UKAEA RESEARCH GROUP

Preprint

MHD STABILITY OF TOROIDAL EQUILIBRIA TO AXISYMMETRIC MODES

F A HAAS
J C B PAPALOIZOU

CULHAM LABORATORY
Abingdon Oxfordshire

1977

MHD STABILITY OF TOROIDAL EQUILIBRIA
TO AXISYMMETRIC MODES

F.A. Haas and J.C.B. Papaloizou[*]

Culham Laboratory, Abingdon, Oxon, OX14 3DB, UK

(Euratom/UKAEA Fusion Association)

Abstract

We consider the stability of an axisymmetric, arbitrary cross-section, arbitrary aspect ratio toroidal plasma, to axisymmetric modes. It is shown that configurations with toroidal current which decreases monotonically towards the boundary, and are maintained by an external magnetic field with negative decay index, must always be unstable in the absence of any external stabilising agencies such as active or passive feedback.

*Now at the Institute of Astronomy, University of Cambridge

I   INTRODUCTION

In this paper we study the MHD stability of an axisymmetric toroidal plasma to axisymmetric ($n = 0$) modes. We consider configurations which are symmetric about the equatorial plane, and which are maintained in equilibrium by current-carrying wires in the vacuum; we assume that conducting walls and shells are either absent, or so far away as not to affect stability. It is further assumed that, during any ensuing perturbations, the current in the wires is maintained constant. In practice, this situation is achieved by coupling the wires in series and including a very large inductance in the circuit. Thus the wires are precluded from influencing the stability. Our results are appropriate to tokamaks and pinches for which the toroidal current density, j_φ, is everywhere positive, that is, there are no current reversals.

The earliest studies of the $n = 0$ mode were concerned with the uniform vertical displacement ($m = 1$) of a circular cross-section plasma of large aspect ratio, the current being essentially uniform. Osovets [1] and Yoshikawa [2] showed the stability for $m = 1$ (rigid displacement) to depend upon the sign of the so-called decay index, that is

$$N = - \frac{R}{H_\perp} \cdot \frac{\partial H_\perp}{\partial R} ,$$

where $H_\perp$ is the externally applied magnetic field perpendicular to the equatorial plane, and R is the distance from the axis of symmetry. These authors showed that for stability N must be positive. It should be noted, however, that the above result is only strictly valid if the plasma is regarded as a point. Assuming the plasma to have an elliptical cross-section of small eccentricity ($b/a \sim 1$), and to be carrying a homogeneous current, then the relation between b/a and N has been shown [3] to be of the form

$$\frac{b}{a} = 1 + \frac{a^2}{R_p^2} \left[\frac{3}{4} \ell n \left(\frac{8R_p}{a} \right) - \frac{17}{16} - N \left(\ell n \left(\frac{8R_p}{a} \right) + \beta_I - \frac{5}{4} \right) \right] ,$$

where $\beta_I = 2 \langle p \rangle / B_\theta^2(a)$ and R_p is the major radius of the plasma. It follows that for stability to vertical rigid displacements

$$\frac{b}{a} - 1 < \frac{a^2}{R_p^2} \left(\frac{3}{4} \ell n \left(\frac{8R_p}{a} \right) - \frac{17}{16} \right). \qquad (1)$$

In the limit of large aspect ratio, this result shows that for values of b/a close to unity, an ellipse with b/a > 1 will be unstable to this mode. This conclusion is supported by the work of Laval et al.[4] and Haas [5]. Applying asymptotic methods to the energy principle, these authors demonstrate an elliptical plasma with a 'flat' current profile to be unstable to m = 1 for all b/a >1. Eq. (1) suggests that for a finite a/R$_p$, an ellipse can be stable for values of b/a larger than one. This conclusion is confirmed by the numerical calculations of Okabayashi and Sheffield [6], and Lackner and MacMahon [7]. In particular, for the 'flat' current profile the latter authors demonstrate stability against m = 1 for values of b/a up to 1.25. The upper limit to b/a is found to depend on the form of current profile. More recently, Wesson and Sykes [8] have studied numerically the actual equations of motion for n = 0 perturbations of an approximately elliptical cross-section plasma with a 'flat' current, and find the motion to be vertical but non-uniform. Thus, in general, the rigid perturbation is not the minimising axisymmetric mode.

