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Preprint

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R BINGHAM
C N LASHMORE-DAVIES

CULHAM LABORATORY
Abingdon Oxfordshire

1977

CLM-P479

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R. Bingham and C.N. Lashmore-Davies
Culham Laboratory, Abingdon, Oxon, OX14 3DB, UK
(Euratom/UKAEA Fusion Association)

ABSTRACT

The problem of the filamentation of an electro-magnetic wave in a plasma has been formulated in terms of the non-linear interaction between the initial plane electro-magnetic wave, the Stokes and anti-Stokes electro-magnetic waves and a density perturbation. Since a perturbation procedure is used the analysis is limited to situations where the total pump wave energy density is a small fraction of the energy density in the undisturbed plasma. A crucial ingredient of the problem is the frequency mis-match between the interacting waves although only those waves are included which satisfy perfect k-matching. The pump wave is treated on the same footing as the other electro-magnetic waves and these waves are treated as distinct throughout the interaction. When the pump amplitude is assumed to remain constant the equations yield the linear threshold and growth rate. The non-linear problem is formulated first with the inclusion of ion inertia. The resulting equations can be shown to lead to the conservation of wave energy, momentum and action densities. By neglecting ion inertia, three coupled non-linear equations (with cubic non-linearities) are obtained for the initial, Stokes and anti-Stokes electro-magnetic waves. These equations have been solved analytically under various assumptions. First, with neglect of damping and spatial dependence the non-linear time evolution is obtained. The behaviour is periodic and is described by Jacobi Elliptic functions. Second, the time evolution has also been obtained with the inclusion of damping and finally, the stationary, spatially dependent, solutions have been obtained. These solutions are, in general, periodic but include solitary wave solutions as a special case.

(Submitted for publication in Nuclear Fusion)

February 1977

I. INTRODUCTION

When an intense wave field is present in a plasma it can give rise to a number of interesting and important non-linear effects. One such class of effects, which is attracting a great deal of attention at the present time, is that of the purely growing instabilities. The first examples of this type are the oscillating two stream instability [1] and the modulational instability of a long wavelength Langmuir wave [2]. Other examples are the self-modulation and filamentation [3 - 7] instabilities of electro-magnetic waves. These instabilities are all of one basic type and can be expected to have similar properties. In fact, in the limit of an infinite wavelength pump, it has been shown that the modulational instability of a Langmuir wave and the oscillating two stream instability are indistinguishable [8].

So far, most effort has been invested in the modulational instability of a Langmuir wave where some very remarkable non-linear properties have been discovered [9]. On the one hand, for the one dimensional problem, soliton solutions have been shown to occur. On the other hand, for two or more dimensions, a spatial collapse of the excited Langmuir wave field has been found [9]. For the models analyzed this collapse proceeds without limit. This result, however, is evidently due to the neglect of ion inertia and dissipation [10]. Similar behaviour has been found for the analogous problem for an electro-magnetic wave [11].

A characteristic feature of these effects is that the initial wave (or pump wave) modifies the properties of the plasma medium in such a way that the frequencies of the excited waves or perturbations are shifted from the values these waves would have in the absence of the pump wave. The non-linear theory of Langmuir wave solitons, and Langmuir wave collapse [9] has considered only the resultant envelope of the pump wave and excited high frequency waves. Consequently the connection between the initial phases of the instability and its later non-linear development remains unclear. In this paper, using the specific example of the filamentation of an electro-magnetic wave, we shall describe a different approach from the ones previously taken [9-11]. Instead of averaging over the pump and high frequency waves we shall formulate the problem in such a way that the pump and excited waves are distinguished from one another throughout the interaction. This procedure shows very clearly the four wave nature of the interaction. We emphasise that although we have analyzed the specific example of electro-magnetic filamentation, the oscillating two stream instability, Langmuir modulational instability and related effects can be treated in a similar manner and one would expect qualitatively similar results.

II. MODEL AND DERIVATION OF NON-LINEAR EQUATIONS

The model we shall use has already been described in detail elsewhere [6]. Here we shall simply give a very brief description for the sake of completeness. The filamentation mechanism is illustrated in figures 1a and 1b $(\omega_o, \underset{\sim}{k}_o)$ is the initial plane electromagnetic pump wave which interacts with a density perturbation $(\Omega, \underset{\sim}{k}_s)$, whose wave number $\underline{k}_s$ is perpendicular to $\underline{k}_o$, and the Stokes electromagnetic wave $(\omega_1, \underline{k}_o - \underline{k}_s)$. The pump wave also couples with the anti-Stokes electromagnetic wave $(\omega_2, \underline{k}_o + \underline{k}_s)$ through the density perturbation $(\Omega, -\underline{k}_s)$. The non-linear wave interaction scheme is chosen by imposing perfect $\underline{k}$-matching. Since the unperturbed frequencies of the Stokes and anti-Stokes waves are given by $\omega_{1,2}^2 = \omega_{pe}^2 + c^2 k_{1,2}^2$ where $\underset{\sim}{k}_{1,2} = \underline{k}_o \mp \underline{k}_s$ no decay instability is possible since $\omega_1 = \omega_2 > \omega_o$.

The configuration of the problem is taken to be the following. $\underline{k}_o = (k_o, 0, 0)$, $\underline{E}_o = (0, 0, E_z)$, $\underline{B}_o = (0, B_y, 0)$ and $\underline{k}_s = (0, k_s, 0)$ where the subscript 'o' refers to the pump electromagnetic wave. In addition, the Stokes and anti-Stokes waves are also assumed to be linearly polarized with their $\underline{E}$-vectors in the z-direction.

We now use a simple (isothermal) two fluid model together with Maxwell's equations to describe the behaviour of perturbations in the presence of the initial electromagnetic wave. The equation for an electromagnetic perturbation $(\omega, \underline{k})$ in the presence of the initial electromagnetic wave and a density perturbation varying as $\exp i(k_s y - \Omega t)$ is

$$\left(k^2 + \frac{\omega_{pe}^2}{c^2} - \frac{\omega^2}{c^2}\right) E_z = - i\omega \mu_o e n_{es} v_z - \mu_o n_o e \nu_e v_z \tag{1}$$

where ω_{pe} is the electron plasma frequency, c the velocity of light in free space, μ_o the magnetic permeability of free space, e the charge of a proton, n_{es} the perturbed electron density, v_z the electron velocity field due to all the electromagnetic waves present, n_o the equilibrium density, ν_e the electron-ion collision frequency and the units are MKS. The first term on the right hand side of equation (1) provides the non-linear coupling to the density perturbation and the second term represents the damping of the electromagnetic wave.

The equation for the density perturbation $(\Omega, \underline{k}_s)$ which is assumed to vary as $\exp i(k_s y - \Omega t)$ is

$$(\Omega^2 - k_s^2 c_s^2) n_{es} = -i n_o \frac{e}{m_i} k_s (\underline{v}_e \times \underline{B})_y - \nu_i \Omega n_{is} \tag{2}$$

where $c_s^2 (= k T_e/m_i)$ is the ion acoustic speed, T_e is the electron temperature, m_i the ion mass, ν_i a phenomenological damping term to simulate the dissipation of the low frequency perturbation and n_{is} is the perturbed ion density. We shall assume that $n_{is} \approx n_{es}$ and use n_{es} to describe the low frequency density perturbation. We have also assumed that the electrons are much hotter than the ions (although this is not essential).

We now write the Stokes and anti-Stokes waves and the density perturbation as

$$E_{1,2}(\underline{x},t) = \mathcal{E}_{1,2}(x,t)\, e^{i(\underline{k}_{1,2}\cdot\underline{x} - \omega_{1,2} t)}$$

$$n_{es}(x,y,t) = N_s(x,t)\, e^{i k_s y}$$

where the slow amplitude variation of the electromagnetic waves is determined by the linear dissipation and the non-linear interaction and the phase factor is due to the linear dispersion. For the density perturbation the time variation is dominated by the non-linear coupling. Using a perturbation procedure on equations (1) and (2) we now obtain equations for $\mathcal{E}_{1,2}$ and N_s. These equations result from imposing perfect $\underline{k}$-matching of the interacting waves but allowing for a small frequency mis-match. The equations are

$$\left(\frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x} + \gamma_T\right) \mathcal{E}_1(x,t) = -i c_{so} N_s^* \mathcal{E}_o e^{-i\delta t} \tag{3}$$

$$\left(\frac{\partial}{\partial t} + v_2 \frac{\partial}{\partial x} + \gamma_T\right) \mathcal{E}_2(x,t) = -i c_{so} N_s \mathcal{E}_o e^{-i\delta t} \tag{4}$$

$$\left(\frac{\partial^2}{\partial t^2} + \gamma_s \frac{\partial}{\partial t} + k_s^2 c_s^2\right) N_s(x,t) = -c_{o1}\left(\mathcal{E}_o \mathcal{E}_1^* e^{-i\delta t} + \mathcal{E}_o^* \mathcal{E}_2 e^{i\delta t}\right) \tag{5}$$

where $v_{1,2} \equiv c^2 k_o/\omega_{1,2}$, $\gamma_T \equiv \nu_e \omega_{pe}^2/2\omega_1^2$, $c_{so} \equiv e^2/4\omega_1 \epsilon_o m_e$[12], $\delta = \omega_o - \omega_1$, ϵ_o is the dielectric constant of free space, m_e the electron mass, γ_s simulates the damping of the density perturbation and $c_{o1} \equiv n_o e^2 k_s^2/2 m_i m_e \omega_o \omega_1$.

In deriving these equations, we have used the fact that $\Omega \ll \omega_o$. The quantity $\mathcal{E}_o$ is the slowly varying amplitude of the pump wave defined by

$$E_o(x,t) = \mathcal{E}_o(x,t)\, e^{i(k_o x - \omega_o t)}$$

We note that when $\mathcal{E}_o$ is treated as a constant field equations (3) - (5) are then linearized and can be solved to give Nishikawa's general dispersion relation [1]. Since δ is necessarily negative for this problem equations (3) - (5) then yield the threshold and initial growth rate for electromagnetic filamentation [4,6].

