)>}{\varphi^{ext}(\underline{k},\omega)} \tag{23}$$

Then, from equation (22), we obtain

$$\chi(\underline{k},\omega) = \frac{-i\frac{\omega_p^2}{4\pi}\iint d\underline{v}\, d\underline{v}'\, G_s(\underline{k},\underline{v},\underline{v}',\omega)\, \underline{k}.\frac{\partial F}{\partial \underline{v}'}(\underline{v}')}{1+i\,\omega_p^2\,\psi(\underline{k})\iint d\underline{v}\, d\underline{v}'\, G_s(\underline{k},\underline{v},\underline{v}',\omega)\, \underline{k}.\frac{\partial F}{\partial \underline{v}'}(\underline{v}')} , \tag{24}$$

where

$$\psi(\underline{k}) \equiv k^{-2}\int \frac{d^3 j}{(2\pi)^3}\, \frac{\underline{k}.\underline{j}}{j^2}\, g_d(\underline{k}-\underline{j}) \tag{25}$$

If $\epsilon(\underline{k},\omega)$, the dielectric function, is defined in terms of the susceptibility in the conventional way,

$$\frac{1}{\epsilon(\underline{k},\omega)} \equiv \frac{4\pi}{k^2}\,\chi(\underline{k},\omega)+1 , \tag{26}$$

we obtain

$$\epsilon(\underline{k},\omega) = \frac{1+i\,\omega_p^2\,\psi(k)\iint d\underline{v}\, d\underline{v}'\, G_s(\underline{k},\underline{v},\underline{v}',\omega)\, \underline{k}.\frac{\partial F}{\partial \underline{v}'}(\underline{v}')}{1+i\,\omega_p^2[\psi(\underline{k})-k^{-2}]\iint d\underline{v}\, d\underline{v}'\, G_s(\underline{k},\underline{v},\underline{v}',\omega)\, \underline{k}.\frac{\partial F}{\partial \underline{v}'}(\underline{v}')} \tag{27}$$

Equation (27) is the main result of this paper.

5. Comparison with other Theories

(a) We can recover the R.P.A. dielectric function by replacing g_d and G_s by the corresponding expressions for a perfect gas, that is

$$g_d(\underline{r}) = 1 \tag{28}$$

(implying

$$g_d(\underline{k}) = (2\pi)^3 \delta(\underline{k}) \tag{29}$$

and hence

$$\psi(\underline{k}) = k^{-2} \qquad) \, , \tag{30}$$

and

$$G_s(\underline{r}-\underline{r}',\underline{v},\underline{v}',t-t') = \delta(\underline{r}-\underline{r}'-\underline{v}'(t-t'))\,\delta(\underline{v}-\underline{v}') \tag{31}$$

or, when Fourier-Laplace transformed,

$$G_s(\underline{k},\underline{v},\underline{v}',\omega) = \frac{\delta(\underline{v}-\underline{v}')}{i(\omega+\underline{k}.\underline{v})} \quad . \tag{32}$$

The approximations (30) and (32) reduce equation (27) to

$$\epsilon(\underline{k},\omega) = 1+\frac{\omega_p^2}{k^2}\int d\underline{v}\ \frac{\underline{k}.\frac{\partial F}{\partial \underline{v}}}{(\omega+\underline{k}.\underline{v})} \quad . \tag{33}$$

(b) If the distinct-particle correlations alone are neglected (by putting $\psi(k) = k^{-2}$) there results a dielectric function similar to those of Dupree and Weinstock:

$$\epsilon(\underline{k},\omega) = 1+i\,\frac{\omega_p^2}{k^2}\iint d\underline{v}\,d\underline{v}'\ G_s(\underline{k},\underline{v},\underline{v}',\omega)\underline{k}.\frac{\partial F(\underline{v}')}{\partial \underline{v}'} \tag{34}$$

It is clear, either on the basis of a formal treatment of the linear random equation (11) (Orszag and Kraichnan (1967), Cook (1974)) or on the basis of physical intuition, that G_s describes diffusion (in a loose sense) in velocity and configuration space. However equation (34) only reduces to the Dupree-Weinstock form if the diffusion in velocity space is ignored, i.e. if G_s has the form

$$G_s(\underline{k},\underline{v},\underline{v}',\omega) = \delta(\underline{v}-\underline{v}')\,R(\underline{k},\underline{v},\omega) \tag{35}$$

(Of course Dupree and Weinstock put forward specific approximations for the resonance function $R(\underline{k},\underline{v},\omega)$.) It is difficult to see how equation (35) could be valid, other than in the guiding centre model.