Recently, there has been considerable interest in obtaining laboratory plasmas of strong ellipticity. Experiments with belt pinches (see eg. [9]) and shaped cross-section tokamaks (see eg. [10]), have the decay index of the vacuum field negative throughout the region occupied by the plasma. In the case of TOSCA [10], stability is achieved by passive feedback. Now the brief theoretical review given above did not consider stability due to external agencies such as passive (conducting walls or wires) or active feedback, and this leads us to the objective of the present paper. We study analytically the n = 0 stability of a general axisymmetric plasma to perturbations which include a non-rigid component; all external stabilising agencies are assumed absent.

Using the energy principle, our procedure is to carry through a program of partial minimisation, and then to use a trial-function. Since the actual minimising perturbation is not rigid [8], we assume a function which includes both rigid and non-rigid components. The final form for the potential energy, δW, comprises contributions from

the decay-index, the shape of the current profile, and the magnitude of j_φ at the plasma-vacuum boundary. Recalling the strictures on symmetry and sign of j_φ given at the beginning of the paper, we derive the following theorem :-

    Any axisymmetric toroidal plasma with toroidal current decreasing monotonically towards the boundary, and maintained in equilibrium by external fields such that the decay index is negative throughout the region occupied by the plasma, is unstable in the absence of active or passive feedback.

The validity of the theorem has been established strictly only for cross-sectional shapes which are simple, but otherwise arbitrary. Strongly indented configurations would require a detailed study of the flux and current density contours before an assessment of their stability to $n = 0$ could be made. The theorem, which is somewhat analogous to that of Earnshaw's in electrostatics, is also true for the academic case of a 'flat' current profile.

    Finally, for completeness, we discuss the straight problem. In this case we consider equilibria which are symmetric about both the equatorial and vertical planes, and we derive the following theorem :-

    Any straight plasma with longitudinal current decreasing monotonically towards the boundary, and maintained in equilibrium by external fields, is unstable in the absence of active or passive feedback.

The analogy with Earnshaw's theorem is closer for this case. We note that this theorem also holds for a uniform current.

II   AXISYMMETRIC MODES IN TOROIDAL PLASMAS

    We adopt the usual coordinate systems appropriate to an axisymmetric toroidal equilibrium, namely, the set (R, φ, Z) based on the axis of symmetry, and the locally orthogonal set (ψ, φ, χ). Following the definitions of Mercier [11], we write

$$\vec{B} = \frac{I(\psi)}{R} \vec{e}_\varphi - \frac{\vec{e}_\varphi \times \nabla \psi}{R} , \qquad (2)$$

where ψ is the total poloidal-flux and $I = R B_\varphi$. The basic MHD equilibrium equation takes the form

$$\nabla \cdot \left(\frac{1}{R^2} \nabla \psi \right) = - \frac{j_\varphi}{R} , \qquad (3)$$

where

$$j_\varphi = Rp'(\psi) + \frac{II'}{R} \qquad (4)$$

and the poloidal current density is given by

$$j_\chi = I' B_\chi . \qquad (5)$$

Since ψ consists of contributions due to currents in the plasma, and currents in the external windings, we can write

$$\psi = \psi_p + \psi_w , \qquad (6)$$

and hence by Eq. (3),

$$\nabla \cdot \left(\frac{1}{R^2} \nabla \psi_w \right) = 0 , \qquad (7)$$

and

$$\nabla \cdot \left(\frac{1}{R^2} \nabla \psi_p \right) = -\frac{j_\varphi}{R} . \qquad (8)$$

We shall restrict ourselves to equilibria which are symmetric about the plane $Z = 0$, and thus

$$\psi_p(Z) = \psi_p(-Z) , \qquad (9)$$

and further

$$\frac{\partial}{\partial Z} \psi_p(Z) = -\frac{\partial}{\partial Z} \psi_p(-Z) . \qquad (10)$$

In general, the potential energy δW resulting from a small displacement $\vec{\xi}$, comprises three terms [12],

$$\delta W = \delta W_p + \delta W_v + \delta W_s ,$$

where

$$\delta W_p = \int_P (Q^2 - \vec{j} \cdot \vec{Q} \times \vec{\xi} + \gamma p (\nabla \cdot \vec{\xi})^2 + \nabla \cdot \xi \, \vec{\xi} \cdot \nabla p) d\tau \qquad (11)$$

and

$$\delta W_v = \int_V \delta \hat{B}^2 \, d\hat{\tau} , \qquad (12)$$

all symbols having their usual meaning. In the present work we consider skin-currents to be absent, and since this implies continuity of $\vec{B}$ at the surface (and hence $p = 0$ at the surface), then $\delta W_s = 0$.