We now wish to describe a non-linear theory of the development of electromagnetic filamentation involving only those waves which take part in the basic interaction. In other words we would like to perform the analogous non-linear calculation for the filamentation instability which Armstrong et al [13] and Galeev and Sagdeev [14] have performed for the decay instability in which the non-linear evolution of only those waves taking part in the instability are included. In order to do this we must add an equation for the slowly varying (on the ω_o time scale) pump amplitude to equations (3) - (5). This equation can also be obtained from equation (1) and is derived in a similar way to equations (3) and (4). The equation for $\mathcal{E}_o(x,t)$ is

$$\left(\frac{\partial}{\partial t} + v_o \frac{\partial}{\partial x} + \gamma_T\right)\mathcal{E}_o(x,t) = -i c_{s1}\left(N_s\, \mathcal{E}_1\, e^{i\delta t} + N_s^*\, \mathcal{E}_2\, e^{i\delta t}\right) \tag{6}$$

where $c_{s1} \equiv e^2/4\omega_o \epsilon_o m_e$ and $v_o \equiv c^2 k_o/\omega_o$.

A complete solution of the non-linear problem involving only the waves taking part in the initial instability requires the simultaneous solution of equations (3) - (6). We have not yet succeeded in obtaining such a solution although we can demonstrate that equations (3) - (6) lead to the conservation of energy, momentum and action densities for the interaction. We shall say more about this later.

We shall now obtain solutions to equations (3) - (6) under various physical restrictions. First, let us consider the spatially independent problem, i.e. the perturbations are not assumed to be localized but to be generated throughout the whole medium. The wave amplitudes then depend only on the time variable. Next, we treat the density perturbation as a driven response to the high frequency electromagnetic fields, i.e. we neglect $\partial^2/\partial t^2$ and $\gamma_s \, \partial/\partial t$ in comparison with $k_s^2 c_s^2$ in equation (5). This approximation is equivalent to the neglect of ion inertia (or the static approximation) by other authors [9-11]. The conditions under which this will be a good approximation can be obtained from a consideration of the minimum threshold. This occurs when $\delta = -\gamma_T$ so that using the result $\delta = -c^2 k_s^2/2\omega_o$ an estimate for k_s can be obtained. With this value we find the plasma conditions, namely density and temperature, and the initial pump amplitude, such that

$$\gamma_T \ll \gamma \ll k_s c_s$$

For a neodymium glass laser, the required conditions are satisfied for $T_e \gtrsim 1$ keV and $n_o \lesssim 10^{19}/\text{cm}^3$ and for a carbon dioxide gas laser when $T_e \gtrsim 1$ keV and $n_o \lesssim 10^{18}/\text{cm}^3$. In other words, the theory we shall give below, applies to the underdense regions of neodymium or carbon dioxide laser created plasmas. The condition on the pump amplitude is consistent with the perturbation requirement $e^2 E_o^2/m_e^2 \omega_o^2 \ll v_{Te}^2$.

Under these conditions we can obtain $N_s(t)$ directly from equation (5)

$$N_s(t) \approx -\frac{c_{o1}}{k_s^2 c_s^2}\left(\mathcal{E}_o \mathcal{E}_1^* e^{-i\delta t} + \mathcal{E}_o^* \mathcal{E}_2 e^{i\delta t}\right) \qquad (7)$$

Substituting this expression for $N_s(t)$ into equations (3), (4) and (6) leads to the following three equations for $\mathcal{E}_o$, $\mathcal{E}_1$ and $\mathcal{E}_2$

$$\frac{\partial \mathcal{E}_o(t)}{\partial t} = i\frac{\Gamma}{\omega_o}\left(|\mathcal{E}_1|^2\mathcal{E}_o + |\mathcal{E}_2|^2\mathcal{E}_o + 2\mathcal{E}_o^* \mathcal{E}_1 \mathcal{E}_2 e^{i2\delta t}\right) \qquad (8)$$

$$\frac{\partial \mathcal{E}_1(t)}{\partial t} = i\frac{\Gamma}{\omega_1}\left(|\mathcal{E}_o|^2\mathcal{E}_1 + \mathcal{E}_o^2 \mathcal{E}_2^* e^{-i2\delta t}\right) \qquad (9)$$

$$\frac{\partial \mathcal{E}_2(t)}{\partial t} = i\frac{\Gamma}{\omega_1}\left(\mathcal{E}_o^2 \mathcal{E}_1^* e^{-i2\delta t} + |\mathcal{E}_o|^2\mathcal{E}_2\right) \qquad (10)$$

where $\Gamma \equiv e^2 \omega_{pe}^2 / 8 m_i m_e \omega_o \omega_1 c_s^2$. Equations (8) - (10) are the ones we shall solve to describe the time evolution of the instability for the restricted range of conditions discussed above. However, before attempting to solve these equations let us consider the conservation relations which are contained in them.

III. CONSERVATION RELATIONS

We have already noted the fact that a characteristic feature of purely growing instabilities is that the pump wave modifies the properties of the plasma medium in such a way that the high frequency excited waves have their frequencies shifted away from their natural values (i.e. their value in the absence of the pump). In fact, the high frequency waves have their frequencies shifted down to that of the pump wave itself. Before attempting to demonstrate the conservation of wave energy density we must first of all obtain an expression for the total wave energy density allowing for this frequency shift due to the presence of the pump wave. We can do this most directly by starting from the following expression for the total energy density of an electro-magnetic wave in a plasma

$$\epsilon = \frac{1}{4} \mu_o \underset{\sim}{H} . \underset{\sim}{H}^* + \frac{1}{4} \epsilon_o \underset{\sim}{E} . \underset{\sim}{E}^* + \frac{1}{4} n_o m_e \underset{\sim}{v} . \underset{\sim}{v}^* \tag{11}$$

where $\underset{\sim}{E} = (0, 0, E_z)$, $\underset{\sim}{H} = (H_x, H_y, 0)$ and $\underset{\sim}{k} = (k_x, k_y, 0)$. Expressing all variables in terms of E_z we can write equation (11) as

$$\epsilon = \left(\frac{ck^2}{\omega^2} + 1 + \frac{\omega_{pe}^2}{\omega^2} \right) \frac{1}{4} \epsilon_o |E_z|^2 . \tag{12}$$

To obtain the final form for the total wave energy density we must evaluate equation (12) at the eigen frequency of the wave. In the presence of the pump wave (ω_o, k_o) the Stokes and anti-Stokes electro-magnetic waves are both excited at the frequency ω_o. Therefore, substituting $\omega = \omega_o$ and $k = k_{1,2}$ in equation (12) we obtain the following expression for the energy density of the excited waves

$$\epsilon_{1,2} = \left(1 + \frac{c^2 k_s^2}{2\omega_o^2} \right) \frac{1}{2} \epsilon_o |E_{1,2}|^2 \tag{13}$$

However, the mis-match parameter δ is given by

$$\delta = - \frac{c^2 k_s^2}{2\omega_o} .$$

We finally obtain the following expression for the total wave energy densities of the excited electro-magnetic waves in the presence of the pump wave

$$\epsilon_{1,2} = \left(1 - \frac{\delta}{\omega_o} \right) \frac{1}{2} \epsilon_o |E_{1,2}|^2 \tag{14}$$

The total energy of the pump wave is given by the usual expression

$$\epsilon_{pump} = \frac{1}{2} \epsilon_o |E_o|^2$$

Armed with equation (14) we can now attempt to demonstrate the various conservation relations from equations (8) - (10). Let us introduce new amplitude variables

$$\alpha_o = \left(\frac{\epsilon_o}{2} \right)^{\frac{1}{2}} \mathcal{E}_0 \; ; \quad \alpha_1 = \left(\frac{\epsilon_o}{2} \right)^{\frac{1}{2}} \mathcal{E}_1 e^{i\delta t} \; ; \quad \alpha_2 = \left(\frac{\epsilon_o}{2} \right)^{\frac{1}{2}} \mathcal{E}_2^* e^{-i\delta t} .$$

The significance of these variables is that $|\alpha_o|^2$ represents the total energy density of the pump wave and the squares of the other amplitudes have the dimensions of energy density. Equations (8) - (10) may now be written

$$\frac{\partial \alpha_o}{\partial t} = \frac{2i\Gamma}{\epsilon_o \omega_o} \left(|\alpha_1|^2 \alpha_o + |\alpha_2|^2 \alpha_o + 2 \alpha_o^* \alpha_1 \alpha_2^* \right) \tag{15}$$

$$\left(\frac{\partial}{\partial t} - i\delta \right) \alpha_1 = \frac{2i\Gamma}{\epsilon_o \omega_1} \left(|\alpha_o|^2 \alpha_1 + \alpha_o^2 \alpha_2 \right) \tag{16}$$

$$\left(\frac{\partial}{\partial t} + i\delta \right) \alpha_2 = - \frac{2i\Gamma}{\epsilon_o \omega_1} \left((\alpha_o^*)^2 \alpha_1 + |\alpha_o|^2 \alpha_2 \right) \tag{17}$$

From equation (15) we obtain

$$\frac{\partial}{\partial t} |\alpha_o|^2 = - \frac{i 4\Gamma}{\epsilon_o \omega_o} \left\{ \alpha_o^2 \alpha_1^* \alpha_2 - (\alpha_o^*)^2 \alpha_1 \alpha_2^* \right\} \tag{18}$$

Similarly from (16) we have

$$\frac{\partial}{\partial t}|\alpha_1|^2 = \frac{i2\Gamma}{\epsilon_o \omega_o\left(1-\frac{\delta}{\omega_o}\right)}\left\{\alpha_o^2 \; \alpha_1^* \alpha_2 - (\alpha_o^*)^2 \; \alpha_1 \; \alpha_2^*\right\} \qquad (19)$$

From these equations we see that

$$\frac{\partial}{\partial t}\left(1-\frac{\delta}{\omega_o}\right)|\alpha_1|^2 = -\frac{1}{2}\frac{\partial}{\partial t}|\alpha_o|^2 \; . \qquad (20)$$

Similarly, from (17) and (18) we have

$$\frac{\partial}{\partial t}\left(1-\frac{\delta}{\omega_o}\right)|\alpha_2|^2 = -\frac{1}{2}\frac{\partial}{\partial t}|\alpha_o|^2 \; . \qquad (21)$$

Combining equations (20) and (21) we obtain

$$\frac{\partial}{\partial t}\left\{|\alpha_o|^2 + \left(1-\frac{\delta}{\omega_o}\right)|\alpha_1|^2 + \left(1-\frac{\delta}{\omega_o}\right)|\alpha_2|^2\right\} = 0 \qquad (22)$$

which we recognise immediately as the conservation of wave energy density for the interaction. The remaining conservation relations follow at once from equation (22). Since both excited waves occur at ω_o we simply divide throughout equation (22) by ω_o giving

$$\frac{\partial}{\partial t}\left\{\frac{|\alpha_o|^2}{\omega_o} + \left(1-\frac{\delta}{\omega_o}\right)\frac{|\alpha_1|^2}{\omega_o} + \left(1-\frac{\delta}{\omega_o}\right)\frac{|\alpha_2|^2}{\omega_o}\right\} = 0 \qquad (23)$$

which is the conservation of action density. The conservation of wave momentum can easily be obtained from equation (23) since we have already imposed perfect $\underset{\sim}{k}$-matching. Equation (23) is equivalent to the Manley-Rowe relations for a three wave interaction. We see clearly from this equation that the basic mechanism of filamentation is a <u>four wave interaction</u> in which two pump "quanta" create two excited wave "quanta" or vice versa.

