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THE STABILITY OF DRIFT WAVES IN AXISYMMETRIC TOROIDAL PLASMAS

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ABSTRACT

The two dimensional eigenvalue equation for electrostatic drift waves in axisymmetric toroidal geometry is investigated. A model version relevant for the Culham Levitron is constructed, and solutions obtained when poloidal variations of shear, curvature and magnetic field are included. General criteria for the existence of localised undamped eigenmodes are established and it is found that for sufficiently strong modulations of various equilibrium quantities, the stabilising effect of magnetic shear is completely nullified. Equivalent criteria are obtained for the large aspect ratio tokamak.

Investigation of the electron Landau resonance in strongly modulated magnetic fields, indicates that for electron drift waves the growth rate will be only logarithmically weaker than in the equivalent slab-model calculation.

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I. INTRODUCTION

In the last few years there has been a considerable amount of theoretical analysis[1-4] of the stability of drift waves in high temperature plasmas. In most of these calculations the authors assume that the plasma geometry is one-dimensional; that is the plasma is modelled by a slab in which only variations in the direction of the density gradient are taken into account. However, as will be shown in this paper the slab model does not adequately describe the stability properties of drift waves in a toroidal plasma. A theory of electrostatic perturbations in axisymmetric toroidal plasmas has been given by Connor and Hastie[5]. These authors obtain a partial differential equation for the potential Φ as a function of the magnetic potential χ (the poloidal variable) and ψ the poloidal flux (radial variable). In the present paper we obtain the drift wave solutions of this equation. These solutions are different in character to solutions of the one-dimensional slab and have different stability properties. In particular we show that the prediction of simple slab theory that sufficient shear stabilises drift waves may be incorrect when variations around the magnetic surfaces are taken into account.

The local field line shear can be clearly identified in the partial differential equation and is $\frac{\partial\nu}{\partial\psi}$ where $\nu = B_T/RB_p^2$. The poloidal variation of $\frac{\partial\nu}{\partial\psi}$ and of the other terms in the equation have been examined for the Culham Levitron and for tokamaks. The local shear in the Levitron is found to vary very strongly with the poloidal variable χ, in fact if one splits the shear into an average part and a periodic part in the form $\frac{\partial\nu}{\partial\psi} = \frac{\partial\bar{\nu}}{\partial\psi} + \frac{\partial\tilde{\nu}}{\partial\psi}\cos\chi$, then the amplitude of the periodic part $\frac{\partial\tilde{\nu}}{\partial\psi}$ is almost as large as the average shear $\frac{\partial\bar{\nu}}{\partial\psi}$. Several other terms in the differential equation are found to vary strongly with χ, the finite larmor radius correction term $k_\perp^2\, a_\perp^2$ and the curvature and ∇B drift terms $k_y\, v_{Dy}$ and $k_x\, v_{Dx}$, the latter are dominating for long wavelengths $k_y\, a_i \ll 1$. By comparing the magnitude of the various terms we are able to reduce the equation in Section III to a simpler model equation for Φ as a function of the poloidal angle θ and radial distance x. The coefficients of this equation being modelled by a constant part plus a θ dependent part which is either a sine or cosine.

For tokamaks, expanding all of the coefficients of the differential equation in powers of the inverse aspect ratio $\varepsilon(= r/R)$ gives a simpler differential equation for Φ as a function of θ and x. This equation is very similar to the model equation for the Levitron; however rather than being a model equation it is exact in the limit if small ε.

When all the varying parts of the shear etc are neglected the model equation is identical to the differential equation of the slab model. The solution of this equation was first obtained in the strong shear limit by Pearlstein and Berk. The solution consists of a set of independent modes which are centred on the mode rational surfaces, the modes are all radiating energy away from their respective mode rational surfaces. This loss of energy means that the wave amplitude would be damped, in the absence of any destabilising effects, and the damping rate is found to increase linearly with the shear strength. The marginal stability criterion is obtained by balancing this damping decrement against the growth terms coming from the electron Landau resonances. The criterion for stability is (in the long wavelength limit) $\frac{r_n}{L_s} > \left(\frac{m_e}{m_i}\right)^{1/3} (k_y a_i)^{4/3}$ and thus for sufficiently strong shear these modes are stable.

When the magnetic field properties such as shear etc vary around the magnetic surface we show in Section III that it is possible to localise a mode in both the poloidal and radial directions. That is modes are found for which the potential decays in all directions in the x, θ plane away from the mode centre. Thus there is no energy radiated from this mode and so it will have no damping decrement, and hence may easily be driven unstable by the electron resonance terms.

Modes which are localised in the θ direction have been discussed previously by Adam, Laval and Pellat(6) and Coppi(3). The first authors used the poloidal variation of the drift terms to localise their mode, while Coppi used the poloidal variation of the finite larmor radius correction terms to achieve localisation. The structure in the radial direction was not calculated in either paper.

In Section III of this paper local solutions of the model equation of Section II, which contains all the significant poloidal variations for both Levitrons and Tokamaks, are found. A condition is derived for the localisation of drift waves in both θ and x. We find that a very small periodic variation of the drift terms is sufficient to localise a mode particuarly for long wavelengths ($k_y a_i << 1$), and indeed the tokamak should be unstable to these long wavelength modes. For the variation of shear alone to localise a mode the amplitude of the periodic part $\tilde{\nu}'$ must exceed $\overline{\nu}'/\sqrt{2}$, where $\overline{\nu}'$ is the average shear strength ($\overline{\nu}' \equiv dq/d\psi$, where q is the safety factor). Although this condition is unlikely to be satisfied in a tokamak it is satisfied on most of the surfaces of the Culham Levitron where as mentioned earlier $\tilde{\nu}' \sim \overline{\nu}'$ (see Fig.1).

The structure of the remainder of the paper is as follows: in Section IV the connection between these local modes and the modes of a toroidal plasma in which there is no poloidal variation is established. Then in Section V the stability of low frequency drift waves whose frequency is less than the ion bounce frequency is discussed. In this limit the ions experience the <u>average</u> magnetic shear and a stability criterion similar to that of the slab model is derived.

Finally in Section VI, the electron landau resonance is examined in detail to determine whether the basic driving terms become weaker in the presence of poloidal variations of equilibrium quantities. We conclude that for electron drift waves the electron landau resonance is only weakly modified in magnitude. Of particular importance is the fact that the mode is now driven by electrons of <u>thermal</u> energy and <u>any</u> pitch angle. Thus subtle neo-classical distortions of the electron distribution function in the neighbourhood of the trapped-passing boundary in velocity space, will have little effect on the mode.

II. REDUCTION OF DIFFERENTIAL EQUATION TO A SIMPLER FORM

In this section suitable model differential equations for the propagation of drift waves in Levitrons and tokamaks are derived. The partial differential equation for electrostatic perturbations in an axisymmetric torus has been given by Connor and Hastie[5]. Their analysis is for vacuum magnetic fields which is appropriate for a levitron, but as we show later in this section the analysis may be trivially modified for a tokamak, however, first we derive a suitable model equation for the Culham Levitron. Using the same notation as Connor and Hastie the equation for ϕ_o (Eq.(57)[5]) for waves whose perturbed potential $\Phi = \phi_o(\psi, \chi)e^{i(\ell\phi - \ell\int^{\chi}\nu(\psi_o)d\chi - \omega t)}$

$$\frac{T_i}{M\,\omega_{ci}^2}\left[(RB_p)^2\frac{\partial^2\phi_o}{\partial\psi^2} - \left(\frac{\ell B}{RB_p}\right)^2\phi_o\right] - \frac{T_i}{M\,\omega^2}B^2\left[\frac{B_p^2}{B^2}\left(\frac{\partial}{\partial\chi} + i\ell\nu'(\psi-\psi_o)\right)\right]^2\phi_o$$

$$-\frac{T_i\,\ell\,B_p^2}{e_i\,\omega}\left[\frac{B^2}{B_p^2}\frac{\partial}{\partial\psi}\left(\frac{1}{B^2}\right)\phi_o - \frac{iRB_T}{\ell}\frac{\partial}{\partial\chi}\left(\frac{1}{B^2}\right)\frac{\partial\phi_o}{\partial\psi}\right]$$

$$+\left(\tau + \frac{\omega_{*e}}{\omega}\right)^{-1}\left(\frac{\omega_{*e}}{\omega} - 1\right)\phi_o = 0 \qquad \ldots (1)$$

where the dependent variables in this equation, χ and ψ are the magnetic potential of the poloidal field and the poloidal flux respectively, defined in the usual manner, $\underline{B}_p = \underline{\nabla}\chi = \underline{\nabla}\psi \wedge \underline{\nabla}\theta$, B_p is the poloidal field, B_T the toroidal field, $\nu' = \partial/\partial\psi\,(B_T/RBp^2)$ is the local field line shear, $\tau = T_e/T_i$ and the electron diamagnetic drift frequency $\omega_e^* = T_e\,\ell n'/en$. In deriving Eq.(1) from Eq.(57) of reference (5) Φ_o was replaced by $\phi_o \exp[i\,\ell\int(\nu + \ )d\chi - iS - i\,\ell\int^{\chi}\nu(\psi_o)d\chi]$, and the sign of ω was changed for convenience. The origin of the terms of Eq.(1) is as follows, the first term is the finite larmor radius correction term $k_\perp^2\,a_\perp^2$, the second term is the ion inertia term $k_\parallel^2\,v_i^2/\omega^2$, the third and fourth terms are the curvature drift terms $k_y\,v_{Dy}/\omega$ and $k_x\,v_{Dx}/\omega$.