Changing to ψ, φ, χ coordinates the plasma energy for $n = 0$ becomes

$$\delta W_p = \int d\psi \, d\chi \left\{ \frac{J|\nabla X|^2}{R^2} + \frac{R^2 G^2}{J} + j_\varphi R \left(\frac{JX}{R^2} \frac{\partial X}{\partial \psi} + \frac{Y}{R^2} \frac{\partial X}{\partial \chi} \right) \right.$$

$$\left. - I' \left(z \frac{\partial X}{\partial \chi} + XG \right) + \frac{\gamma p}{J} \left[\frac{\partial}{\partial \psi} (JX) + \frac{\partial Y}{\partial \chi} \right]^2 + Xp'(\psi) \left[\frac{\partial}{\partial \psi} (JX) + \frac{\partial Y}{\partial \chi} \right] \right\},$$

$$(13)$$

where $\quad X = R B_\chi \xi_\psi$, $\qquad Y = \frac{\xi_\chi}{B_\chi}$, $\qquad z = \frac{\xi_\varphi}{R}$,

$$G = \frac{\partial}{\partial \psi} \left(\frac{JXI}{R^2} \right) + I \frac{\partial}{\partial \chi} \left(\frac{Y}{R^2} \right) - \frac{\partial z}{\partial \chi} \qquad (14)$$

and J, the Jacobian, is given by $\quad J = R(|\nabla \psi| \, |\nabla \chi|)^{-1}$. Since we are only concerned with $n = 0$, the φ-integration will be ignored throughout.

The above form for δW_p is the same as that used by Jukes [13] to investigate axisymmetric perturbations for the wall-on-the-plasma case. For completeness we repeat his initial steps, and then modify and extend the analysis to cater for the free-boundary problem.

Minimising δW_p with respect to z, we are led to the equation

$$G - \frac{J I' X}{R^2} = \frac{J}{R^2} f(\psi), \qquad (15)$$

where f is an arbitrary function. Now the integral $\int d\chi \left\{ - I' \left(\frac{\partial X}{\partial \chi} z + XG \right) \right\}$ in Eq. (13) can be written as

$$\int d\chi \left\{ - \frac{G^2 R^2}{J} + \frac{Jf^2}{R^2} + II'X \left(\frac{\partial}{\partial \psi} \left(\frac{JX}{R^2} \right) + \frac{\partial}{\partial \chi} \left(\frac{Y}{R^2} \right) \right) \right\}, \qquad (16)$$

where we have used Eq. (15) and noted the periodicity in χ for all quantities. Similarly, using Eq. (4), the integrals

$$\int d\psi \, d\chi \left\{ j_\varphi R \left(\frac{XJ}{R^2} \frac{\partial X}{\partial \psi} + \frac{1}{R^2} \frac{\partial X}{\partial \chi} \right) + p'(\psi) X \left[\frac{\partial}{\partial \psi} (JX) + \frac{\partial Y}{\partial \chi} \right] \right\}$$

become

$$- \int d\psi \, d\chi \left\{ XII' \left[\frac{\partial}{\partial \psi} \left(\frac{XJ}{R^2} \right) + \frac{\partial}{\partial \chi} \left(\frac{Y}{R^2} \right) \right] + JX^2 \left[\frac{(II')'}{R^2} + p''(\psi) \right] - \frac{\partial}{\partial \psi} \left(\frac{J j_\varphi X^2}{R} \right) \right\}$$

$$(17)$$

- 5 -

Substitution of (16) and (17) into Eq. (13) leads to

$$\delta W_p = \int d\psi \, d\chi \left\{ \frac{J|\nabla X|^2}{R^2} + \frac{Jf^2}{R^2} + \frac{\gamma p}{J}\left(\frac{\partial}{\partial \psi}(JX) + \frac{\partial Y}{\partial \chi} \right)^2 \right.$$

$$\left. - JX^2\left[\frac{(II')'}{R^2} + p'' \right] + \frac{\partial}{\partial \psi}\left(\frac{J\,j_\varphi}{R} X^2 \right) \right\}. \qquad (18)$$