Finally, it is worth noting that although the above conservation relations have been obtained from equations (8) - (10), exactly the same results can be obtained from equations (3) - (6). In other words these conservation relations do not depend on the neglect of the ion inertia (or the static approximation for the ions). This is demonstrated in the appendix. We also see that in the conservation relations the unperturbed energy density for the pump wave appears. This is evidently due to the fact that the Stokes, anti-Stokes and density perturbations drive the pump wave resonantly whereas the Stokes and anti-Stokes waves are driven off resonance resulting in their frequency shift.

IV THE NON-LINEAR SOLUTION

(i) Time Dependent, Spatially Independent case.

We shall now obtain an exact solution of the non-linear equations (8) - (10). In order to solve these equations we follow a procedure similar to that of Armstrong et al [13]. We therefore write the complex amplitudes $\mathcal{E}_n(t)$ as follows

$$\mathcal{E}_n(t) = a_n(t)\, e^{i\varphi_n(t)} \quad ; \quad n = 0,\ 1 \text{ or } 2$$

where $a_n(t)$ and $\varphi_n(t)$ are real functions of time. Substituting these expressions into equations (8) - (10) we obtain

$$\frac{\partial a_o}{\partial t} = \frac{2\Gamma}{\omega_o} a_o\, a_1\, a_2 \sin\theta(t) \tag{24}$$

$$\frac{\partial a_1}{\partial t} = -\frac{\Gamma}{\omega_1} a_o^2\, a_2 \sin\theta(t) \tag{25}$$

$$\frac{\partial a_2}{\partial t} = -\frac{\Gamma}{\omega_2} a_o^2\, a_1 \sin\theta(t) \tag{26}$$

where

$$\theta(t) \equiv 2\,\varphi_o(t) - \varphi_1(t) - \varphi_2(t) - 2\,\delta t$$

$$\frac{\partial \varphi_o}{\partial t} = \frac{\Gamma}{\omega_o}\left(a_1^2 + a_2^2\right) + \frac{2\Gamma}{\omega_o} a_1 a_2 \cos\theta(t) \tag{27}$$

$$\frac{\partial \varphi_1}{\partial t} = \frac{\Gamma}{\omega_1} a_o^2 + \frac{\Gamma}{\omega_1}\frac{a_o^2 a_2}{a_1}\cos\theta(t) \tag{28}$$

$$\frac{\partial \varphi_2}{\partial t} = \frac{\Gamma}{\omega_1} a_o^2 + \frac{\Gamma}{\omega_1}\frac{a_o^2 a_1}{a_2}\cos\theta(t) \tag{29}$$

Using equations (27) - (29) we obtain the following equation for $\theta(t)$:

$$\frac{\partial\theta}{\partial t} = -2\delta + 2\Gamma\left(\frac{a_1^2}{\omega_o} + \frac{a_2^2}{\omega_o} - \frac{a_o^2}{\omega_1}\right) + \cot\theta\,\frac{\partial}{\partial t}\,\ell n\,(a_o^2\,a_1\,a_2) \tag{30}$$

Now equations (24) - (26) lead to the conservation relation which we write as a constant of the motion

$$\frac{a_o^2}{\omega_o} + \frac{\omega_1}{\omega_o^2}\,a_1^2 + \frac{\omega_1}{\omega_o^2}\,a_2^2 = W \tag{31}$$

and which defines the constant W. It is now convenient to normalize the amplitudes and transform the time variable as follows

$$a_o = (2\omega_o W)^{\frac{1}{2}}\,u_o$$

$$a_1 = \left(\frac{\omega_o^2}{\omega_1}W\right)^{\frac{1}{2}}u_1$$

$$a_2 = \left(\frac{\omega_o^2}{\omega_1}W\right)^{\frac{1}{2}}u_2$$

and

$$\tau = 2\Gamma\frac{\omega_o}{\omega_1}W\,t$$

In terms of these variables, equation (31) can be written

$$2u_o^2 + u_1^2 + u_2^2 = 1 \tag{32}$$

We also obtain the additional constants of the motion from the normalized versions of equations (24) - (26)

$$u_o^2 + u_1^2 = m_1 \tag{33}$$

$$u_o^2 + u_2^2 = m_2 \tag{34}$$

$$u_2^2 - u_1^2 = m_3 \tag{35}$$

which define the constants m_1, m_2 and m_3. With the aid of equations

(24), (25), (30) and (32) we obtain the following result

$$u_o^2 u_1 u_2 \cos\theta(t) + \frac{\Delta}{2} u_o^2 - \frac{u_1^2}{2} - u_o^4 = \Lambda \qquad (36)$$

where Λ is a constant defined by this equation and $\Delta \equiv -\delta\omega_1/\Gamma\omega_o W$.

We shall take as our initial conditions

$$u_o(o) \neq 0 \;;\quad u_1(o) = 0 \;;\quad u_2(o) \neq 0$$

Λ may now be expressed as

$$\Lambda = \frac{\Delta}{2} u_o^2(o) - u_o^4(o). \qquad (37)$$

Obtaining $\sin\theta(t)$ from equation (36) [15], substituting it into equation (25) and eliminating u_o and u_2 with the aid of equations (30) and (32) results in the following equation involving only u_1

$$\frac{\partial}{\partial\tau}\left(\frac{u_1^2}{2}\right) = \left[u_1^2 (m_1 - u_1^2)^2 (m_3 + u_1^2) - \left\{\Lambda - \frac{\Delta}{2}(m_1 - u_1^2) + (m_1 - u_1^2)^2 + \frac{u_1^2}{2}\right\}^2\right]^{\frac{1}{2}} \qquad (38)$$

Since we have chosen $u_1(o) = 0$, m_1 and m_3 are identified as $m_1 = u_o^2(o)$ and $m_3 = u_2^2(o)$. Substituting equation (37) into equation (38) we obtain

$$\frac{\partial u_1}{\partial\tau} = \sqrt{\Delta}\, AB \left[\left(1 - \frac{u_1^2}{A^2}\right)\left(1 + \frac{u_1^2}{B^2}\right)\right]^{\frac{1}{2}} \qquad (39)$$

where
$$A^2 = \frac{b}{2\Delta} + \frac{1}{2}\left(\frac{b^2}{\Delta^2} + \frac{4c}{\Delta}\right)^{\frac{1}{2}}$$

$$B^2 = -\frac{b}{2\Delta} + \frac{1}{2}\left(\frac{b^2}{\Delta^2} + \frac{4c}{\Delta}\right)^{\frac{1}{2}}$$

and
$$b = m_1^2 + 2m_1\Delta - (1 + \Delta)^2/4$$

$$c = m_1^2 m_3 .$$

Putting $\psi = u_1/A$, equation (39) becomes

$$\frac{\partial\psi}{\partial\tau} = \Delta^{\frac{1}{2}} B \left[(1-\psi^2)(1+\frac{A^2}{B^2}\psi^2) \right]^{\frac{1}{2}} \tag{40}$$

Noting the result

$$\frac{d}{d\tau} \text{cn}(\tau, k) = -k' \left[(1-\text{cn}^2)\left(1+\frac{k^2}{k'^2}\text{cn}^2\right)\right]^{\frac{1}{2}}$$

where $\text{cn}(\tau, k)$ is a Jacobi Elliptic function and k and k' its modulus and complementary modulus respectively, - it is now a simple matter to verify that the solution of equation (40), satisfying the initial condition $u_1(o) = 0$, is

$$u_1(\tau) = A\,\text{cn}\,(K(k) - \beta\tau, k) \tag{41}$$

where $$k = A/(A^2 + B^2)^{\frac{1}{2}}$$

and $$\beta = \Delta^{\frac{1}{2}}(A^2 + B^2)^{\frac{1}{2}}.$$

Equation (41) is the desired solution for the Stokes electromagnetic wave. The corresponding solutions for the anti-Stokes and pump waves can be obtained from equations (35) and (33) respectively. As is to be expected, the solution shows a periodic behaviour in time. The maximum amplitude of the excited electromagnetic waves is given by the quantity A and the time $\tau_{max.}$ for the waves to reach this value is given by $\tau_{max.} = K(k)/\beta$.

The phases of the individual waves can also be calculated explicitly. From equation (28) we easily obtain

$$\varphi_1(\tau) = \left(\frac{\Delta}{2} + \frac{1}{2} - m_1\right)\tau + \varphi_1(o), \tag{42}$$

and from equation (27)

$$\varphi_o(\tau) = \frac{m_3}{2}\tau + \varphi_o(o) + \left(\frac{\Delta}{2} + \frac{1}{2} - m_1\right)\int \frac{u_1^2}{(m_1 - u_1^2)}\,d\tau. \tag{43}$$

The integral on the right hand side of equation (43) can be written in a standard form whose values are tabulated [16,17]. The remaining phase $\varphi_2(\tau)$ can be obtained from a knowledge of φ_1, φ_o and equation (36). Knowing the phases we may finally obtain an expression for the density perturbation resulting from the filamentation phenomenon

$$\frac{N_s(\tau)}{n_o} = -\frac{1}{\sqrt{2}}\left(\frac{\omega_o}{\omega_1}\right)^{3/2} \frac{v_o^2}{v_{Te}^2}\left(1+\frac{\omega_1}{\omega_o}\frac{m_3}{m_1}\right)\times\left\{u_o\,u_1\cos\left(\varphi_o-\varphi_1+\frac{\Delta}{2}\tau\right)\right.$$

$$\left. + u_o\,u_2\cos\left(\varphi_o-\varphi_2+\frac{\Delta}{2}\tau\right)\right\} \qquad (44)$$

where $v_o = e\,E_o/m_e\,\omega_o$ and $v_{Te} = (k\,T_e/m_e)^{\frac{1}{2}}$.