In Eq.(1) the parameters R, B, B_p, ν' are all functions of χ and ψ and so the coefficients of Eq.(1) are also functions of χ and ψ. The usual slab approximation is to neglect all dependence on χ of these coefficients but retain the simple ψ dependence in the shear term and also in ω_{*e} in the last term of Eq.(1). In the following treatment we shall expand ψ about a magnetic surface ψ_o as in the slab model but the χ dependence of the coefficients is modelled by an average part plus a periodic part which will be taken as a sine or cosine. First, however, we transform equation (1) into a form in which the second derivatives have constant coefficients. We multiply equation (1) by $(\hat{B}/\hat{R}\,\hat{B}_p)^2$ where $\hat{B} = B/B_o$, $\hat{R} = R/R_o$, $\hat{B}_p = B_p/B_o$, and B_o, B_{po}, R_o are constant fields and a scale length characteristic of the configuration, and introduce new independent variables L and x given by

$$x = (\psi-\psi_o)/R_oB_{po} \quad ; \quad L = \int^{\chi} \frac{\hat{R}d\chi}{B_p} . \qquad \ldots (2)$$

Here L is a stretched length in the poloidal direction, and x measures length in the radial direction (i.e. along the density gradient). Then defining a (constant) larmor radius and wavelength by

$$a_i^2 = \frac{T_i}{m_i\,\omega_{co}^2} \quad ; \quad k = \frac{\ell\, B_o}{R_o\, B_{po}} , \qquad \ldots (3)$$

equation (1) becomes

$$a_i^2 \frac{\partial^2\phi_o}{\partial x^2} - \left(\frac{r_n\, B_{po}}{ka_i\,\tau\, B_o}\right)^2 \left[\frac{\partial}{\partial L} + i\,\ell\,\frac{B_p}{\hat{R}}\frac{\partial\nu}{\partial x}x - \frac{1}{2}\frac{\partial g}{\partial L}\right]^2 \phi_o + i\,\frac{r_n}{k\tau}\frac{\partial}{\partial L}(\ln \hat{B}^2)\,\frac{1}{\hat{R}\hat{B}_p}\frac{\partial\phi_o}{\partial x}$$

$$- \left[k^2a_i^2\left(\frac{\hat{B}}{R^2B_p^2}\right)^2 + \frac{r_n}{\tau}\frac{\partial \ln B^2}{\partial x}\left(\frac{\hat{B}}{RB_p}\right)^2 + \left(\frac{r_n}{ka_i\tau}\frac{B_{po}}{B_o}\right)^2\left(\frac{1}{2}\frac{\partial^2 g}{\partial L^2} - \frac{1}{4}\left(\frac{\partial g}{\partial L}\right)^2\right)\right.$$

$$\left. - \frac{\left(\frac{\omega_{*e}}{\omega} - 1\right)}{1+\tau}\left(\frac{\hat{B}}{RB_p}\right)^2\right]\phi_o = 0 \qquad \ldots (4)$$

where $r_n^{-1} = \frac{1}{n}\frac{dn}{dx}$ and we have chosen $R_oB_o = RB_T$.

Now we split the coefficients of equation (4) into an average and a periodic part and <u>model</u> the variation on L as follows.

$$\frac{r_n}{\tau}\frac{\partial}{\partial L}(\ln \hat{B}^2)\,\frac{1}{\hat{R}\hat{B}_p} = -\,\tilde{\mu}\sin\theta \qquad \ldots (5)$$

$$\left(\frac{\hat{B}}{R^2B_p^2}\right)^2 = \bar{\alpha} + \tilde{\alpha}\cos\theta \qquad \ldots (6)$$

$$\frac{r_n}{\tau}\frac{\partial(\ln \hat{B}^2)}{\partial x}\left(\frac{\hat{B}}{RB_p}\right)^2 = -\bar{\beta} + \tilde{\beta}\cos\theta \qquad \ldots (7)$$

$$\frac{L_o\ \ell\ B_p}{2\pi\ k\ \hat{R}}\ \frac{\partial\nu}{\partial x} = \bar{\gamma} + \tilde{\gamma}\cos\theta \qquad \ldots (8)$$

$$\frac{L_o}{2\pi}\ \frac{\partial g}{\partial L} = \tilde{g}\sin\theta \qquad \ldots (9)$$

$$\left(\frac{\hat{B}}{RB_p}\right)^2 = \bar{\alpha}_1 + \tilde{\alpha}_1\cos\theta \qquad \ldots (10)$$

where $\theta = 2\pi L/L_o$ with $L_o = \oint \hat{R}\ \frac{d\chi}{B_p}$.

For comparison with these forms the actual functions of equations (5)-(10) are shown in Figs.1-5 for a typical magnetic surface in the Culham Levitron. With the above forms equation (4) becomes

$$a_i^2\frac{\partial^2\phi}{\partial x^2} + iA\sin\theta\ a_i\frac{\partial\phi}{\partial x} - D\left(\frac{\partial}{\partial\theta} + iSx/a_i\right)^2\phi$$

$$+ (\Omega - \Gamma x^2/a_i^2 + E\cos\theta + F\cos 2\theta)\phi = 0 \qquad \ldots (11)$$

where

$$A = \frac{-1}{ka_i}(\tilde{\mu} + \tilde{\gamma}\ k^2a_i^2) \qquad \ldots (12)$$

$$D = (2\pi\ r_n\ B_{po}/ka_i\ \tau\ B_o\ L_o)^2 \qquad \ldots (13)$$

$$S = ka_i\bar{\gamma} \qquad \ldots (14)$$

$$E = \tilde{\beta} - k^2a_i^2\ \tilde{\alpha} + \frac{\delta\omega}{\omega_{*e}(1+\tau)}\ \tilde{\alpha}_1 + \frac{1}{2}\tilde{g}\ D \qquad \ldots (15)$$

$$\Omega = -\frac{\delta\omega}{\omega_{*e}(1+\tau)}\ \bar{\alpha}_1 - \bar{\beta} - k^2a_i^2\ \bar{\alpha} - F \qquad \ldots (16)$$

$$F = -\frac{1}{2}\ ka_iA\tilde{\gamma} - \frac{1}{8}\tilde{g}^2\ D \qquad \ldots (17)$$

$$\Gamma(\theta) = a_i^2(\bar{\alpha}_1 + \tilde{\alpha}_1\cos\theta)/(1+\tau)s^2 \qquad \ldots (18)$$

where we have expanded $\omega_{*e} = \omega_{*eo}\ (1 - x^2/s^2)$.

In deriving equation (11) the periodic part of the shear was separated from the average by transforming to the new potential $\phi = \phi_o\exp(ikx\tilde{\gamma}\sin\theta)\exp(\tilde{g}/2\cos\theta)$. Solutions of equation (11) are given in the next two sections. In the remainder of this section the corresponding equation is derived for a large aspect ratio tokamak.