Consideration of Y indicates the minimising perturbation to be incompressible, and this yields the constraint

$$\oint X J \, d\chi = 0 . \qquad (19)$$

A further minimisation is achieved by setting $f = 0$. Using Eqs. (14) and (15), this leads to the additional constraint

$$\oint \frac{X J}{R^2} \, d\chi = 0 . \qquad (20)$$

The form for δW_p now becomes

$$\delta W_p = \int J \, d\chi \, d\psi \left\{ \frac{|\nabla X|^2}{R^2} - X^2\left[\frac{\partial}{\partial \psi}\left(\frac{j_\varphi}{R} \right) \right] \right\}_R - \int_b d\chi \, \frac{J\,j_\varphi}{R} X^2 , \qquad (21)$$

where $\frac{\partial}{\partial \psi}$ is to be carried out holding R constant. Our definitions indicate ψ to increase inwards from the plasma boundary, and hence the sign of the last term in Eq. (21) is negative.

We now minimise δW_v subject to the constraint $\nabla . \vec{\delta B} = 0$. Thus we can write

$$\vec{\delta B} = \frac{\vec{e}_\varphi \times \nabla X_v}{R} + \vec{\delta B}_\varphi , \qquad (22)$$

and the Euler equation for $\vec{\delta B}$ is $\nabla \times \vec{\delta B} = 0$. It is straightforward to show that the minimising X_v must satisfy the equation

$$\nabla . \left(\frac{1}{R^2} \nabla X_v \right) = 0 , \qquad (23)$$

for axisymmetric perturbations, and also that $\vec{\delta B}_\varphi = 0$. At the plasma-vacuum boundary we require the condition

$$\vec{n} . \vec{\delta B} = \vec{n} . \vec{Q} , \qquad (24)$$

to be satisfied. It follows from our definitions that condition (24) becomes

$$X = X_v . \qquad (25)$$

The total energy can be expressed as

$$\delta W = \int_P d\tau \left\{ \frac{|\nabla X|^2}{R^2} - X^2 \left[\frac{\partial}{\partial \psi} \left(\frac{j_\varphi}{R} \right) \right] \right\} - \int_b d\chi \frac{JX^2}{R} j_\varphi + \int_v d\hat{\tau} \frac{|\nabla X_v|^2}{R^2} , \quad (26)$$

where $d\tau = J\, d\psi\, d\chi$. This form is analogous to that given by Laval et al. [4] for the cylindrical approximation.

To continue further we take the trial-function $X = \alpha \dfrac{\partial \psi_P}{\partial Z}$ throughout the plasma, and $X_v = \alpha \dfrac{\partial \psi_P}{\partial Z}$ in the vacuum, α being a constant. By Eq. (10) this choice ensures that the constraints (19) and (20) are trivially satisfied; the trial-function for the vacuum region clearly satisfies the condition that X_v vanish at infinity. From Eq. (8)

$$\nabla \cdot \left(\frac{1}{R^2} \nabla \psi_P \right) = 0 , \quad (27)$$

in the vacuum region, and hence $X_v = \alpha \dfrac{\partial \psi_P}{\partial Z}$ is a solution of Eq. (23). Using Eqs. (6), (23) and (25), together with Gauss' theorem, δW becomes

$$\delta W = \int_b dS \frac{X}{R^2} [\vec{n} \cdot \nabla X]_v^P - \int_b \frac{j_\varphi}{R} X^2 J\, d\chi + \alpha^2 \int_b j_\varphi \frac{\partial \psi_w}{\partial Z} dR$$

$$- \alpha^2 \int_P d\tau \frac{j_\varphi}{R} \frac{\partial^2 \psi_w}{\partial Z^2} - \alpha^2 \int_P d\tau \left(\frac{\partial \psi_w}{\partial Z} \right)^2 \left[\frac{\partial}{\partial \psi} \left(\frac{j_\varphi}{R} \right) \right]_R , \quad (28)$$