As an illustration of the above solutions figure 2 shows the normalized amplitudes of the pump and excited electromagnetic waves (u_o, u_1 and u_2) as a function of the normalized time τ. In figure 3 we show the density perturbation associated with these amplitudes. It can be seen that there is a density depletion when the excited waves interfere constructively and an accumulation when these waves interfere destructively. Figures 2 and 3 have been plotted for a value of $\Delta = 0.1$. Using the minimum threshold condition to estimate k_s we obtain the result $\Delta = 4\,\nu_e\,v_{Te}^2/\omega_o\,v_o^2$. Choosing $T_e = 1$ keV and $n_o = 10^{18}/cm^3$ for a carbon dioxide laser we then find that figures 2 and 3 describe a situation where the initial laser power is 3×10^{13} watts/cm^2 or an electric field $E_o \approx 10^8$ volts/metre. Other choices of n_o and T_e imply different laser powers.

For the time evolution problem we have just solved there is, of course, a spatial periodicity perpendicular to the direction of the initial wave. This is given by the assumed form of the density perturbation $\exp(i\,k_s\,y)$. This then means that, perpendicular to the direction of propagation of the pump wave, the Stokes, anti-Stokes and density perturbations vary periodically in space. The density is depleted where the excited fields are strong and vice versa. Superimposed on this spatial behaviour is the overall non-linear behaviour in time. Thus, for a given point, the Stokes and anti-Stokes waves will first reach a maximum and the density a maximum depleted level. Subsequently the density will reach a maximum accummulation level at which time the excited electromagnetic waves, although at an extremum, interfere destructively. The cycle of values which the perturbations follow at any point are determined by the non-linear solution and the spatial variation $\exp(i\,k_s\,y)$.

We have already noted that we can easily obtain the maximum amplitudes of the Stokes and anti-Stokes waves from our solution. Table I gives the maximum amplitude of the Stokes wave (u_1) as a function of the mis-match parameter Δ and the initial value of the anti-Stokes field (we have assumed $u_1(0) = 0$. We have again taken Δ as given by the minimum threshold condition i.e. $\Delta = 4\nu_e v_{Te}^2/\omega_o v_o^2$. Table I shows that the maximum value of the electric field of the Stokes electromagnetic wave (and hence the anti-Stokes wave) changes only very slowly as the mis-match parameter changes through four orders of magnitude for fixed initial conditions. The maximum value is virtually independent of Δ for $\Delta \ll 1$ and only begins to fall for $\Delta \lesssim 1$. Once Δ increases beyond unity, however, the maximum amplitude of the excited waves drops very rapidly so that for $\Delta = 3$ the excited amplitudes never attain more than a tiny fraction of the initial pump amplitude. Thus $\Delta \approx 3$ is a critical value above which there is virtually no filamentation.

Table I also shows that the maximum amplitudes of the scattered electromagnetic waves are almost independent of the initial conditions. However, as the initial value of the anti-Stokes wave is increased the time taken for it (and also the Stokes wave) to reach its maximum value is reduced. We would like to emphasise that these conclusions apply only to the underdense regions of the plasma.

(ii) Solution with Damping

In the above solutions for the electric field amplitudes $\mathcal{E}_o$, $\mathcal{E}_1$ and $\mathcal{E}_2$ it was assumed that the initial value of the pump wave was sufficiently far above threshold such that the damping terms could be ignored. We now relax this restriction and include the damping on the electromagnetic waves. Equations (8) - (10) then become

$$\left(\frac{\partial}{\partial t} + \gamma_T\right) \mathcal{E}_o(t) = \frac{i\Gamma}{\omega_o} \left(|\mathcal{E}_1|^2 \mathcal{E}_o + |\mathcal{E}_2|^2 \mathcal{E}_o + 2\mathcal{E}_o^* \mathcal{E}_1 \mathcal{E}_2 \, e^{i2\delta t}\right) \tag{45}$$

$$\left(\frac{\partial}{\partial t} + \gamma_T\right) \mathcal{E}_1(t) = \frac{i\Gamma}{\omega_1} \left(|\mathcal{E}_o|^2 \mathcal{E}_1 + \mathcal{E}_o^2 \mathcal{E}_2^* \, e^{-i2\delta t}\right) \tag{46}$$

$$\left(\frac{\partial}{\partial t} + \gamma_T\right) \mathcal{E}_2(t) = \frac{i\Gamma}{\omega_1} \left(\mathcal{E}_o^2 \mathcal{E}_1^* \, e^{-i2\delta t} + |\mathcal{E}_o|^2 \mathcal{E}_2\right) \tag{47}$$

Since we assume the damping is weak and since $\omega_o \approx \omega_1$ we have taken $\gamma_{To} = \gamma_{T1,2} = \gamma_T \equiv \frac{\nu_e}{2} \frac{\omega_p^2}{\omega_o^2}$. With this approximation we can solve equations (45) - (47) as follows. We first introduce the quantities G_o, G_1, and G_2 through the relations

$$\mathcal{E}_o = G_o \, e^{-\gamma_T t}$$

$$\mathcal{E}_1 = G_1 \, e^{-\gamma_T t}$$

and

$$\mathcal{E}_2 \, e^{i2\delta t} = G_2 \, e^{-\gamma_T t}$$

In terms of these variables, equation (46) (for example) becomes

$$e^{-\gamma_T t} \frac{\partial G_1}{\partial t} = i \frac{\Gamma}{\omega_1} \left(|G_o|^2 G_1 + G_o^2 G_2^* \right) e^{-3\gamma_T t} \tag{48}$$

If we now use the transformation [13]

$$\tau = \frac{1}{2\gamma_T} \left(1 - e^{-2\gamma_T t}\right)$$

$G_o(t)$, $G_1(t)$ and $G_2(t)$ are written as $W_o(\tau)$, $W_1(\tau)$ and $W_2(\tau)$ and equation (48) becomes

$$\frac{d W_1}{d\tau} = \frac{i\Gamma}{\omega_1} \left(|W_o|^2 W_1 + W_o^2 W_2^* \right) \tag{49}$$

Similarly, equations (45) and (47) become

$$\frac{d W_o}{d\tau} = \frac{i\Gamma}{\omega_o} \left(|W_1|^2 W_o + |W_2|^2 W_o + 2W_o^* W_1 W_2 \right) \tag{50}$$

$$\left(\frac{d}{d\tau} - i2\delta \right) W_2 = \frac{i\Gamma}{\omega_1} \left(W_o^2 W_1^* + |W_o|^2 W_2 \right) \tag{51}$$

Equations (49) - (51) are now in the same form as equations (8) - (10). We can therefore immediately write down the solutions for the electric field amplitudes in the presence of damping. The expression for $|\mathcal{E}_1|$ is

$$|\mathcal{E}_1(t)| = A' \, \mathrm{cn} \left(K(k) - \frac{\beta'}{2\gamma_T} \left(1 - e^{-2\gamma_T t}\right) , k \right) e^{-\gamma_T t} \tag{52}$$

where $A' = \left(\frac{\omega_o^2}{\omega_1} W \right)^{\frac{1}{2}} A$, $\beta' = 2\beta\Gamma \frac{\omega_o}{\omega_1} W$ and A and β were defined in the previous section. The form of this solution is similar to a solution of the damped non-linear Schrödinger equation obtained by Nicholson and Goldman [18]. The corresponding expressions for $\mathcal{E}_o$ and $\mathcal{E}_2$ can be obtained from equation (52) and the equations corresponding to (33) and (34). We note from the solution for $\mathcal{E}_1$ that the effect of damping not only produces an exponential decay of the amplitude but also a lengthening of the period of oscillation. As the amplitude tends to zero the period tends to infinity.

(iii) Spatially Dependent Case, Stationary Solutions

In order to describe the spatial behaviour of the filamentation process we return to equations (3) - (6). Again, neglecting ion inertia as for the spatially independent case, we obtain the following equations

$$\left(\frac{\partial}{\partial t}+v_o\frac{\partial}{\partial x}\right)\mathcal{E}_o(x,t) = \frac{i\Gamma}{\omega_o}\left(|\mathcal{E}_1|^2\mathcal{E}_o + |\mathcal{E}_2|^2\mathcal{E}_o + 2\mathcal{E}_o^*\,\mathcal{E}_1\,\mathcal{E}_2\,e^{i2\delta t}\right) \tag{53}$$

$$\left(\frac{\partial}{\partial t}+v_1\frac{\partial}{\partial x}\right)\mathcal{E}_1(x,t) = \frac{i\Gamma}{\omega_1}\left(|\mathcal{E}_o|^2\mathcal{E}_1 + \mathcal{E}_o^2\,\mathcal{E}_2^*\,e^{-i2\delta t}\right) \tag{54}$$

$$\left(\frac{\partial}{\partial t}+v_1\frac{\partial}{\partial x}\right)\mathcal{E}_2(x,t) = \frac{i\Gamma}{\omega_1}\left(\mathcal{E}_o^2\,\mathcal{E}_1^*\,e^{-i2\delta t} + |\mathcal{E}_o|^2\,\mathcal{E}_2\right) \tag{55}$$

Where we have used the fact that $v_{2x} = v_{1x} \equiv v_1$. We have also neglected the damping terms although these can be included in the manner just described.