For the tokamak we again use an electrostatic treatment and this is valid for $\beta < m_e/m_i$. In practice β exceeds this limit and the electromagnetic corrections should be added. For expediency these corrections will be ignored here. The differential equation for electrostatic perturbations in a non-vacuum axisymmetric toroidal magnetic field is similar to Eq.(1) except $\frac{d\chi}{Bp^2}$ is replaced by $Jd\chi$ and $\nu = IJ/R^2$, where J is the Jacobian of the transformation from cartesian coordinates to χ, ψ, θ coordinates and is given by the differential equation

$$\frac{\partial}{\partial\psi} \ln (J\, B_p^2) = - \left(\frac{dp}{d\psi} + \frac{I}{R^2} \frac{dI}{d\psi} \right) / B_p^2 \qquad \ldots (19)$$

For tokamaks of arbitrary cross section this equation (tokamak equivalent of Eq.(1)) for the potential may be treated in the same manner as Eq.(1) for the Levitron. However in the case of a large aspect ratio tokamak of circular cross section a more detailed analytic treatment is possible. Expanding all of the coefficients in the inverse aspect ratio, and retaining only leading terms in r/R we obtain the following differential equation for ϕ_o

$$a_i^2 \frac{\partial^2\phi_o}{\partial x^2} - k^2 a_i^2 \left[1 - 2 \left(\frac{r}{R_o} + 2\Delta' \right) \cos\theta \right] \phi_o - \left(\frac{r_n}{R_o q \tau k a_i} \right)^2 \left[\frac{\partial}{\partial\theta} + ikx \left(\frac{rq'}{q} \right) \left(1 - \gamma\cos\theta \right) \right]^2 \phi_o$$

$$+ 2i \frac{r_n}{R_o \tau k} \sin\theta \frac{\partial\phi_o}{\partial x} + \frac{2r_n}{R_o \tau} \cos\theta\, \phi_o - \left(\frac{r_n}{R_o q \tau k a_i} \right)^2 \left(\frac{r}{R_o} + \frac{1}{2}\Delta' \right) \cos\theta\, \phi_o$$

$$- \frac{\left(\frac{\omega_{*e}}{\omega} - 1 \right)}{(1+\tau)} \left[1 - 2 \left(\Delta' + \frac{r}{R_o} \right) \cos\theta \right] \phi_o = 0 \qquad \ldots (20)$$

where $k \equiv \ell q/r$, $q = rB_o/R_oB_\theta$, Δ is the radial shift in the magnetic surfaces satisfying

$$\frac{d}{dr} (r\, B_\theta^2\, \Delta') = rB_\theta^2/R_o - 2r^2/R_o\, (dp/dr) \qquad \ldots (21)$$

and the prime in the above equation and in Eq.(20) denotes d/dr. In deriving Eq.(20) radial derivatives on equilibrium quantities were neglected in comparison with those acting on ϕ_o. In the ion inertia term the quantity

$\frac{rq'}{q}(1 - \gamma\cos\theta)$ contains the local shear, and γ is given by

$$\gamma = \frac{r}{R_o}\left[2 + 3\frac{R_o}{r}\Delta' + 2\left(\frac{rq'}{q}\right)^{-1}\left(1 - \frac{rp'}{B_\theta^2} - \frac{R_o}{r}\Delta'\right)\right] \qquad \ldots (22)$$

Thus the appropriate equation for drift waves in a large aspect ratio tokamak may also be put in the form of Eq.(11) with the constants A, D, E, F and S having the following interpretation:

$$A = \frac{1}{ka_i}\left[\frac{2r_n}{R_o\tau} + k^2a_i^2\,\gamma\left(\frac{rq'}{q}\right)\right] \quad ; \quad D = \left(\frac{r_n}{R_o q\tau ka_i}\right)^2$$

$$E = \frac{2r_n}{R_o\tau} + 2k^2a_i^2\left(\frac{r}{R_o} + 2\Delta'\right) - \left(\frac{r_n}{R_o q\tau\, ka_i}\right)^2\left(\frac{r}{R_o} + \frac{1}{2}\Delta'\right) + \frac{2}{(1+\tau)}\left(\frac{\delta\omega}{\omega_{*e}}\right)\left(\Delta' + \frac{r}{R_o}\right)$$

$$F = \frac{r_n}{R_o\tau}\gamma\left(\frac{rq'}{q}\right) + \frac{1}{2}\gamma^2\left(\frac{rq'}{q}\right)^2 k^2a_i^2 \quad ; \quad S = ka_i\left(\frac{rq'}{q}\right) \qquad \ldots (23)$$

$$\Omega = -k^2a_i^2 - \frac{r_n}{R_o\tau}\gamma\left(\frac{rq'}{q}\right) - \frac{\delta\omega}{\omega_{*e}(1+\tau)}$$

where $\omega = \omega_{*e} + \delta\omega$, and $r_n = \left(\frac{1}{n}\frac{dn}{dr}\right)^{-1}$

To summarise, in this section we have derived a partial differential equation (Eq.(11)) for the propagation of drift waves in an axisymmetric torus. The parametric dependence of the coefficients of this equation have been derived for the Levitron Eqs. (12)-(18) and for a large aspect ratio tokamak (Eq.(23)).

III. LOCALISED SOLUTIONS OF THE MODEL EQUATION

In this section solutions of the model equation (11) are found which are localised in θ and in x. We shall try and localise about an angle $\theta = \theta_o$ and so expand all of the coefficients of equation (11) in $(\theta - \theta_o)$ keeping terms of up to $(\theta - \theta_o)^2$. After removing the shear term by the substitution $\phi = \hat{\phi}\exp(-iSx\theta/a_i)$ equation (11) becomes

$$a_i^2 \frac{\partial^2\hat{\phi}}{\partial x^2} + iQ\theta\, a_i \frac{\partial\hat{\phi}}{\partial x} - D \frac{\partial^2\hat{\phi}}{\partial\theta^2} + i\, A \sin\theta_o a_i \frac{\partial\hat{\phi}}{\partial x}$$

$$+ [W - \Gamma x^2/a_i^2 + U\theta + V\theta^2]\,\hat{\phi} = 0 \qquad \ldots (24)$$

where

$$Q = A \cos\theta_o - 2S$$

$$U = (AS - E)\sin\theta_o + 2F \sin 2\theta_o$$

$$W = \Omega + E\cos\theta_o + F\cos 2\theta_o \qquad \ldots (25)$$

$$V = (AS - E/2)\cos\theta_o - S^2 - 2F\cos 2\theta_o$$

and we have replaced $(\theta - \theta_o)$ by θ to simplify the notation. We substitute the form $\exp(a\,\theta^2/2 + b\,x^2/2\,a_i^2 + c\theta x/a_i + e\theta + fx/a_i)$ and then by comparing the coefficients of the θ^2, x^2, θx, x, θ and constant term show that this is indeed a solution with a-f given by the following set of equations,

$$c^2 + iQc - D\,a^2 + V = 0 \qquad \ldots (26)$$

$$2cb + iQb - 2Dac = 0 \qquad \ldots (27)$$

$$b^2 - Dc^2 - \Gamma = 0 \qquad \ldots (28)$$

$$2bf - 2Dce + iA \sin\theta_o\, b = 0 \qquad \ldots (29)$$

$$2cf - iQf - 2Dea + U + iAc\sin\theta_o = 0 \qquad \ldots (30)$$

$$f^2 + b - D(e^2 + a) + iAf\sin\theta_o + W = 0 \qquad \ldots (31)$$

The above set of nonlinear equations can be solved in the following manner. The first three equations (26)-(28) are decoupled from the remainder, and may be solved for a, b and c. Equations (29) and (30) are a pair of linear simultaneous equations for e and f in terms of a, b, c and hence may be readily solved. Then finally the eigenvalue W is obtained from equation (31). To determine whether the mode is properly localised the coefficients of the quadratic terms a, b and c are only required. These are as follows

$$a^2 = \frac{(\Gamma^{\frac{1}{2}} - Z)^2 - DQ^2/4}{D^2(\Gamma^{\frac{1}{2}} - Z)^2} \; Z^2 \qquad \ldots (32)$$

$$b^2 = \Gamma\{(\Gamma^{\frac{1}{2}} - Z)^2 - DQ^2/4\}/(\Gamma^{\frac{1}{2}} - Z)^2 \qquad \ldots (33)$$

$$c = - iQ\Gamma^{\frac{1}{2}}/2(\Gamma^{\frac{1}{2}} - Z) \qquad \ldots (34)$$

where $$Z = \{D(V + Q^2/4)\}^{\frac{1}{2}} \qquad \ldots (35)$$

The conditions on a, b and c, for ϕ to decay in every direction from the mode centre are

$$\text{Real (a)} < 0 \qquad \ldots (36)$$

and $$\text{Real (a)} \times \text{Real (b)} > (\text{Real (c)})^2 \qquad \ldots (37)$$

This result was obtained by putting $\theta = r\cos\delta$, $x = \sin\delta$ and assuming that $\phi \to 0$ as $r \to \infty$ for all δ. If the above inequalities are satisfied then since there is no outgoing energy in any direction the eigenvalue W (and hence Ω must be real. This last point may also be independently verified by using Eq.(31) to show that the imaginary part of W is zero when inequalities (36) and (37) are satisfied.