where $[\vec{n} \cdot \nabla X]_v^P$ denotes the jump in $\vec{n} \cdot \nabla X$ at the boundary, and $\vec{n}$ is the outward unit normal. Differentiating Eq. (8) with respect to Z and then integrating it through the boundary, we obtain

$$[\vec{n} \cdot \nabla X]_v^P = \alpha \left(\frac{j_\varphi}{B_\chi} \cdot \frac{\partial \psi}{\partial Z} \right)_b , \quad (29)$$

which is equivalent to the perturbed pressure balance equation. Substituting Eq. (29) into Eq. (28) leads to the result

$$\delta W = - \alpha^2 \int_P d\tau \frac{j_\varphi}{R} \frac{\partial^2 \psi_w}{\partial Z^2} - \alpha^2 \int_P d\tau \left(\frac{\partial \psi_w}{\partial Z} \right)^2 \left[\frac{\partial}{\partial \psi} \left(\frac{j_\varphi}{R} \right) \right]_R - \alpha^2 \int_b d\chi \frac{J j_\varphi}{R} \left(\frac{\partial \psi_w}{\partial Z} \right)^2 .$$

$$(30)$$

We now discuss this result, and begin by showing the first integral to be related to the decay index. In order to fix our ideas concerning signs, we consider the illustrative example given in the figure. This shows a toroidal plasma maintained by an external vertical field combined with that of a quadrupole. Thus, using our previous definitions, we write the decay index as

$$N = - \frac{R}{|B_z^w|} \frac{\partial}{\partial R} |B_z^w| , \qquad (31)$$

and it follows that δW can be expressed in the form

$$\delta W = \alpha^2 \int_P d\tau \, N \, \frac{j_\varphi |B_z^w|}{R} - \alpha^2 \int_b d\chi \, \frac{J j_\varphi}{R} \left(\frac{\partial \psi_w}{\partial Z} \right)^2 - \alpha^2 \int_P d\tau \left(\frac{\partial \psi_w}{\partial Z} \right)^2 \left(\frac{\partial}{\partial \psi} \left(\frac{j_\varphi}{R} \right) \right)_R .$$

$$(32)$$

We observe that, for an equilibrium with j_φ everywhere positive, and the decay index negative throughout the region occupied by the plasma, then the first integral is destabilising. In fact, $N < 0$ implies a vertically elongated plasma. The second integral is destabilising if j_φ at the boundary is positive. The third integral is destabilising if $\frac{\partial}{\partial \psi} j_\varphi \geq 0$ along each plane $R = \text{constant}$. Alternatively, since

$$\left. \frac{\partial j_\varphi}{\partial Z} \right|_R = \left. \frac{\partial \psi}{\partial Z} \right|_R \left. \frac{\partial j_\varphi}{\partial \psi} \right|_R , \qquad (33)$$

it follows that this integral is negative if $\frac{1}{B_R} \left. \frac{\partial j_\varphi}{\partial Z} \right|_R \leq 0$. To keep the argument straightforward, we now discuss these inequalities for plasmas with cross-sections of simple, but otherwise arbitrary shape. For a 'flat' current the j_φ contours are vertical planes, and $\left. \frac{\partial j_\varphi}{\partial \psi} \right|_R = \frac{\partial j_\varphi}{\partial Z} = 0$ throughout the plasma. For a current profile which has a positive and constant j_φ around the boundary, and a single maximum on the $Z = 0$ plane, the above inequalities are clearly satisfied; this also covers the case $j_\varphi = 0$ at the boundary. We can also discuss configurations with two current maxima, for example, the doublet. In this case, for currents which decrease away from each maximum, the conditions $\left. \frac{\partial j_\varphi}{\partial \psi} \right|_R \geq 0$ and $\frac{1}{B_R} \frac{\partial j_\varphi}{\partial Z} \leq 0$ are again satisfied.

Thus we can conclude that the third integral is destabilising for currents which decrease monotonically towards the plasma boundary, and hence the theorem stated in the introduction. It is likely that our theorem is also valid for equilibria which are strongly distorted or indented. However, for such an equilibrium, it would be necessary to examine the particular ψ and j_φ contours before passing judgment as to its stability against $n = 0$.