It is convenient to replace the amplitude $\mathcal{E}_2$ by the new amplitude $\mathcal{E}_2\,e^{i2\delta t} \equiv \alpha_2$. We now look for stationary solutions to equations (53) - (55) where the amplitudes $\mathcal{E}_o$, $\mathcal{E}_1$ and α_2 are functions only of the coordinate ξ where $\xi \equiv x - ut$ and where u is a velocity as yet unspecified. Writing

$$\mathcal{E}_o(\xi,t) = a_o(\xi,t)\ e^{i\varphi_o(\xi,t)}$$

$$\mathcal{E}_1(\xi,t) = a_1(\xi,t)\ e^{i\varphi_1(\xi,t)}$$

$$\alpha_2(\xi,t) = a_2(\xi,t)\ e^{i\varphi_2(\xi,t)}$$

and applying the condition for a stationary solution

$$\frac{\partial a_o}{\partial t} = \frac{\partial a_1}{\partial t} = \frac{\partial a_2}{\partial t} = 0$$

we obtain the following equations

$$(v_o - u)\,\frac{\partial a_o}{\partial \xi} = \frac{2\Gamma}{\omega_o}\, a_o a_1 a_2 \sin\theta \tag{56}$$

$$(v_1 - u)\,\frac{\partial a_1}{\partial \xi} = \frac{-\Gamma}{\omega_1}\, a_o^2 a_2 \sin\theta \tag{57}$$

$$(v_1 - u)\,\frac{\partial a_2}{\partial \xi} = \frac{-\Gamma}{\omega_1}\, a_o^2 a_1 \sin\theta \tag{58}$$

$$(v_o - u)\,\frac{\partial \varphi_o}{\partial \xi} = \frac{\Gamma}{\omega_o}(a_1^2 + a_2^2) + \frac{2\Gamma}{\omega_o}\, a_1 a_2 \cos\theta \tag{59}$$

$$(v_1 - u)\,\frac{\partial \varphi_1}{\partial \xi} = \frac{\Gamma}{\omega_1}\, a_o^2 + \frac{\Gamma}{\omega_1}\, a_o^2\,\frac{a_2}{a_1}\cos\theta \tag{60}$$

$$(v_1 - u)\,\frac{\partial \varphi_2}{\partial \xi} = 2\delta + \frac{\Gamma}{\omega_1}\, a_o^2 + \frac{\Gamma}{\omega_1}\, a_o^2\,\frac{a_1}{a_2}\cos\theta \tag{61}$$

where $\theta(\xi) \equiv 2\varphi_0 - \varphi_1 - \varphi_2$ and φ_0, φ_1 and φ_2 depend only on ξ for a stationary solution. Using equations (56) - (61) we obtain the following constants of the motion

$$a_o^2 + \frac{\omega_1 V_1}{\omega_o V_o}\, a_1^2 + \frac{\omega_1 V_1}{\omega_o V_o}\, a_2^2 = W \tag{62}$$

$$a_o^2 + \frac{2\omega_1 V_1}{\omega_o V_o}\, a_1^2 = \text{const.} \tag{63}$$

$$a_o^2 + \frac{2\omega_1 V_1}{\omega_o V_o}\, a_2^2 = \text{const.} \tag{64}$$

$$a_2^2 - a_1^2 = \text{const.} \tag{65}$$

where $V_o \equiv v_o - u$ and $V_1 \equiv v_1 - u$. We now proceed as for the spatially independent case. Corresponding to equation (36) we obtain

$$a_o^2 a_1 a_2 \cos\theta - W a_1^2 - \frac{1}{2}\frac{\omega_o V_o}{\omega_1 V_1}\, a_o^4 - \frac{\delta}{V_1}\frac{\omega_o V_o}{2\Gamma}\, a_o^2 = \Lambda \tag{66}$$

where Λ is a constant. Using equations (57) and (66) we obtain

$$-\frac{\omega_1 V_1}{\Gamma}\,\frac{d}{d\xi}\left(\frac{a_1^2}{2}\right) = \left[(a_o^2 a_1 a_2)^2 - \left(\Lambda + W a_1^2 + \frac{\omega_o V_o}{2\omega_1 V_1}\, a_o^4 + \frac{\delta\, \omega_o V_o}{2\Gamma V_1}\, a_o^2 \right)^2 \right]^{\frac{1}{2}} \tag{67}$$

In order to solve equation (67) for a_1^2 we express a_o and a_2 in terms of a_1 with the aid of equations (63) and (65). We are free to choose the origin for the ξ coordinate and look for a solution such that a_1 has a maximum at $\xi = 0$. Referring to equation (57) this is easily done by choosing $\theta(0) = 0$ (or $n\pi$). For $\theta(0) = 0$ we see from equations (56) and (58) and (59) - (61) that when $a_1(0)$ is a maximum, $a_2(0)$ will also be a maximum and $a_o(0)$ will be a minimum provided

$$a_1(0)\, a_2(0) > \frac{1}{2} \frac{\omega_o}{\omega_1} \frac{V_o}{V_1} a_o^2(0)$$

and V_o and V_1 are both positive. This choice also fixes the value of the constant Λ in terms of the amplitudes at $\xi = 0$.

Putting $a_1^2 = u$, equation (67) can be put in the form

$$\frac{du}{d\xi} = -K(-u^3 + c_2 u^2 + c_1 u + c_o)^{\frac{1}{2}} \qquad (68)$$

where $K \equiv 2\Gamma|\alpha|/\omega_1 V_1$, $\alpha = 4\omega_1 V_1 \delta\omega_1/\omega_o V_o \Gamma$ and we take $V_o > 0$, $V_1 > 0$. The constants c_o, c_1 and c_2 are given by the following expressions

$$c_2 = -\frac{4\omega_1 V_1}{\omega_o V_o} \frac{A}{|\alpha|}(m_2 - m_1) + \frac{A^2}{|\alpha|} - \frac{1}{|\alpha|}\left(W - 2A - \frac{\delta\omega_1}{\Gamma}\right)^2$$

$$- \frac{4}{|\alpha|} \frac{\omega_1 V_1}{\omega_o V_o} \left(\Lambda + \frac{1}{2} \frac{\omega_o V_o}{\omega_1 V_1} A^2 + \frac{\delta}{V_1} \frac{\omega_o V_o}{2\Gamma} A\right)$$

$$c_1 = \frac{A^2}{|\alpha|}(m_2 - m_1) - \frac{2}{|\alpha|} \left(\Lambda + \frac{1}{2} \frac{\omega_o V_o}{\omega_1 V_1} A^2 + \frac{\delta}{V_1} \frac{\omega_o V_o}{2\Gamma} A\right) \left(W - 2A - \frac{\delta\omega_1}{\Gamma}\right)$$

$$c_o = -\frac{1}{|\alpha|} \left(\Lambda + \frac{1}{2} \frac{\omega_o V_o}{\omega_1 V_1} A^2 + \frac{\delta}{V_1} \frac{\omega_o V_o}{2\Gamma} A\right)^2$$

where $A \equiv m_o + \frac{2\omega_1 V_1}{\omega_o V_o} m_1$. We have chosen the following conditions at $\xi = 0$

$$a_o^2(0) = m_o \; ; \; a_1^2(0) = m_1 \; ; \; a_2^2(0) = m_2$$

We now write the cubic expression on the right hand side of equation (68) $-u^3 + c_2 u^2 + c_1 u + c_o \equiv (u - u_1)(u - u_2)(u_3 - u)$ where u_1, u_2 and u_3 are the real roots of this cubic and are ordered such that

$$u_3 > u_2 \geq u_1 \;.$$

Equation (68) can now be written

$$\frac{du}{d\xi} = - K[(u - u_1)(u - u_2)(u_3 - u)]^{\frac{1}{2}} \tag{69}$$

With the aid of the transformation $y^2 = u_3 - u$ the solution of equation (69) is easily found to be

$$a_1^2(\xi) = u_2 + (u_3 - u_2)\, cn^2(\beta\xi\, ,\, k) \tag{70}$$

where $\beta \equiv \frac{K}{2}(u_3 - u_1)^{\frac{1}{2}}, \; k^2 = (u_3 - u_2)/u_3 - u_1)$

and u_1 , u_2 and u_3 depend on the values of the amplitudes at $\xi = 0$ and the mis-match parameter δ (δ of course depends on the pump amplitude and the plasma parameters). We note from equation (70) that the value of a_1^2 at $\xi = 0$ is given by u_3 . We may therefore put $u_3 = m_1$ on the right hand side of equation (69). With the aid of equations (63) and (65) the solutions for a_2^2 and a_o^2 are the following

$$a_2^2(\xi) = m_2 - (m_1 - u_2)\, sn^2(\beta\xi\, ,\, k) \tag{71}$$

$$a_o^2(\xi) = m_o + \frac{2\omega_1 V_1}{\omega_o V_o}(m_1 - u_2)\, sn^2(\beta\xi\, ,\, k) \tag{72}$$

We note that since we have identified u_3 with m_1 , the remaining two roots u_1 and u_2 are easily obtained as the real roots of a quadratic equation. The solutions for the amplitudes given by equations (70) - (72) show that, in general, the stationary solutions for the filamentation process are periodic in space corresponding to successive focussing and de-focussing of the incident light wave. The filamentation length ℓ over which the electromagnetic energy changes from a uniform distribution (corresponding to the pump wave) to the maximum non-uniform filamentary distribution is given by $\ell = K(k)/\beta$ where $K(k)$ is the complete Jacobi Elliptic integral of the first kind.

In addition to these spatially periodic solutions for the electric field amplitudes we may also have pulse-like solutions. These will occur when $u_1 = u_2$ since then the solution to equation (69) is proportional to $\mathrm{sech}^2\, \beta\xi$. As we have already mentioned, since u_3 has been identified with m_1 , u_1 and u_2 are the roots of a quadratic which is easily obtained from the right hand sides of equations (68) and (69). The roots u_1 and u_2 are then given by

$$u_{1,2} = \frac{1}{2}\left\{c_2 - m_1 \pm \left[(c_2 - m_1)^2 - \frac{4c_o}{m_1}\right]^{\frac{1}{2}}\right\} \tag{73}$$

For a pulse like or solitary wave solution u_1 and u_2 must be coincident. Since $-4c_o/m_1 \geq 0$ the condition for a solitary wave solution is

$$c_2 - m_1 = 0 \tag{74}$$

and

$$c_o = 0 \tag{75}$$

For these conditions, it then follows that both a_1^2 and a_2^2 will be pulse or solitary wave solutions which vanish as $\xi \to \infty$ and that a_o^2 will take on its minimum value where a_1^2 and a_2^2 have their maxima and will tend to a constant value asymptotically. It turns out that equations (74) and (75) can only be satisfied by the following values of the parameters

$$m_1 = m_2 \tag{76}$$

and

$$\frac{|\delta|\,\omega_1}{\Gamma m_o} = 2 \tag{77}$$

These conditions give the result $V_1/V_o = \frac{1}{2}\frac{m_o}{m_1}\frac{\omega_o}{\omega_1}$ which fixes the velocity

$$u \approx \frac{\left(v_1 - \frac{1}{2}\frac{m_o}{m_1}\frac{\omega_o}{\omega_1}v_o\right)}{\left(1 - \frac{1}{2}\frac{m_o}{m_1}\frac{\omega_o}{\omega_1}\right)}$$

at which the stationary field distribution moves through the plasma. For consistency $1/\sqrt{2} < m_o/m_1 < 1$. The first condition (eqn. (76)) means that the Stokes and anti-Stokes waves are equal in amplitude for all ξ. The second condition is equivalent to $v_o^2/v_{Te}^2 = 2\nu_e/\omega_1$ where v_o is the amplitude of the electron quiver velocity in the field of the pump wave at $\xi = 0$. Since the asymptotic value of the pump amplitude is twice its value at the origin the pump amplitude never exceeds the instability threshold. This is consistent with the fact that the Stokes and anti-Stokes waves tend asymptotically to zero. The explicit forms of these solutions are

$$a_1^2(\xi) = a_2^2(\xi) = m_1 \operatorname{sech}^2(\beta\xi) \tag{78}$$

$$a_o^2(\xi) = m_o + \frac{2\omega_1 V_1}{\omega_o V_o} m_1 \tanh^2(\beta\xi) \tag{79}$$

These solitary wave solutions are limiting cases of periodic solutions where the period becomes infinite. In view of the very restrictive conditions which must be satisfied it seems clear that the periodic solutions are by far the most significant for this problem.