To obtain a necessary criterion for a real eigenvalue using (32)-(37) in terms of the parameters D, V, Γ etc is rather involved and probably best done numerically. However one can obtain a sufficient condition from these equations for a real eigenvalue and this is

$$\Gamma^{\frac{1}{2}} > |D^{\frac{1}{2}}Q/2| - |Z| \; , \quad \text{for} \; Z^2 > 0 \; . \qquad \ldots (38)$$

The above conditions are both satisfied when

$$V \equiv (AS - E/2) \cos \theta_o - 2F \cos 2\theta_o - S^2 > 0 \qquad \ldots (39)$$

Although the above criterion is sufficient for the mode to decay in all directions and the eigenvalue to be real, the condition that the mode is properly localised in θ, that is the amplitude is exponentially small at the angle $\theta - \theta_o = \pi$, is

$$|a\,\pi^2/2| \gg 1 \qquad \ldots (40)$$

This latter condition is slightly more stringent than inequality (39) and we check that it is also satisfied for the Levitron and tokamak examples of the last part of this section.

When Γ is small compared to the parameters D, Z etc, that is when the scale length of ω_{*e} is large, simple expressions for a, b and c may be obtained from Eqs.(32)-(34),

$$a = - V^{\frac{1}{2}}/D^{\frac{1}{2}} \quad ; \quad b = - \Gamma^{\frac{1}{2}} \left\{ \frac{V}{V + Q^2/4} \right\}^{\frac{1}{2}} \qquad \ldots (41)$$

$$c = iQ\,\Gamma^{\frac{1}{2}} \Big/ \left\{ D(V + Q^2/4) \right\}^{\frac{1}{2}} \qquad \ldots (42)$$

We now return to the criterion (39) for a real eigenvalue and determine what amplitude of varying part of the shear etc is required for a real eigenvalue. The parameters A, E, F in Eq.(39) are all functions of the periodic parts of the shear, curvature drift and finite larmor radius correction terms, so when $A = E = F = 0$ then $V = - S^2$ (note S is proportional to the average shear), the eigenvalue Ω has an imaginary part and the solution is damped. However for sufficiently large A, E or F one can obtain $V > 0$ and localised solutions with a real eigenvalue. Maximising V w.r.t. θ_o gives the following criterion for a localised mode.

$$V_{max} = |AS - E/2 - 2F| - S^2 > 0 \qquad \text{if } \left|\frac{2AS - E}{16F}\right| > 1$$

$$V_{max} = \frac{(2AS - E)^2}{16F} + 2F - S^2 > 0 \qquad \text{if } \left|\frac{2AS - E}{16F}\right| < 1. \qquad \ldots (43)$$

For the Levitron condition (43) is satisfied on most magnetic surfaces for all wavelengths and so one would expect these local drift waves to be easily driven unstable by the diamagnetic or ohmic currents. For short wavelengths $k_\perp a_i \sim 1$ the varying shear is the dominant term in Eq.(43) and this condition then reduces to $\tilde{\nu}' > \bar{\nu}'/\sqrt{2}$ or expressing this in terms of shear lengths $\tilde{L}_s < \sqrt{2}\,\bar{L}_s$. From Fig.1 it can be seen that the above condition is easily satisfied since $\tilde{\nu}' \sim 0.9\,\bar{\nu}'$.

For the tokamak $F = O(\varepsilon^2)$ (see equation (23)) and then the criterion for a real eigenvalue Eq.(43) reduces to

$$|AS - E/2| > S^2$$

or in more familiar notation

$$k^2 a_i^2 (rq'/q)^2 < \left|\frac{r_n}{R_o \tau}[2rq'/q - 1] + k^2 a_i^2 [\gamma (rq'/q)^2 - \Delta']\right| \qquad \ldots (44)$$

In this expression the first pair of terms are of order ε and originate from the two drift terms $\underset{\sim}{k}.\underset{\sim}{V}_D/\omega$ in equation (1). The third term is of order $\varepsilon\, k^2 a_i^2$ and originates from the modulated shear, while the last term is also of order $\varepsilon\, k^2\, a_i^2$ and originates from the modulation of the finite larmor radius $(k_y^2\, a_i^2)$ term. Clearly long wavelength modes are more readily localised, and for these modes the drift terms are more effective in localising the mode than the other terms. For long wavelength modes the criterion for the existence of localised undamped modes is

$$\left(\frac{rq'}{q}\right)^2 (k^2 a_i^2) < \frac{r_n}{R_o \tau}\left|1 - 2rq'/q\right| \qquad \ldots (45)$$

For parabolic density and current profiles this criterion is always satisfied near the axis, and for sufficiently long wavelengths may be satisfied across most of the minor radius.

Two validity conditions must also be satisfied before we conclude that such a mode exists. First we have assumed throughout that $\omega\tau_i > 2\pi$ where τ_i is the ion transit time. This requires that

$$k\,a_i > \frac{r_n}{R_o\, q\tau} \qquad \ldots (46)$$

which is compatible with inequality (45). In addition we must satisfy inequality (40), i.e.

$$k^2 a_i^2 \left(\frac{R_o\,\tau q}{r_n}\right)^2 \left[\frac{r_n}{R_o\,\tau}\left|1 - 2rq'/q\right| - k^2 a_i^2 (rq'/q)^2\right] \gtrsim 1 \quad \ldots (47)$$

which is somewhat more stringent than (46). (A simpler version of this criterion arises in a one-dimensional treatment given by Cheung and Horton[(8)]). Taking ka_i as an adjustable parameter, we find that (45) and (47) are compatible if

$$rq'/q < q\ \left|1 - 2rq'/q\right| \qquad \ldots (48)$$

which for a typical profile of $q \simeq (1 + 4\, r^2/a^2)$ is satisfied for $0 < r/a \leqslant .2$ and $.5 < r/a < 1$. The 'stable' band for intermediate values of r/a is quite general, since (45) can never be satisfied in the neighbourhood of $rq'/q = \frac{1}{2}$.

IV. CONNECTION WITH THE ONE-DIMENSIONAL SLAB MODES

In this section the connection between the localised modes of Section III and the modes of the slab model is discussed. That is as the amplitude of the periodic terms of equation (11) is reduced how does the structure of the mode change. From equation (39) we see that as the amplitude of the periodic terms A, E, F are reduced V becomes less than zero and from Eqs. (41) and (42), a becomes pure imaginary

$$a = -\,i\ |V|^{\frac{1}{2}}/D^{\frac{1}{2}}$$

and b and c can be either real or imaginary depending on the sign of $V + Q^2/4$. Thus the mode is not localised in the θ direction and is radiating energy away from the mode centre in that direction; in the radial direction, x, the mode can be of the local type or radiating type depending on the sign of $V + Q^2/4$ which can usually take either sign by changing θ_o.

Since both modes are not local in the θ direction one's first thought is that the modes have to be periodic in θ. However this is not necessarily the case, since the analysis in Section III does not take into account ion Landau damping. Indeed if the mode is Landau damped before $\theta - \theta_o = \pi$ then it is again possible to have a mode of the radiating type which is local in θ. The condition for the existence of this mode is $a_i S \pi > 1$ which in terms of the average shear length etc, is

$$a_i r\pi/\bar{L}_s > 1 \qquad \ldots (49)$$

The damping decrement of this mode is obtained from Eq.(31) and is

$$\gamma/\omega_{*e} = - r_n/\bar{L}_s \qquad \ldots (50)$$

This mode is similar to the quasi mode discussed by Roberts and Taylor[7], and has the same damping decrement as the Pearlstein and Berk slab modes but the outgoing energy is radiated in the poloidal direction rather than the x-direction.