(c) If the self-correlations are neglected (by replacing G_s by its perfect gas limit as in (a) above) equation (27) reduces to

$$\epsilon(\underline{k},\omega) = \frac{1+\omega_p^2\,\psi(k)\displaystyle\int d\underline{v}\,\frac{\underline{k}.\frac{\partial F}{\partial \underline{v}}}{(\omega+\underline{k}.\underline{v})}}{1+\omega_p^2[\psi(k)-k^{-2}]\displaystyle\int d\underline{v}\,\frac{\underline{k}.\frac{\partial F}{\partial \underline{v}}}{(\omega+\underline{k}.\underline{v})}} \tag{36}$$

This is identical to the dielectric function found by Singwi et al (1968) (their equation (14)). (Note that Singwi et al use functions $S(k)$ and $G(k)$ which are related to our $\psi(k)$ and $g_d(r)$ as follows

$$S(k) = 1+n\int d\underline{r}\,e^{i\underline{k}.\underline{r}}(g_d(\underline{r})-1) \tag{37}$$

$$-G(k) = k^2(\psi(k)-k^2)\ .) \tag{38}$$

(d) The dielectric function found by Ichimaru (1974) is also closely related to ours. Ichimaru was attempting to calculate not $\epsilon(\underline{k},\omega)$ but $g_d(\underline{r})$ for a turbulent plasma (which he did by truncating the BBGKY hierarchy at the level of triple correlations, using Kirkwood's approximation). However in the course of this analysis a function occurs which Ichimaru interprets as a dielectric function. It is

$$\epsilon(\underline{k},\omega) = 1+\frac{\omega_p^2}{k^2}\,t_k\int d\underline{v}\,\frac{\underline{k}.\frac{\partial F}{\partial \underline{v}}}{(\omega+\underline{k}.\underline{v})} \tag{39}$$

where

$$t_k = 1+\frac{1}{n}\int\frac{d^3j}{(2\pi)^3}\,\frac{\underline{k}.\underline{j}}{j^2}\,S(\underline{k}-\underline{j}) \tag{40}$$

This result was obtained earlier by Hubbard (1967).

We can obtain equations (39) and (40) from equation (36) by ignoring in the denominator the distinction between $\psi(k)$ and k^{-2} and expressing the $\psi(k)$ which occurs in the numerator in terms of $S(k)$, the structure factor, by means of equation (37).

(e) We turn now to theories designed to explain collective motions in classical liquids. As explained in the introduction, these theories share with the theories of turbulent, dense or low-temperature plasmas the common feature of the important role assigned to interparticle correlations. Moreover since the shape of the intermolecular potential plays no role in these theories they can be turned into theories of the response function of plasmas by substitution of the Coulomb potential. We shall discuss only two theories, that of Singwi et al (1970) and that of Hubbard and Beeby (1969).

Singwi et al begin by writing the susceptibility in the form

$$\chi(\underline{k},\omega) = \frac{\chi_s(\underline{k},\omega)}{1 - 4\pi h(k)\,\chi_s(\underline{k},\omega)} \tag{41}$$

(where we have altered their notation to avoid confusion). $\chi_s(\underline{k},\omega)$ and $h(k)$ are regarded as unknown functions.

This is of the same form as our equation (24) and it will be seen that our choice for $\chi_s(\underline{k},\omega)$ is

$$\chi_s(\underline{k},\omega) = - i\frac{\omega_p^2}{4\pi}\int d\underline{v}\int d\underline{v}'\; G_s(\underline{k},\underline{v},\underline{v}',\omega)\,\underline{k}.\frac{\partial F}{\partial \underline{v}'}(\underline{v}') \tag{42}$$

Singwi et al use the fluctuation - dissipation theorem to express $h(k)$ in terms of $S(k)$ and $\chi_s(\underline{k},0)$. Of course this procedure works only for systems in thermodynamic equilibrium. The only relevant comparison we can make is between their choice for $\chi_s(\underline{k},\omega)$ and ours (equation 42).