Having solved for the amplitudes of the electro-magnetic waves the density perturbation can be obtained with the aid of equation (7) as was done for the spatially independent solution. For this we also need to know the phases φ_o, φ_1 and φ_2.

$\varphi_1(\xi)$ can be obtained from equations (60) and (66) and is given by the expression

$$\begin{aligned}\varphi_1(\xi) = \varphi_1(0) + \frac{\Gamma}{\omega_1 V_1} \Bigg\{ & \left[W - \frac{\delta\omega_1}{\Gamma} - \left(m_o + \frac{2\omega_1 V_1}{\omega_o V_o} m_1 \right) \right] \xi \\ & + \frac{1}{m_1 \beta} \left[\Lambda + \frac{\delta\omega_o V_o}{2\Gamma V_1} \left(m_o + \frac{2\omega_1 V_1}{\omega_o V_o} m_1 \right) + \frac{\omega_o V_o}{2\omega_1 V_1} \left(m_o + \frac{2\omega_1 V_1}{\omega_o V_o} m_1 \right)^2 \right] \\ & \times \Pi \left(\beta\xi, \frac{m_1 - u_2}{m_1}, k \right) \Bigg\}\end{aligned} \tag{80}$$

Similarly, $\varphi_2(\xi)$ is obtained from equations (61) and (66) and is given by

$$\begin{aligned}\varphi_2(\xi) = \varphi_2(0) + \frac{\Gamma}{\omega_1 V_1} \Bigg\{ & \left[W - \left(m_o + \frac{2\omega_1 V_1}{\omega_o V_o} m_2 \right) + \frac{\delta\omega_1}{\Gamma} \right] \xi \\ & + \frac{1}{m_2 \beta} \left[\Lambda + W(m_1 - m_2) + \frac{\omega_o V_o}{2\omega_1 V_1} \left(m_o + \frac{2\omega_1 V_1}{\omega_o V_o} m_2 \right)^2 \right. \\ & \left. + \frac{\delta\omega_o V_o}{2\Gamma V_1} \left(m_o + \frac{2\omega_1 V_1}{\omega_o V_o} m_2 \right) \right] \Pi \left(\beta\xi, \frac{m_1 - u_2}{m_2}, k \right) \Bigg\}\end{aligned} \tag{81}$$

where $\Pi(\varphi, \alpha^2, k)$ is the incomplete elliptic integral of the third kind [16]. Knowing $\varphi_1(\xi)$ and $\varphi_2(\xi)$, $\varphi_o(\xi)$ can then be determined from equation (66). The density perturbation can then be obtained as for the time dependent solution and can be expected to have a variation in space similar to its time behaviour.

V. CONCLUSIONS

We have presented a non-linear theory of the development of the filamentation of a plane electromagnetic wave in a plasma. As we have already emphasised the underlying mechanism of the filamentation phenomenon is closely related to other non-linear effects such as the oscillating two stream instability and the Langmuir modulational instability. In contrast to other workers we have described the non-linear development by distinguishing the incident (or pump) wave and the Stokes and anti-Stokes waves throughout the interaction. In other words we have solved the coupled non-linear equations for the interaction of three slowly varying wave envelopes rather than a single non-linear equation for the resultant wave envelope (the approach which leads to the non-linear Schrödinger equation).

The non-linear theory that we have given takes account only of those waves which are responsible for the initial instability. These waves are selected by imposing perfect k-matching but allowing for a small frequency mis-match. In view of the fact that the waves considered are so strongly correlated (remember that the properties of the density perturbation and the excited electromagnetic waves are determined by the pump wave) it may be that the non-linear terms considered are the dominant ones, at least for light energy densitites small compared with the plasma energy density.

The non-linear calculation has been divided into three parts. In the first, the spatial variation has been neglected, and only the time evolution is obtained. This solution is the analogue of the exact non-linear solutions for the interaction of three waves [13,14] and is also periodic, as is to be expected. In order to obtain these solutions, we neglected ion inertia in the low frequency perturbation (the static approximation). With the aid of the resulting equations we have demonstrated the four wave nature of the filamentation interaction (and hence similar phenomena) and have shown explicitly how the pump wave and frequency shifted Stokes and anti-Stokes waves conserve energy, momentum and action densities throughout the interaction. In the appendix, we have derived these results including the effect of ion inertia. Evidently the ion perturbation acts as a virtual wave, and does not account for any of the energy - as might be expected of a zero frequency mode!

In the second part we have solved the equations with the inclusion of a damping term on the electromagnetic waves. The effect of the damping is two fold. As expected, it results in an exponential decay of the amplitudes with time. However, it also causes an increase in the period of the non-linear oscillations of the amplitudes.

In the third part we have given the stationary wave solutions for the spatial development of filamentation. Again we find that the general solution is periodic in space corresponding to a filamentation of the incident, uniform wave, followed by regeneration of the initial distribution, followed by further filamentation and so on. We also find solitary wave solutions as a special limiting case of the periodic solution i.e. when the period tends to infinity. For this case, the pump wave takes the form of an envelope hole soliton and the Stokes and anti-Stokes waves are both single soliton solutions which tend asymptotically to zero. Evidently these solutions require the asymptotic value of the pump wave not to be greater than the instability threshold.

Envelope soliton solutions which result from the non-linear Schrödinger equation arise as a result of a balance between the non-linear growth and the dispersion of the wave envelope. However, in our calculation, the pump, Stokes and anti-Stokes wave envelopes are assumed to propagate without dispersion. Instead, the dispersive nature of the interaction is accounted for through the different group velocities of the wave envelopes. We would like to emphasize, however, that since the soliton solutions result only under very special conditions, the periodic behaviour may be expected to be more significant as a contribution to the interpretation of filamentation in a plasma. We conclude by giving two examples of the length required for a plane electromagnetic wave to reach maximum intensity in a given filament. The maximum amplitude of the excited electromagnetic waves is achieved in a distance ℓ (the filamentation length) given by $\ell = \frac{K(k)}{\beta}$. In a recent experiment by Spalding et al. [19] filamentation lengths between $40\,\mu m$ - $100\,\mu m$ were measured in a CO_2 laser produced plasma. Using their experimental results [19, 20] for the density n_e ($= 10^{18} cm^{-3}$) and temperature Te (= 1keV) and CO_2 laser intensity I ($\approx 10^{13}$ Watts/cm^3) we obtain values of the filamentation length ℓ for this plasma of $35\,\mu m$ - $70\,\mu m$ which are in fair agreement with the observed values.

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APPENDIX

We have already demonstrated the conservation of energy, momentum and action densities assuming that ion inertia or the low frequency response was negligible. We shall now prove these conservation relations including the effect of ion inertia.

For the purpose of this argument it is convenient to describe the low frequency response by two first order equations [6] rather than a single second order equation. The non-linear equations for the wave amplitudes are as follows

$$\frac{\partial \mathcal{E}_1}{\partial t} = - i c_{so} \mathcal{E}_o \left\{ (N_s^+)^* e^{-i(\delta - \omega_s)t} + (N_s^-)^* e^{-i(\delta + \omega_s)t} \right\} \tag{A.1}$$

$$\frac{\partial \mathcal{E}_2}{\partial t} = - i c_{so} \mathcal{E}_o \left\{ N_s^+ e^{-i(\delta + \omega_s)t} + N_s^- e^{-i(\delta - \omega_s)t} \right\} \tag{A.2}$$

$$\frac{\partial \mathcal{E}_o}{\partial t} = - i c_{s1} \left\{ \left(\mathcal{E}_1 N_s^+ + \mathcal{E}_2 (N_s^-)^* \right) e^{i(\delta - \omega_s)t} + \left(\mathcal{E}_1 N_s^- + \mathcal{E}_2 (N_s^+)^* \right) e^{i(\delta + \omega_s)t} \right\} \tag{A.3}$$

$$\frac{\partial N_s^+}{\partial t} = - i c_{o1} \left\{ \mathcal{E}_o \mathcal{E}_1^* e^{-i(\delta - \omega_s)t} + \mathcal{E}_o^* \mathcal{E}_2 e^{i(\delta + \omega_s)t} \right\} \tag{A.4}$$

$$\frac{\partial N_s^-}{\partial t} = i c_{o1} \left\{ \mathcal{E}_o \mathcal{E}_1^* e^{-i(\delta + \omega_s)t} + \mathcal{E}_o^* \mathcal{E}_2 e^{i(\delta - \omega_s)t} \right\} \tag{A.5}$$

where we consider the spatially independent problem again for simplicity and we have neglected the damping as before. Equations (A.1) - (A.5) are equivalent to equations (3) - (6). When the non-linear coupling is negligible N_s^+ and N_s^- represent ion acoustic waves which propagate in the positive and negative y-direction respectively. The coupling coefficients c_{so} and c_{s1} have already been defined and $c_{o1} = n_o e^2 k_s^2 / 4 m_e m_i \omega_o \omega_1 \omega_s$.