When the inequality given by Eq.(49) is not satisfied the solutions must be periodic in θ. The relationship between the localised modes of Section III and the periodic modes of a torus in which variations around the magnetic surface are weak has recently been discussed by Taylor[9]. Using a somewhat simpler model equation than Eq.(11) a solution is found in the large modulation limit which is the fourier transform in θ of the localised solutions obtained in Section III of our paper. The condition for an undamped mode is the same as the localisation criterion of Section III (Eq.(39)). As the amplitude of the periodic term is reduced it is shown that the mode splits into a set of modes with a mode centred on each mode rational surface radiating energy in the radial direction away from their respective mode rational surfaces. In the limit of zero amplitude modulation the modes are completely uncoupled.

Ion Landau damping can prevent the modes on different rational surfaces from coupling together to form the localised mode and the criterion for this is very similar to the reverse of the inequality given in Eq.(49). Thus the picture that emerges is that as the amplitude of the periodic term is reduced these localised or ballooning modes go over to quasi modes if inequality, (49) is satisfied or to a set of weakly coupled Pearlstein and Berk outgoing energy modes centred on the mode rational surfaces when inequality (49) is not satisfied. The effect of the weak coupling of Pearlstein and Berk modes is to reduce the shear induced damping from its slab value.

V. LOW FREQUENCY LIMIT FOR DRIFT WAVES

In Section III it was found that long wavelength drift waves are particularly susceptible to localisation caused by modulation of the drift terms in equation (1). However the eigenvalue equation used the approximation $\omega\tau_i > 2\pi$ in calculating the ion response, and this condition is violated for sufficiently long wavelengths, namely when

$$k a_i < 2\pi r_n / L_c \tau \qquad \ldots (51)$$

where L_c denotes the connection length.

For very long wavelengths (satisfying (51)) we use the results of ref.5 in the $\omega\tau_i < 1$ approximation. Writing the perturbed potential Φ in the form

$$\Phi = \phi\, e^{-i\omega t + i\ell(\phi - \int^{\chi} \nu(\psi_o)\, d\chi)}$$

the perturbed charge density for ions is given by

$$\begin{aligned}
\rho_i = \frac{e^2 n}{T_i} \Bigg\{ - \phi + \left(1 - \frac{\omega_{*i}}{\omega}\right) \frac{1}{2} \int_o^{1/B} <\phi> \frac{dy}{h} \Bigg\} \\
+ \frac{3}{2} \frac{e^2 n}{T_i} \left(1 - \frac{\omega_{*i}}{\omega}\right) \Bigg\{ \frac{T_i}{M} \left(\frac{2\pi(\ell q - m)}{\omega}\right)^2 \int_o^{1/Bm} <\phi> \frac{dy}{h t_o^{\,2}} + \frac{\omega_{*i}}{\omega} \left(\frac{n'}{n}\right)^{-1} \int_o^{1/B} \frac{dy}{h} \frac{<\phi>}{t_o} \left(\frac{\partial}{\partial\psi} \oint \frac{B^2 h}{B_p^{\,2}} d\chi\right) \\
- \frac{\omega_{*i}}{\omega} \left(\frac{n'}{n}\right)^{-1} R B_T q' \int_o^{1/Bm} \frac{dy}{t_o} <\phi> + \frac{1}{4} \frac{T_i}{M} \int_o^{1/B} \frac{y\, dy}{h} \Bigg[\frac{B}{\omega_{ci}^2} (R B_p)^2 \left\langle \frac{\partial^2 \phi}{\partial x^2} \right\rangle \\
+ \left\langle \frac{B}{\omega_{ci}^2} (R B_p)^2 \frac{\partial^2 \phi}{\partial x^2} \right\rangle \Bigg] \Bigg\}
\end{aligned} \qquad \ldots (52)$$

while the electron charge density is the same as in Section III,

$$\rho_e = \frac{e^2 n}{T_e}\left\{-\phi + \left(1 - \frac{\omega_{*e}}{\omega}\right)\frac{1}{2}\int_{1/Bm}^{1/B} <\phi> \frac{dy}{h}\right\} \qquad \ldots (53)$$

In equations (52) and (53), $h = (1 - yB)^{\frac{1}{2}}/B$, $t_o = \oint \frac{d\chi}{B_p^2 h}$

In equation (52) the ion inertia now contains the average shear q', and is integrated over passing ions only. The principal curvature contributes to the drift term $\underset{\sim}{k}.\underset{\sim}{V}_D/\omega$, but the geodesic curvature has zero average over a transit.

As in the intermediate frequency case we determine a first approximation to the eigenvalue from the dominant terms of the charge neutrality equation.

In leading order this is

$$-\phi_o(1+\tau) + \left(\tau + \frac{\omega_{*e}}{\omega}\right)\frac{1}{2}\int_0^{1/B} <\phi_o> \frac{dy}{h} + \left(1 - \frac{\omega_{*e}}{\omega}\right)\frac{1}{2}\int_{1/Bm}^{1/B} <\phi_o> \frac{dy}{h} = 0 \qquad \ldots (54)$$

If B is not modulated in χ equation (54) reduces to

$$\left(\tau + \frac{\omega_{*e}}{\omega}\right)\oint \phi_o \frac{d\chi}{B_p^2} = \phi_o(1+\tau)\oint \frac{d\chi}{B_p^2} \qquad \ldots (55)$$

and since the left hand side of this equation is independent of χ, it follows that the only solution is the flute solution

$$\phi_o = \overline{\Phi} \; ; \; \omega = \omega_{*e} \qquad \ldots (56)$$

If B is modulated in χ, the flute solution (equation (56)) remains valid, but other solutions with $0 < \omega < \omega_{*e}$ become possible. We discuss these later, but for the moment examine the flute/drift-wave in detail. In next order, expanding $\omega = \omega_{*e} + \delta\omega$, and after annihilating the $\delta\hat{\Phi}$ terms by integrating $\oint d\chi/B_p^2$, we obtain a one-dimensional eigenvalue equation for $\overline{\Phi}(x)$, $\delta\omega$;

$$\phi_o \frac{\delta\omega}{\omega_{*e}} \oint \frac{d\chi}{B_p^2}\left[1 - (1 - B/B_m)^{\frac{1}{2}}\right] + (1+\tau)\left\{-\frac{\partial^2\phi_o}{\partial x^2}\frac{T_i}{m_i}\oint\frac{R^2 d\chi}{\omega_c^2} + \frac{\phi_o}{\tau}\left(\frac{n'}{n}\right)^{-1}\frac{\partial}{\partial\psi}\oint\frac{d\chi}{B_p^2}\right.$$

$$\left. - \frac{3}{2}\frac{T_i}{M}\left(\frac{2\pi(\ell q - m)}{\omega}\right)^2 \phi \int_0^{1/B_m}\frac{dy}{t_o} - \frac{3}{2}\phi_o\left(\frac{n'}{n}\right)^{-1} RB_T q' \int_0^{1/B_m}\frac{dy}{t_o}\oint\frac{d\chi}{B_p^2}\right\} = 0$$

... (57)

Comparing this one-dimensional equation with the equation obtained in slab geometry we may write down the effective $\frac{r_n}{L_s}$ parameter (i.e. the quantity which determines the damping) as

$$\frac{r_n^2}{L_s^2} = \frac{3}{2}\left(2\pi q' \frac{n}{n'}\right)\oint\frac{R^2 d\chi}{B^2}\int_0^{1/B_m}\frac{dy}{t_o}\left\{\oint\frac{d\chi}{B_p^2}\left[1 - (1 - B/B_m)^{\frac{1}{2}}\right]\right\}^{-2} \quad \dots (58)$$

For the large aspect ratio tokamak, this reduces to

$$\frac{r_n}{L_s} \simeq \left(r_n \frac{rq'}{R_o q^2}\right)\left[1 - \frac{2}{\pi}(2\, r/R_o)^{\frac{1}{2}}\right]^{-1} \quad \dots (59)$$

indicating slightly stronger damping than in the 'equivalent' slab problem.

As in the analysis of Pearlstein and Berk, balancing the damping due to the outward energy flux (proportional to r_n/L_s above) against the growth due to the electron landau resonance results in a stability criterion with $\frac{r_n}{L_s} \sim (\frac{m}{M})^{1/3}(ka_i)^{4/3}$. When B is modulated the electron driving term may be somewhat weaker than in a slab calculation, but as shown in Section VI this is not a strong effect.