They propose that for real ω

$$I_m\,\chi_s(\underline{k},\omega) = -\frac{\pi n e^2}{k_B T}\,\omega\, g_s(\underline{k},\omega)\ , \tag{43}$$

where $g_s(\underline{k},\omega)$ is the Fourier-Fourier transform of the self-correlation function, k_B is Boltzmann's constant and the factor e^2 has been inserted to compensate for its absence in our definition of $\chi(\underline{k},\omega)$. Equation (43) is a relationship between the "self" quantities analogous to the fluctuation-dissipation theorem.

$g_s(\underline{k},\omega)$ is defined as

$$g_s(\underline{k},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \int d\underline{r}\, e^{i\underline{k}.\underline{r}+i\omega t}\, g_s(\underline{r},t) \ . \tag{44}$$

(Note the factor 2π and the fact that a Fourier-Fourier transform has been taken.) The self-correlation function can be obtained from $G_s(\underline{r},\underline{v},\underline{v}',t)$ by integrating over the final velocity and over the initial velocity weighted with the velocity distribution function:

$$g_s(\underline{r},t) = \int d\underline{v} \int d\underline{v}'\, G_s(\underline{r},\underline{v},\underline{v}',t)\, F(\underline{v}') \tag{45}$$

Combining equations (44) and (45) and using a symmetry property of G_s we can easily show that the right hand side of equation (43) is

$$-\frac{\pi n e^2}{k_B T}\omega\, g_s(\underline{k},\omega) = -\frac{n e^2}{k_B T}\int d\underline{v}\int d\underline{v}'\, \omega\, F(\underline{v}')\, \mathrm{Re}\, G_s(\underline{k},\underline{v},\underline{v}',\omega) \tag{46}$$

The imaginary part of equation (42) is, if $F(\underline{v})$ is chosen to be a Maxwellian with temperature T ,

$$I_m\, \chi_s(\underline{k},\omega) = \frac{n e^2}{k_B T}\int d\underline{v}\int d\underline{v}'\, (\underline{k}.\underline{v}')\, F(\underline{v}')\, \mathrm{Re}\, G_s(\underline{k},\underline{v},\underline{v}',\omega) \tag{47}$$

Comparing equations (46) and (47) we see that equation (43) is satisfied only if G_s contains (for real ω) a term $\delta(\omega+\underline{k}.\underline{v}')$, i.e. only if G_s is replaced by its perfect gas limit. Thus the present theory and that of Singwi et al do not agree even for systems in thermodynamic equilibrium, except when correlations are sufficiently unimportant that both theories reduce to the R.P.A. theory.

(f) A theory in which the fluctuation-dissipation theorem (or analogues thereof) is not used in the calculation of $\chi(\underline{k},\omega)$, and which can thus be applied to systems not in thermodynamic equilibrium, is that of Hubbard and Beeby (1969). The method used by these authors is radically different from that used in this paper and we outline it briefly below.

Hubbard and Beeby begin by considering Newton's equations of motion for the trajectory functions $\underline{r_i}(t)$. In the presence of an infinitesimal external potential,

$$-i\,\underline{k}\,e\,\varphi^{ext}(\underline{k},\omega)\,e^{-i\underline{k}.\underline{r}-i\omega t} ,$$

the trajectories are changed to $\underline{r_i}(t)+\underline{\Delta r_i}(t)$ and the equations of motion are

$$m\frac{d^2}{dt^2}\left(\underline{r_i}(t)+\underline{\Delta r_i}(t)\right) = -\sum_{j\neq i}\underline{\nabla}V(\underline{r_i}(t)+\underline{\Delta r_i}(t)-\underline{r_j}(t)-\underline{\Delta r_j}(t))$$
$$-i\,\underline{k}\,e\,\varphi^{ext}(\underline{k},\omega)\,e^{-i\underline{k}.[\underline{r_i}(t)+\underline{\Delta r_i}(t)]-i\omega t} \qquad (48)$$

where $V(\underline{r})$ is the interparticle potential (the Coulomb potential, in the present case).