We now define the new amplitudes

$$\alpha_o = \left(\frac{\epsilon_o}{2} \right)^{\frac{1}{2}} \mathcal{E}_o \quad ; \quad \alpha_1 = \left(\frac{\epsilon_o}{2} \right)^{\frac{1}{2}} \mathcal{E}_1^* \quad ; \quad \alpha_2 = \left(\frac{\epsilon_o}{2} \right)^{\frac{1}{2}} \mathcal{E}_2 \, e^{i2\delta t}$$

$$\alpha_{s1} = \left(\frac{m_i c_s^2}{2 n_o}\right)^{\frac{1}{2}} N_s^+ \; e^{i(\delta - \omega_s)t} \quad ; \quad \alpha_{s2} = \left(\frac{m_i c_s^2}{2 n_o}\right)^{\frac{1}{2}} N_s^- \; e^{i(\delta + \omega_s)t}$$

such that the modulus squared of these amplitudes is an energy density. In terms of these amplitudes, equations (A.1) - (A.5) become

$$\frac{\partial \alpha_1}{\partial t} = i\, c'_{so}\, \alpha_o^* \,(\alpha_{s1} + \alpha_{s2}) \tag{A.6}$$

$$\left(\frac{\partial}{\partial t} - i2\delta\right)\alpha_2 = -ic'_{so}\, \alpha_o \,(\alpha_{s1} + \alpha_{s2}) \tag{A.7}$$

$$\frac{\partial \alpha_o}{\partial t} = -ic'_{s1}\,(\alpha_1^* \alpha_{s1} + \alpha_2\, \alpha_{s2}^* + \alpha_1^* \alpha_{s2} + \alpha_2\, \alpha_{s1}^*) \tag{A.8}$$

$$\left(\frac{\partial}{\partial t} - i(\delta - \omega_s)\right)\alpha_{s1} = -\, i\, c'_{o1}(\alpha_o\, \alpha_1 + \alpha_o^* \alpha_2) \tag{A.9}$$

$$\left(\frac{\partial}{\partial t} - i(\delta + \omega_s)\right)\alpha_{s2} = i\, c'_{o1}(\alpha_o\, \alpha_1 + \alpha_o^* \alpha_2) \tag{A.10}$$

where

$$c'_{so} = \left(\frac{2 n_o}{m_i c_s^2}\right)^{\frac{1}{2}} c_{so}$$

$$c'_{s1} = \left(\frac{2 n_o}{m_i c_s^2}\right)^{\frac{1}{2}} c_{s1}$$

$$c'_{o1} = \frac{2}{\epsilon_o}\left(\frac{m_i c_s^2}{2 n_o}\right)^{\frac{1}{2}} c_{o1}$$

Defining the real amplitude and phase functions as follows

$$\alpha_n(t) = a_n(t)\, e^{i\varphi_n(t)} \quad ; \quad \alpha = 0,\ 1,\ 2$$

$$\alpha_{s1,2}(t) = a_{s1,2}\; e^{i\varphi_{s1,2}(t)}$$

we obtain the following equations

$$\frac{\partial a_1}{\partial t} = c'_{so}(a_o\, a_{s1} \sin\theta_1 + a_o\, a_{s2} \sin\theta_2) \tag{A.11}$$

$$\frac{\partial a_2}{\partial t} = c'_{so}(a_o\, a_{s1} \sin\theta_3 + a_o\, a_{s2} \sin\theta_4) \tag{A.12}$$

$$\frac{\partial a_o}{\partial t} = - c'_{s1}(a_o\, a_{s1} \sin\theta_1 + a_2\, a_{s2} \sin\theta_4 + a_1\, a_{s2} \sin\theta_2 + a_2\, a_{s1} \sin\theta_3) \tag{A.13}$$

$$\frac{\partial a_{s1}}{\partial t} = c'_{o1}\,(a_o\, a_1 \sin\theta_1 - a_o\, a_2 \sin\theta_3) \tag{A.14}$$

$$\frac{\partial a_{s2}}{\partial t} = - c'_{o1}(a_o\, a_1 \sin\theta_2 - a_o\, a_2 \sin\theta_4) \tag{A.15}$$

$$\frac{\partial \varphi_1}{\partial t} = c'_{so}\left(\frac{a_o\, a_{s1}}{a_1} \cos\theta_1 + \frac{a_o\, a_{s2}}{a_1} \cos\theta_2\right) \tag{A.16}$$

$$\frac{\partial \varphi_2}{\partial t} = 2\delta - c'_{so}\left(\frac{a_o\, a_{s1}}{a_2} \cos\theta_3 + \frac{a_o\, a_{s2}}{a_2} \cos\theta_4\right) \tag{A.17}$$

$$\frac{\partial \varphi_o}{\partial t} = - c'_{s1}\left(\frac{a_1\, a_{s1}}{a_o} \cos\theta_1 + \frac{a_2\, a_{s2}}{a_o} \cos\theta_4 + \frac{a_1\, a_{s2}}{a_o} \cos\theta_2 + \frac{a_2\, a_{s1}}{a_o} \cos\theta_3\right) \tag{A.18}$$

$$\frac{\partial \varphi_{s1}}{\partial t} = \delta - \omega_s - c'_{o1}\left(\frac{a_o\, a_1}{a_{s1}} \cos\theta_1 + \frac{a_o\, a_2}{a_{s1}} \cos\theta_3\right) \tag{A.19}$$

$$\frac{\partial \varphi_{s2}}{\partial t} = \delta + \omega_s + c'_{o1}\left(\frac{a_o\, a_1}{a_{s2}} \cos\theta_2 + \frac{a_o\, a_2}{a_{s2}} \cos\theta_4\right) \tag{A.20}$$

where

$$\theta_1 = \varphi_o + \varphi_1 - \varphi_{s1}$$

$$\theta_2 = \varphi_o + \varphi_1 - \varphi_{s2}$$

$$\theta_3 = \varphi_o - \varphi_2 + \varphi_{s1}$$

$$\theta_4 = \varphi_o - \varphi_2 + \varphi_{s2}$$

Next, we calculate the time rate of change of the sum of the squares of the amplitudes using equations (A.11) - A.16) giving

$$\frac{\partial}{\partial t}\left(a_o^2 + a_1^2 + a_2^2 + a_{s1}^2 + a_{s2}^2\right) = \frac{\delta}{\omega_o}\frac{\partial}{\partial t}a_1^2 + \frac{\delta}{\omega_o}\frac{\partial}{\partial t}a_2^2 + \frac{\partial}{\partial t}a_{s1}^2 + \frac{\partial}{\partial t}a_{s2}^2 \quad (A.21)$$

where we have made use of the two relations

$$c'_{so} - c'_{s1} = c'_{so}\frac{\delta}{\omega_o} \qquad \text{and} \qquad c'_{o1} = c'_{so}\frac{\omega_s}{\omega_o} .$$

Equation (A.21) expresses the conservation of the total wave energy densities for the filamentation interaction and can be written in the form obtained earlier

$$\frac{\partial}{\partial t}\left\{a_o^2 + \left(1 - \frac{\delta}{\omega_o}\right)a_1^2 + \left(1 - \frac{\delta}{\omega_o}\right) a_2^2\right\} = 0 . \quad (A.22)$$

Equation (A.21) shows that the density perturbation evidently does not account for any of the energy in the interaction, and acts as a 'virtual' wave. This is consistent with the zero frequency nature of the perturbation.

Other constants of the motion analogous to those derived when ion inertia was neglected can also be obtained from equations (A.11) - (A.20). It is straight forward to obtain these relations and we therefore give only the results which are as follows.

$$\frac{\partial}{\partial t}\left(\frac{\omega_s}{\omega_1}a_o^2 + \frac{2\,\omega_s}{\omega_o}a_1^2 + a_{s2}^2 - a_{s1}^2\right) = 0 \quad (A.23)$$

$$\frac{\partial}{\partial t}\left(\frac{\omega_s}{\omega_1}a_o^2 + \frac{2\,\omega_s}{\omega_o}a_2^2 + a_{s1}^2 - a_{s2}^2\right) = 0 \quad (A.24)$$

$$\frac{\partial}{\partial t}\left\{\frac{\omega_s}{\omega_o}\left(a_1^2 - a_2^2\right) + a_{s2}^2 - a_{s1}^2\right\} = 0 \qquad (A.25)$$

$$\frac{\partial}{\partial t}\left\{a_o\, a_1\, a_{s1} \cos\theta_1 + a_o\, a_1\, a_{s2} \cos\theta_2 + a_o\, a_2\, a_{s1} \cos\theta_3 + a_o\, a_2\, a_{s2} \cos\theta_4 + \frac{1}{2}\frac{\omega_s}{c'_{o1}}\left(a_{s1}^2 + a_{s2}^2\right) - \frac{1}{2}\frac{\delta}{c'_{s1}}\, a_o^2\right\} = 0 \qquad (A.26)$$

Table I

showing $\{u_1^2(\tau)\}_{MAX}$ as a function of $u_2^2(o)$ and Δ

$u_2^2(o)$ \ Δ	0.0001	0.001	0.01	0.1	0.5	1.0	3.0	10.0
2×10^{-6}	0.4999998	0.499998	0.4975	0.475	0.375	0.249999	6.66×10^{-7}	2.5×10^{-8}
2×10^{-4}	0.499	0.4996	0.497	0.475	0.3747	0.2499	6.66×10^{-5}	2.5×10^{-6}
2×10^{-3}	0.498	0.498	0.496	0.474	0.372	0.249	6.6×10^{-4}	2.49×10^{-5}
2×10^{-2}	0.485	0.4848	0.482	0.46	0.349	0.24	5.7×10^{-3}	2.4×10^{-4}