Thus in the low frequency limit the electron drift wave remains susceptible to shear stabilisation even in the presence of strongly modulated shear or curvature drift. As noted above, the possibility of other low frequency modes, with $\omega < \omega_{*e}$, and a modulated eigenfunction, exists when B is modulated in χ. At first sight it might appear that strong coupling between different mode rational surfaces might occur for such modes resulting in extended undamped modes as found when $\omega\tau_i > 1$. In the low frequency limit, however, strong ion damping of each mode occurs before the first mode rational surface is encountered, so that such coupling becomes impossible.

We conclude that the criterion $\omega\tau_i < 1$ is effectively a stability criterion for drift waves in strongly sheared systems.

VI. ELECTRON LANDAU RESONANCE AND THE ELECTRON APPROXIMATION NEAR A MODE RATIONAL SURFACE

In the determination of the two dimensional mode structure and the oscillatory (or damped) eigenvalue the perturbed electron charge density was taken, in Section III, to be

$$\rho_e = -\frac{e^2 n}{T_e}\phi_o + \frac{e^2 n}{T_e}\left(1 - \frac{\omega_{*e}}{\omega}\right)\frac{1}{2}\int_{1/B_m}^{1/B} \langle\phi_o\rangle dy/h \qquad \ldots (60)$$

Near to a mode rational surface this approximation breaks down, and in a narrow radial layer there are additional complex terms of similar magnitude which give rise to an imaginary contribution to the eigenvalue. To determine the growth rate we require to evaluate these terms. In addition we wish to investigate the possibility that the additional real terms present a narrow barrier to the outgoing Pearlstein-Berk solutions when these exist, and reflect part of the outgoing energy, thus modifying the damping. This effect is entirely independent of the direct coupling between Pearlstein-Berk modes caused by modulation effects.

From ref.5 (equations (28),(38)) we see that the additional contribution to ρ_e occurring in the vicinity of a mode rational surface is:

$$\delta\rho_e = -\frac{e^2}{T_e}2\pi\int_p \frac{Bd\mu d\kappa}{[2(\kappa-\mu B)]^{\frac{1}{2}}}\frac{F_o[\omega-\omega_{*e}(1-3\eta/2 + \frac{m\kappa}{T}\eta)]}{[\cos 2\pi\ell q - \cos\omega\tau_e]}\oint_{\chi}^{\chi+\chi_o}\frac{Bd\chi'\phi_o(\chi')}{B_p^2[2(\kappa-\mu B)]^{\frac{1}{2}}} A_p(\chi,\chi') \qquad \ldots (61)$$

where

$$A_p(\chi,\chi') = \omega\tau_e\, e^{+i\ell x\int_{\chi}^{\chi'}\nu' d\chi}$$

$$\tau_e = \oint\frac{B\,d\chi}{B_p^2[2(\kappa-\mu B)]^{\frac{1}{2}}} \quad ; \quad \eta = \frac{\partial\,\ell n T_e}{\partial\,\ell n\, n}$$

the quantity x is the distance (in flux coordinates) from the central mode rational surface. F_o is the Maxwellian distribution, and we have used the fact that $\omega\tau_e << 1$, $(\ell q-n) << 1$ in obtaining the expression for A_p.

Introducing the pitch angle variable $y = u/\varepsilon$ we may carry out the energy integration in equation (61) to obtain

$$\delta\rho_e = -\frac{e^2 n}{T_e}\int_0^{1/B_m}\frac{dy}{h}\, a^2\left\{\left(1-\frac{\omega_{*e}}{\omega}\right)\left[1+aZ(a)\right]+\frac{\omega_*}{\omega}\eta\left[\left(\frac{3}{2}-a^2\right)(1+aZ)-\frac{1}{2}\right]\right\}\Lambda_N \quad \ldots (62)$$

where

$$a^2 = \frac{m}{2T}\left(\frac{\omega\oint\frac{d\chi}{B_p^2 h}}{2\pi(\ell q-m)}\right)^2 \quad \text{and}$$

$$\Lambda_N = \left(\oint_{\chi}^{\chi+\chi_o}\frac{d\chi'}{B_p^2 h}\,\phi_o(\chi')\, e^{+i\ell x_N\int_{\chi}^{\chi'}\nu'\, d\chi}\right)\left(\oint\frac{d\chi}{B_p^2 h}\right)^{-1}$$

with x_N denoting distance to the Nth mode rational surface, and $Z(a) = \pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}\frac{e^{-t^2}dt}{t-a}$ is the plasma dispersion function.

The imaginary contribution to $\delta\rho_e$, which we shall require in order to calculate the growth rate, is given by

$$\mathfrak{Im}\,\delta\rho_e = -\frac{e^2 n}{T}\sqrt{\pi}\int_0^{1/B_m}\frac{dy}{h}\, a^3 e^{-a^2}\left\{\left(1-\frac{\omega_{*e}}{\omega}\right)+\left(\frac{3}{2}-a^2\right)\frac{\omega_*}{\omega}\eta\right\}\Lambda_N \quad \ldots (63)$$

which contains the familiar $a^3 \simeq (\omega/k_{\parallel} V_{Te})^3$ dependence typical of the landau resonance in general geometry.

When B is not modulated (ie. $\partial B/\partial\chi = 0$) the pitch angle integration can also be performed explicitly in equation (62) with the result,

$$\delta\rho_e = -\frac{e^2 n}{T}\Lambda_N\left\{a_o Z(a_o)\left[1-\frac{\omega_{*e}}{\omega}(1-\eta/2)\right]+\frac{\omega_{*e}}{\omega}\eta\, a_o^2\,(1+a_o Z(a_o))\right\} \quad \ldots (64)$$

where $a_o = a(y=0)$.

and the imaginary part is now

$$\mathfrak{Im}\,\delta\rho_e = -\frac{e^2 n}{T}\sqrt{\pi}\,\Lambda_N\, a_o\, e^{-a_o^2}\left\{\left[1-\frac{\omega_{*e}}{\omega}(1-\eta/2)\right]-a_o^2\frac{\omega_{*e}}{\omega}\eta\right\} \quad \ldots (65)$$

which contains the familiar $a_o \simeq (\omega/k_{\parallel} V_{Te})$ dependence of the landau resonance in slab geometry. However as we shall see the apparent sharp decrease (from a_o to a^3, where a_o and a are small over most of the radial extent of the mode) in the imaginary contribution to $\delta\rho_e$ when $\partial B/\partial\chi \neq 0$, is misleading. After radial integration across the mode the difference between the imaginary contributions is only logarithmic in the small parameter $\omega\tau_e << 1$.

Before determining drift wave growth rates in the general geometry ($\partial B/\partial\chi \neq 0$) and slab-like ($\partial B/\partial\chi = 0$) cases, we examine the amplitude factor Λ_N.

The quantity Λ_N determines the transparency of the N^{th} mode rational surface to the eigenmode, and it is the only quantity in equation (62) which distinguishes one mode rational surface from another. On the N^{th} mode rational surface $\ell \times \oint \nu' d\chi = N$; thus, ignoring the variation of both B and B_p, we may model Λ_N by taking

$$\Lambda_N \simeq \hat{\Lambda}_N = \frac{1}{2\pi} \oint d\theta\ \phi(\theta, x_N)\ e^{-iN\bar{L}_s \int_0^\theta \frac{d\theta}{L_s(\theta)}} \qquad \ldots (66)$$

and with $\phi = \phi_n(x) e^{in\theta}$, $\frac{1}{L_s(\theta)} = \frac{1}{\bar{L}_s}(1 - \gamma\cos\theta)$
$\hat{\Lambda}_N$ is finally given by

$$\Lambda_N = \sum_n \phi_n (x_N) J_{n-N}(\gamma N) \quad . \qquad \ldots (67)$$

Thus even if Φ is purely flute-like ($\Phi = \Phi_o(x)$), $\hat{\Lambda}$ is in general non-zero so that each mode rational surface presents a complex barrier to the propagation of outgoing solutions of the Pearlstein-Berk type when these exist. Only if each of B, B_p, L_s and Φ is un-modulated is the true slab limit obtained:-

$$\left.\begin{aligned} \Lambda_o &= \phi_o \\ \Lambda_N &= 0 \quad , \quad N \neq 0 \end{aligned}\right\} \qquad \ldots (68)$$

in which the central mode rational surface is unique.