III AXISYMMETRIC MODES IN STRAIGHT PLASMAS

We now consider the large aspect-ratio or straight case. Setting $R = R_o + x$ we take R_o to be a very large constant. The basic equilibrium equation now becomes

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = - R_o \, j_\varphi \, , \tag{34}$$

where

$$j_\varphi = R_o \, p'(\psi) + \frac{II'(\psi)}{R_o} \, . \tag{35}$$

We confine ourselves to configurations which are symmetric about both the $z = 0$ and $x = 0$ planes. Thus as well as the conditions (9) and (10), we now have

$$\psi_p(x) = \psi_p(-x) \, , \tag{36}$$

and

$$\frac{\partial}{\partial x} \psi_p(x) = - \frac{\partial}{\partial x} \psi_p(-x) \, . \tag{37}$$

These additional conditions allow us to introduce the new trial-function $X = \alpha \, \dfrac{\partial \psi_p}{\partial x}$ in the plasma, and $X_v = \alpha \, \dfrac{\partial \psi_p}{\partial x}$ in the vacuum region. Thus we obtain

$$\delta W_x = - \, \alpha^2 \int_P j_\varphi \frac{\partial^2 \psi_w}{\partial x^2} \, dx \, dz - \alpha^2 \int_P \left(\frac{\partial \psi_w}{\partial x} \right)^2 \frac{dj_\varphi}{d\psi} \, dx \, dz - \alpha^2 \int_b \frac{d\chi}{|\nabla\psi| \, |\nabla\chi|} \, j_\varphi \left(\frac{\partial \psi_w}{\partial x} \right)^2 \tag{38}$$

Using the previous trial-function, we obtain

$$\delta W_z = - \, \alpha^2 \int_P j_\varphi \frac{\partial^2 \psi_w}{\partial z^2} \, dx \, dz - \alpha^2 \int_P \left(\frac{\partial \psi_w}{\partial z} \right)^2 \frac{dj_\varphi}{d\psi} \, dx \, dz - \alpha^2 \int_b \frac{d\chi}{|\nabla\psi| \, |\nabla\chi|} \, j_\varphi \left(\frac{\partial \psi_w}{\partial z} \right)^2 \, . \tag{39}$$

Adding Eqs. (38) and (39), and noting that

$$\frac{\partial^2 \psi_w}{\partial x^2} + \frac{\partial^2 \psi_w}{\partial z^2} = 0 \, , \tag{40}$$

we derive

$$\delta W_x + \delta W_z = -\alpha^2 \int_P (\nabla \psi_w)^2 \frac{dj_\varphi}{d\psi} \, dx \, dz - \alpha^2 \int_b \frac{d\chi}{|\nabla \psi| \, |\nabla \chi|} \, j_\varphi \, (\nabla \psi_w)^2 \tag{41}$$

Taking $j_\varphi \geq 0$, we observe that for $\frac{dj_\varphi}{d\psi} \geq 0$ everywhere, the right hand side of Eq. (41) is negative. Hence either δW_x or δW_z (or both) must be negative, and thus the equilibrium is unstable. As before, since ψ increases inwards, it follows that any straight system for which j_φ decreases monotonically towards the boundary, is unstable. We note that for the case of uniform current, that is $\frac{dj_\varphi}{d\psi} = 0$, instability is due to the boundary term alone.

IV   DISCUSSION

We have established a theorem for the situation where the decay index (of the vacuum field) is negative throughout the region occupied by the plasma. In the case of positive j_φ we have demonstrated the adverse effects of a monotonically decreasing current and j_φ finite at the surface. We have not been able to draw any definite conclusions for the case of current reversal. A cursory examination of Eq. (32), however, indicates the possibility of beneficial effects in the presence of reversed current.

So far we have made no attempt to interpret the trial-functions $X = \alpha \frac{\partial \psi_p}{\partial Z}$ and $X_v = \alpha \frac{\partial \psi_p}{\partial Z}$. We note that had we assumed $X = \alpha \frac{\partial \psi}{\partial Z}$ in the plasma, then α is directly interpretable as a uniform vertical displacement, ξ_z. Taking $X_v = \alpha \frac{\partial \psi}{\partial Z}$ in the vacuum, however, is unsatisfactory for two reasons : firstly, ψ becomes singular in the vicinity of the wires, and secondly, X_v is not a minimising solution. This is confirmed by a comparison with the special case of the large aspect ratio ellipse. Thus inserting these test functions in our general formalism leads to $\delta W = 0$, whilst in the case of the large aspect ratio ellipse, δW is negative [4, 5].