The next step is to linearise equation (48) and use it to form an equation for the (linear) density perturbation,

$$\delta\rho(\underline{k},t) = \sum_i e^{-i\underline{k}.\underline{r_i}(t)}[-i\underline{k}.\underline{\Delta r_i}(t)] \qquad (49)$$

Hubbard and Beeby found

$$\delta\rho(\underline{k},t) = -\frac{e}{m}k^2\varphi^{ext}(\underline{k},\omega)\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt''\sum_i e^{i\underline{k}.[\underline{r_i}(t)-\underline{r_i}(t'')]}\,e^{-i\omega t''}$$
$$-\frac{1}{m}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt''\sum_i e^{i\underline{k}.\underline{r_i}(t)}\sum_j i\underline{k}.\underline{\underline{\Phi}}_{ij}(t'').\underline{\Delta r_j}(t'') , \qquad (50)$$

where

$$\underline{\underline{\Phi}}_{ij}(t) \equiv \underline{\nabla}\underline{\nabla} V(\underline{r}_i(t) - \underline{r}_j(t)) \tag{51}$$

for $i \neq j$, and

$$\underline{\underline{\Phi}}_{ii}(t) = - \sum_{j \neq i} \underline{\nabla}\underline{\nabla} V(\underline{r}_i(t) - \underline{r}_j(t)) \tag{52}$$

When equation (50) is averaged the first term on the right hand side gives a term proportional to the self-correlation function $g_s(\underline{r},t)$. The second term is more troublesome and to deal with it Hubbard and Beeby used the following decoupling procedure (in fact their procedure is rather more longwinded, but amounts to the same thing)

$$< e^{i\underline{k}.\underline{r}_i(t)} \Phi_{ij}(t'') \Delta r_j(t'') > = < e^{i\underline{k}.[\underline{r}_i(t) - \underline{r}_j(t)]} \Phi_{ij}(t'') e^{i\underline{k}.\underline{r}_j(t'')} \Delta \underline{r}_j(t'') > \tag{53}$$

$$\approx < e^{i\underline{k}.[\underline{r}_i(t) - \underline{r}_j(t'')]} \Phi_{ij}(t'') > < e^{i\underline{k}.\underline{r}_j(t'')} \Delta \underline{r}_j(t'') > \tag{54}$$

This approximation does not seem to have a clear physical interpretation.

The first factor in angular brackets on the right hand side of equation (54) involves the probability that (i) at $\tau = 0$ there is a particle at the origin and another at $\underline{r}'$ and (ii) at time $\tau = t$ the particle at the origin has moved to $\underline{r}$. Hubbard and Beeby assume that this probability is approximately $g_s(\underline{r},t)\, g_d(\underline{r}')$.

After making these approximations they find a susceptibility which, for the one-species isotropic plasma considered in this paper, we may write in the form

$$\chi^{HB}(\underline{k},\omega) = \frac{\chi_s^{HB}(\underline{k},\omega)}{1 - 4\pi \psi^{HB}(k)\, \chi_s^{HB}(\underline{k},\omega)} \tag{55}$$

where

$$\chi_s^{HB}(\underline{k},\omega) = -\frac{\omega_p^2}{4\pi} k^2 \int_0^\infty dt \int d\underline{r}\, t\, g_s(\underline{r},t)\, e^{-i\underline{k}.\underline{r}-i\omega t} \tag{56}$$

and

$$\psi^{HB}(k) = \frac{1}{k^2}\cdot\frac{k^\alpha k^\beta}{4\pi k^2} \int d\underline{r} \left(\nabla^\alpha \nabla^\beta \frac{1}{r}\right)\left(1 - e^{-i\underline{k}.\underline{r}} \right) g_d(\underline{r}) \tag{57}$$

Let us first compare the self-response functions of the present paper and Hubbard and Beeby. Performing the integrals in equation (56) and using equation (45) we obtain

$$\chi_s^{HB}(\underline{k},\omega) = i\frac{\omega_p^2}{4\pi} k^2 \frac{\partial}{\partial\omega} \int d\underline{v} \int d\underline{v}'\, G_s(\underline{k},\underline{v},\underline{v}',\omega)\, F(\underline{v}') \tag{58}$$