We estimate the growth rate using a perturbation method[10,11]. Writing the complete charge neutrality equation in the form

$$L(\omega)\phi + iR(\omega)\phi = 0 \qquad \dots (69)$$

where R denotes the electron landau resonance terms, and L is the operator defined by equation (1). Then writing $\omega = \omega_o + i\gamma$, $\phi = \phi_o + \phi_1$, where

$$L(\omega_o)\phi_o = 0 \qquad \dots (70)$$

we obtain the following result for the Landau growth γ.

$$2\,\mathrm{Im}\,\frac{(\delta\omega)}{\omega_{*e}}\langle\phi_1\phi_o^*\rangle + \frac{\gamma}{\omega_{*e}}\langle|\phi_o|^2\rangle + \langle\phi_o^* R\phi_o\rangle = 0 \qquad \dots (71)$$

where $\langle\ \rangle$ denotes $\int d\psi \oint \frac{d\chi}{B_p^2}$ and $\mathrm{Im}\,\delta\omega$ is the damping decrement due to shear when the modes are of the Pearlstein-Berk type. Now we shall finally be interested in the regime where γ is of order $\mathrm{Im}\,\delta\omega$ so that the $\langle\Phi_1\Phi_o^*\rangle$ term may be neglected in evaluating γ. Thus

$$\frac{\gamma}{\omega_{*e}} = -\langle\phi_o^* R\,\phi_o\rangle/\langle|\phi_o|^2\rangle \qquad \dots (72)$$

where

$$\langle\phi_o^* R\,\phi_o\rangle = \sqrt{\pi}\int d\psi\int_0^{1/B_m} dy\, e^{-a^2} a^3\left\{\mathrm{Re}\,\frac{\delta\omega}{\omega_{*e}} + \eta\left(\frac{3}{2}-a^2\right)\right\}|\Lambda_N|^2\oint\frac{d\chi}{B_p^2 h}$$

$$\simeq \sqrt{\pi}\sum_{N=-N_c}^{+N_c}\int_{-x_1}^{+x_1} R_o B_o dx\int_0^{1/B_m} dy\, e^{-a^2} a^3\left\{\mathrm{Re}\,\frac{\delta\omega}{\omega_{*e}} + \eta\left(\frac{3}{2}-a^2\right)\right\}|\Lambda_N|^2\oint\frac{d\chi}{B_p^2 h} \qquad \dots (73)$$

where the summation is over the mode rational surfaces, and the integration $\int dx$ is the integration through each mode rational surface with $2x_1$ the distance between consecutive surfaces. The integer N_c denotes the surface at which strong ion landau damping intervenes. If the mode is radially localised, $|\Lambda_N|^2 \to 0$ exponentially as N increases and the truncation at N_c is redundant.

Converting the x integration in equation (73) into integration over a, using

$$a = \delta \frac{x_1}{x} \quad \text{with} \quad \delta = 2 \sqrt{\frac{m}{2T}} \left(\frac{\omega \oint \frac{d\chi}{B_p{}^2 h}}{2\pi} \right)$$

we find

$$2\int_o^{x_1} dx\, a^3 e^{-a^2} = 2x_1\, \delta(y) \int_{\delta(y)}^{\infty} a e^{-a^2}\, da \simeq x_1\, \delta(y)$$

In this expression $\delta(y) << 1$ for all y except in an exponentially narrow region close to $y = 1/B_m$, which we may neglect. Thus we may approximate as shown, and note that the dominant contribution has come from $a \sim O(1)$, and all y values. In other words the resonant electrons are of thermal energy, and arbitrary pitch angle, but occur in a narrow region of real space close to a mode rational surface ($a \sim O(1) \Rightarrow x/x_1 << 1$).

Thus the growth rate is finally given by

$$\frac{\gamma}{\omega_{*e}} \simeq \sqrt{\pi}\, x_1 \left\{ \int_o^{1/B_m} dy \left[\frac{R_e \delta\omega}{\omega_{*e}} + {}^{\eta}/_{2} - \eta\delta^2 \right] \delta e^{-\delta^2} \oint \frac{d\chi}{B_p{}^2 h} \sum_N |\Lambda_N|^2 \right\} \langle |\phi_o|^2 \rangle^{-1}$$

$$\simeq O\left(\frac{\omega\tau_e}{2\pi} \right) \qquad \dots (74)$$

It is instructive to carry out the same procedure in the slab-approximation. Although we could obtain the relevant result by direct integration of expression (74) over the pitch angle variable, we return to expression (73) and perform the x integration, transforming first to $a_o = \delta_o\, x_1/x$ where $\delta_o = \delta(y = 0)$. Now the relevant integral is

$$2\int_o^{x_1} dx\, a_o e^{-a_o{}^2} = x_1\, \delta_o \int_{\delta_o^2}^{\infty} e^{-t} \frac{dt}{t} \simeq -\, x_1\, \delta_o\, \ln \delta_o^2 \qquad \dots (75)$$

and

$$\frac{\gamma}{\omega_{*e}} \simeq \sqrt{\pi}\, x_1\, \delta_o \sum_N |\Lambda_N|^2 \oint \frac{d\chi}{B_p^2} \left[\left(\mathrm{Re}\, \frac{\delta\omega}{\omega_{*e}} + \eta/2 \right) \left(|\ln \delta_o^2| - .5771 \right) - \eta \right]$$

$$\times \langle |\phi_o|^2 \rangle^{-1}$$

The slab resonance is therefore logarithmically larger than the landau resonance in configurations with a modulated magnetic field, and significantly is driven by a different class of electrons, those with $a_o \simeq \delta_o << 1$. In contrast to the result obtained in general geometry the resonant electrons therefore occur in a narrow region of velocity space (small $v_{\parallel}$) and are widely distributed in real space. The strength of the slab resonance is due to the relative abundance of slow electrons with $v_{\parallel} < v_{crit}$ [these are $O(\frac{v_{crit}}{v_T})$], compared to the paucity of long transit-time electrons in general geometry. In the latter case these may be of low energy, in which case there are $O(\tau_o/\tau_c)^3$ with transit times τ longer than τ_c, or of the 'just passing' variety ($y \simeq 1/B_{max}$) in which case there are $O(e^{-(\tau_c/\tau_o)})$ with transit times longer than τ_c [where τ_o denotes the transit time of a thermal electron with $y = 0$]. In the transition from the general geometry result to the slab result it is the narrow band around $y \simeq 1/B_{max}$ which broadens to provide the strong slab resonance. The criterion which must be satisfied for the slab-like result to be valid is

$$\delta B/B < \left(\frac{\omega\tau_{eo}}{2\pi} \right)^2 \qquad \ldots (76)$$

which certainly is not satisfied in the Culham Levitron. For the large aspect ratio tokamak this criterion takes the form

$$r/R < k^2 a_i^2 \frac{m}{M} \tau \, (R_o q/r_n)^2 \qquad \ldots (77)$$

which again is typically violated.

We conclude that for electron drift waves in Tokamaks and Levitrons the landau resonance is of the general geometry type, is only logarithmically weaker than that found in a slab model, and is driven by electrons of thermal energy and arbitrary pitch angle. Consequently we are also forced to conclude that subtle neo-classical distortions of the electron distribution function will have little effect on the drift wave growth rates. These conclusions are essentially opposite to those of Coppi and Rem(4).

Finally, we note that whereas in the true slab model the summation over mode rational surfaces in equation (75) reduces to the zero-order term only

$$\sum_N |\Lambda_N|^2 = |\Lambda_o|^2 = |\phi_o|^2$$

modulation of B, B_p or L_s, results in finite contributions from $2N_c + 1$ mode rational surfaces where N_c may be a large integer $(N_c \sim \frac{ka_i\ L_c}{2\pi\ r_n}\tau)$.

CONCLUSION

We have found that sufficiently strong modulation of many equilibrium quantities as functions of the poloidal angle θ can result in the localisation of drift modes in both θ and radius, thus rendering shear stabilisation ineffective in axisymmetric toroidal geometry. In particular, rather strong variations of the local shear quantity $(\frac{1}{J}\frac{\partial\nu}{\partial\psi})$ or of the product RB_p can remove the shear damping inherent in a one-dimensional treatment. Variation of $|B|$ makes itself felt mainly through the geodesic drift, and a similar effect results from modulation of the principal curvature drift. Small variations of these quantities are sufficient to localise long wavelength drift modes, so that even in a large aspect ratio tokamak of circular cross-section the toroidal curvature may result in the appearance of unstable drift modes over part of the minor radius.