We now investigate the nature of the trial-functions used in the present paper. In particular, we consider the equations

$$\nabla \cdot \vec{\xi} = 0 \qquad (42)$$

and

$$\vec{\xi} \cdot \nabla \psi = \alpha \frac{\partial}{\partial Z} (\psi - \psi_w) \qquad (43)$$

in the neighbourhood of the magnetic axis. We denote all quantities at the magnetic axis $(R = R_o , Z = 0)$ by the subscript zero, and define $x = R - R_o$. Thus, Taylor expanding $\vec{\xi}$, ψ and ψ_w, and taking into account the vertical symmetry of the stream-functions, it is straightforward to derive

$$\left(\frac{\partial \xi_x}{\partial Z} \right)_o \left(\frac{\partial^2 \psi}{\partial x^2} \right)_o + \left(\frac{\partial \xi_z}{\partial x} \right)_o \left(\frac{\partial^2 \psi}{\partial Z^2} \right)_o + \xi_{zo} \left(\frac{\partial^3 \psi}{\partial Z^2 \partial x} \right)_o$$

$$= \alpha \left(\left(\frac{\partial^3 \psi}{\partial Z^2 \partial x} \right)_o - \left(\frac{\partial^3 \psi_w}{\partial Z^2 \partial x} \right)_o \right) , \qquad (44)$$

where α can now be expressed as

$$\alpha = \xi_{zo} \left[1 - \left(\frac{\partial^2 \psi_w}{\partial Z^2} \right)_o \left(\frac{\partial^2 \psi}{\partial Z^2} \right)_o^{-1} \right]^{-1} . \qquad (45)$$

Thus if we regard ξ_{zo} and $\left(\frac{\partial \xi_x}{\partial Z} \right)_o$ as arbitrarily prescribed quantities, then we can derive $\left(\frac{\partial \xi_z}{\partial x} \right)_o$ from Eq. (44). Finally, we deduce that

$$\vec{\xi} = \left[\xi_o + x \left(\left(\frac{\partial \xi_z}{\partial x} \right)_o + \left(\frac{\partial \xi_x}{\partial Z} \right)_o \right) \right] \vec{e}_z + \left(\frac{\partial \xi_x}{\partial Z} \right)_o \vec{e}_\varphi \times \vec{r} \qquad (46)$$

where

$$\vec{r} = x \vec{e}_x + Z \vec{e}_z . \qquad (47)$$

Thus we observe that $\vec{\xi}$ in the vicinity of the magnetic axis comprises an arbitrary vertical rigid displacement, combined with an arbitrary rigid rotation, as well as a shear in the vertical direction, the magnitude of which is itself determined by the rotation and displacement.

- 11 -

V   CONCLUSIONS

We have considered the stability of an axisymmetric, arbitrary cross-section, arbitrary aspect ratio toroidal plasma, to axisymmetric modes. We have shown that configurations with toroidal current which decreases monotonically towards the boundary, and are maintained by an external magnetic field with negative decay index, must always be unstable in the absence of any external stabilising agencies such as active or passive feedback.

REFERENCES

[1] OSOVETS, S.M., in Fizika plazmy i problema upravljaemyh termojadernyhi reackcij (Plasma Physics and the Problems of Controlled Thermonuclear Reactions) 2, Izd. Akad. Nauk SSSR, Moscow (1958) 238.

[2] YOSHIKAWA, S., Phys. Fluids 7 (1964) 278.

[3] MUKHOVATOV, V.S., SHAFRANOV, V.D. Nucl. Fusion 11 (1971) 605.

[4] LAVAL, G., PELLAT, R., SOULE, J.S., Phys. Fluids 17, 4 (1974) 835.

[5] HAAS, F.A., Nucl Fusion 15 (1975) 407.

[6] OKABAYASHI, M., SHEFFIELD, G., Nucl Fusion 14 (1974) 263.

[7] LACKNER, K., MacMAHON, A.B., Nucl. Fusion 14 (1974) 575.