This should be compared with our result (equation 42):-

$$\chi_s^c(\underline{k},\omega) = -i\frac{\omega_p^2}{4\pi} \int d\underline{v} \int d\underline{v}'\, G_s(\underline{k},\underline{v},\underline{v}',\omega)\, \underline{k}.\frac{\partial F(\underline{v}')}{\partial \underline{v}'} \tag{59}$$

The difference is

$$\chi_s^c(\underline{k},\omega) - \chi_s^{HB}(\underline{k},\omega) = i\frac{\omega_p^2}{4\pi} k^2 \int d\underline{v} \int d\underline{v}'\, F(\underline{v}') \left(\frac{\underline{k}}{k^2}\cdot\frac{\partial}{\partial \underline{v}'} - \frac{\partial}{\partial\omega} \right) G_s(\underline{k},\underline{v},\underline{v}',\omega) \tag{60}$$

It is clear that the two expressions for χ_s coincide only when G_s takes its limiting free-particle form (equation 32).

In comparing the local field corrections, $\psi^{HB}(k)$ and $\psi_k^c(k)$, it is convenient to use $S(\underline{k})$ rather than $g_d(\underline{k})$, in order to avoid divergences arising from the long range of the Coulomb potential (alternatively a uniform neutralising background may be employed). In terms of $S(\underline{k})$ the local field correction of Hubbard and Beeby is

$$\psi^{HB}(\underline{k}) = k^{-2}\left\{ 1 + \frac{1}{n}\int \frac{d^3j}{(2\pi)^3} \frac{(\underline{k}.j)^2}{k^2 j^2} \left[S(\underline{k}-\underline{j}) - S(-j) \right] \right\} , \tag{61}$$

whereas we found (cf. equation (40))

$$\psi^{c}(\underline{k}) = k^{-2} \left\{ 1 + \frac{1}{n} \int \frac{d^3 j}{(2\pi)^3} \frac{\underline{k}.\underline{j}}{j^2} S(\underline{k} - \underline{j}) \right\} \tag{62}$$

These expressions coincide only when the distinct particle correlations are neglected, i.e. $S(\underline{k}) = 1$.

6. Discussion

Clearly the forms of the dielectric function discussed in section 5 parts (a) - (d), though originally derived by a variety of methods, may be obtained by approximating equation (27) in various ways.

The second theory of Singwi et al (section 5(e)) can be applied only to systems in thermodynamic equilibrium. In addition the central assumption (that a fluctuation-dissipation-like relationship obtains between $\chi_s(\underline{k},\omega)$ and $g_s(\underline{k},\omega)$) is of unproven value even in thermodynamic equilibrium.

The Hubbard-Beeby theory and the theory advanced in this paper are very similar in principle though they differ significantly in detail. Since in both treatments various statistical assumptions must be made it is not surprising that the results should turn out rather differently.

The discrepancy between $\psi^{c}(k)$ and $\psi^{HB}(k)$ is not too significant. It is in the decoupling procedures leading to these functions that both theories are at their weakest. In defence of $\psi^{c}(k)$ it can be argued that the decoupling which leads to it has a clear physical interpretation with a certain plausibility. On the other hand it has been shown (Niklasson 1974) that in the limit of very high ω (i.e. the very short time response) $\chi(\underline{k},\omega)$ is exactly given by

$$\chi(\underline{k},\omega) = \frac{\chi_s^{(o)}(\underline{k},\omega)}{1 - 4\pi\psi^{HB}(k)\,\chi_s^{(o)}(\underline{k},\omega)} , \tag{63}$$

where $\chi_s^{(o)}(\underline{k},\omega)$ is the self-response function of a perfect gas.

The discrepancy between $\chi_s^c(\underline{k},\omega)$ and $\chi_s^{HB}(k,\omega)$ is more serious because it persists even in a model situation (independent particles in a prescribed random potential) where there are no distinct particle correlations and because of the importance of orbit diffusion effects in theories of plasma turbulence.

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