When the poloidal modulations are insufficiently strong to establish a localised drift mode, application of a different method of solution due to Taylor(9) shows that a mode of the Pearlstein-Berk type persists, but now with a damping decrement which is smaller than that found in the unmodulated slab model. In this case there may be some additional reflection of the outgoing energy from the mode rational surfaces, reducing still further the shear-induced damping.

A detailed study of the electron landau resonance shows that this is only marginally weaker than in a slab-model analysis, and that the main contribution comes from a restricted region of real space, near to a mode rational surface where $\omega/k_{\parallel} \simeq V_{Te}$ and from the bulk of electrons with thermal energies. We conclude that the electron landau resonance will be relatively insensitive to small distortions of the electron distribution function.

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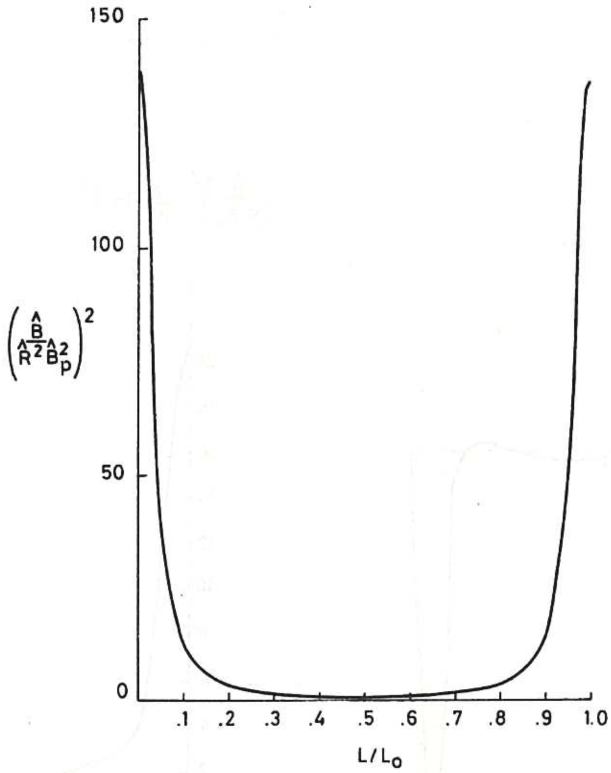

Fig.1 Variation of the finite larmor radius correction term with the poloidal variable L, normalised to be unity at $L/L_o = 0.5$ ($\theta = 0$).

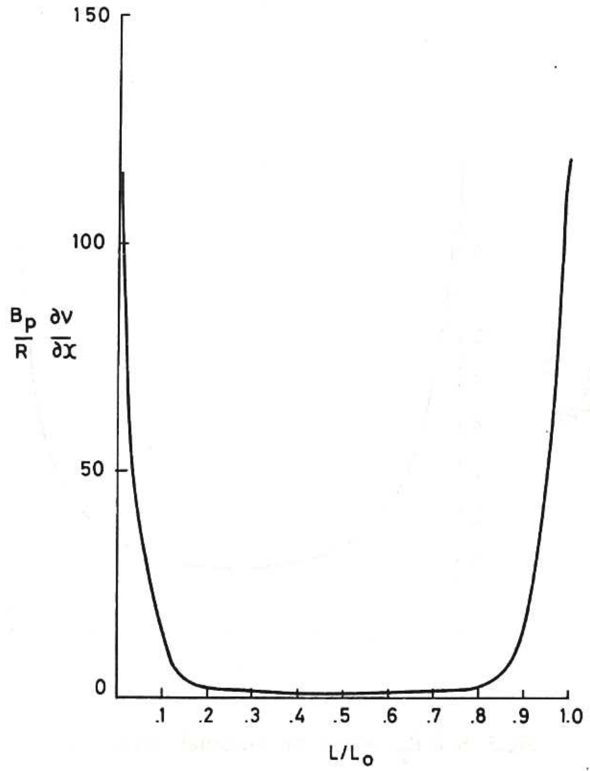

Fig.2 Variation of the shear strength with the poloidal variable L, normalised to be unity at $L/L_o = 0.5$ ($\theta = 0$).

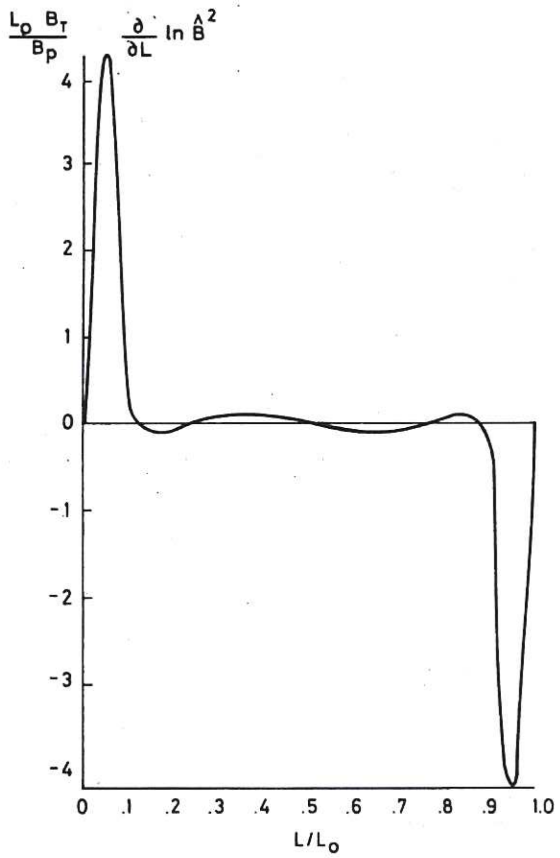

Fig.3 Variation of geodesic curvature with the poloidal variable L.

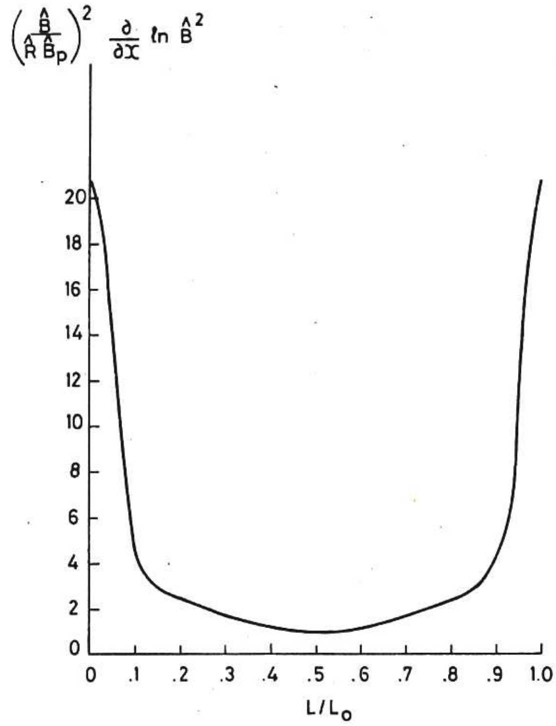

Fig.4 Variation of the principal curvature with the poloidal variable L.

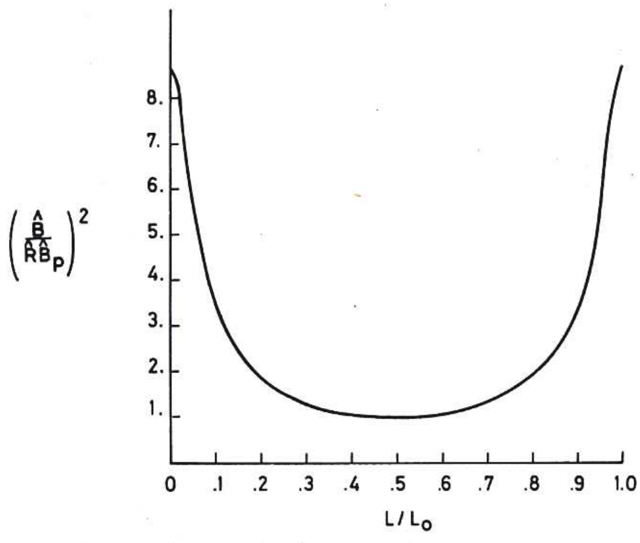

Fig.5 $\hat{B}/\hat{R}\,\hat{B}_p$ against the poloidal variable L.