[8] WESSON, J.A., SYKES, A., in Plasma Physics and Controlled Nuclear Fusion Research (Proc. 5th. Int. Conf., Tokyo, 1974), IAEA-CN-33/A 12 - 3.

[9] GRUBER, O., KRAUSE, H., MAST, F., WILHELM, R., in Plasma Physics and Controlled Nuclear Fusion Research (Proc. 6th. Int. Conf., Berchtesgaden, 1976). IAEA-CN-35/A10 - 2.

[10] CIMA, G., ROBINSON, D.C., THOMAS, C.Ll., WOOTTON, A.J., in Plasma Physics and Controlled Nuclear Fusion Research (Proc. 6th. Int. Conf., Berchtesgaden, 1976). IAEA-CN-35/A10 - 5.

[11] MERCIER, C., Nucl. Fusion 1 (1960) 47.

[12] BERNSTEIN, I.B., FRIEMAN, E.A., KRUSKAL, M.D., KULSRUD, R.M., Proc. Roy. Soc. A 244 (1958) 17.

[13] JUKES, J.D., Nucl. Fusion 10 (1970) 131
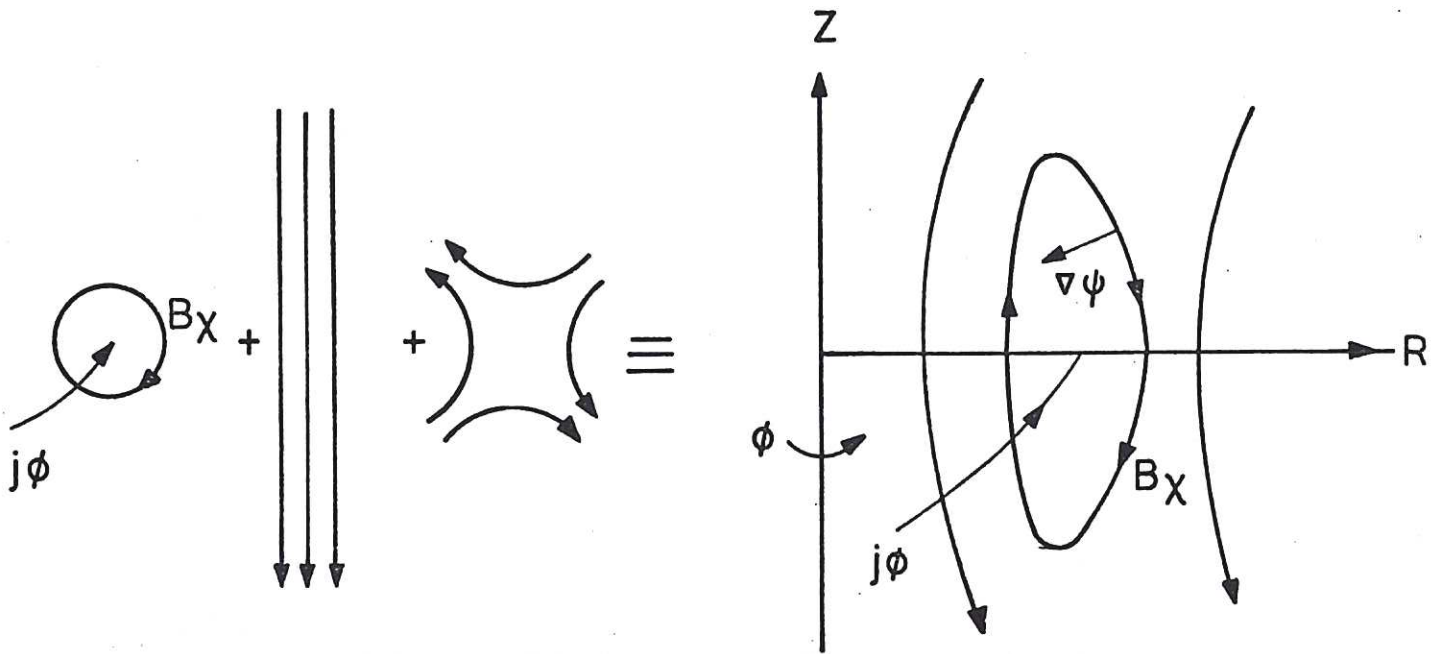
Example of a vertically elongated plasma maintained in equilibrium by external fields.