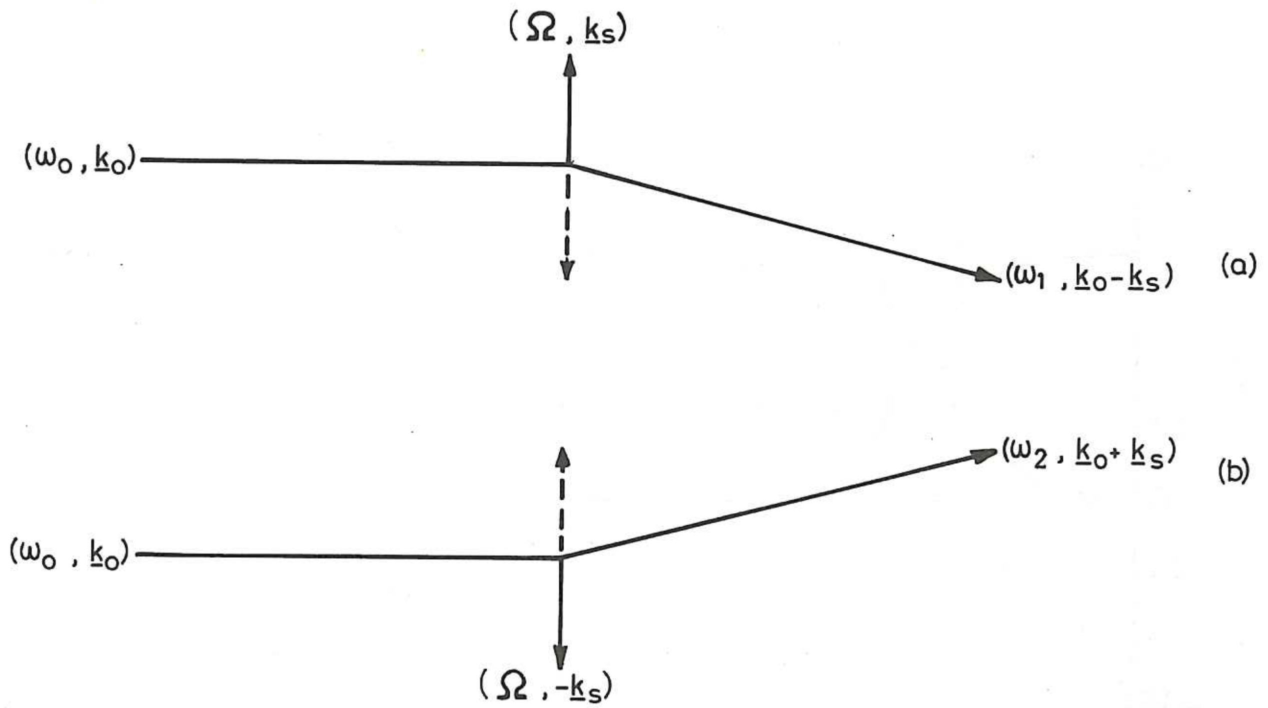

Figure 1 Configuration of wave vectors for filamentation
(a) $(\omega_o, \underline{k}_o)$ is the incident electromagnetic wave, $(\omega_1, \underline{k}_o - \underline{k}_s)$ is the Stokes electromagnetic wave and $(\Omega, \underline{k}_s)$ is a density perturbation;
(b) $(\omega , \underline{k}_o + \underline{k}_s)$ is the anti-Stokes electromagnetic wave and $(\Omega, \underline{k}_s)$ is a density perturbation.

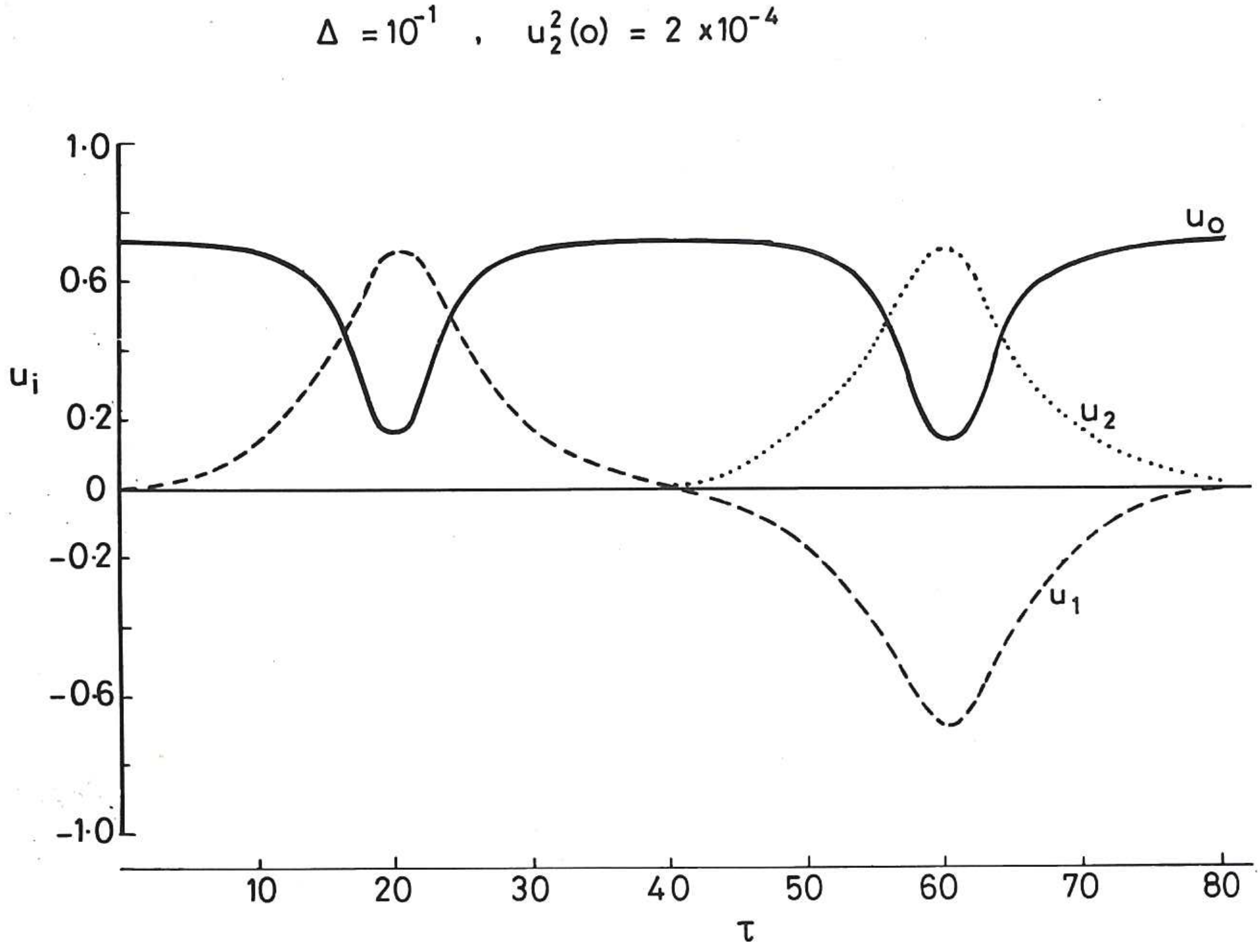

Figure 2 Normalized amplitudes of the incident u_o, Stokes u_1 and anti-Stokes u_2, electromagnetic waves as a function of the normalized time coordinate τ for $\Delta = 10^{-1}$ and $u_2^2(o) = 2 \times 10^{-4}$.

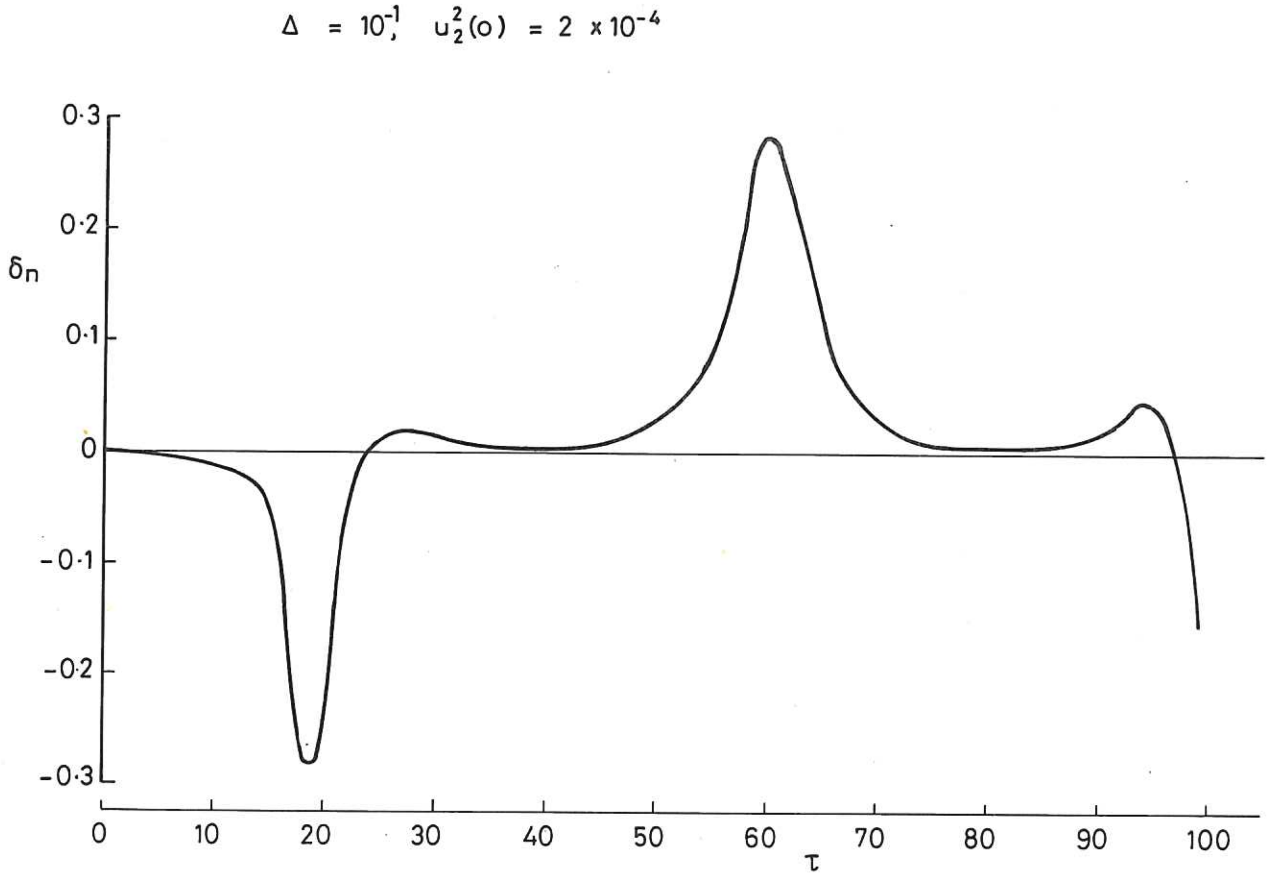

Figure 3 The associated density perturbation (as a fraction of the equilibrium density) plotted as a function of the normalized time τ for $\Delta = 10^{-1}$ and $u_2^2(o) = 2 \times 10^{-4}$.