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APPLICATION OF THE NOVIKOV-FURUTSU THEOREM TO THE RANDOM ACCELERATION PROBLEM

by

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Abstract

The problem of calculating the average propagator of a particle accelerated by a turbulent electric field is treated in a new way. Using a theorem of Novikov and Furutsu, the formulation of the problem is transformed into one in which the random field does not appear explicitly. A hierarchy of approximations is devised which converges on a closed equation for the propagator. This leads to a Dupree-like theory, but with short-time, rather than long-time, propagators in the resonance functions.

Some more speculative material is presented in an appendix.

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1. Introduction

In calculations of turbulent transport and in theories of plasma turbulence based on the concept of "resonance broadening" a crucial rôle is played by the averaged single particle propagator Γ, defined by

$$\Gamma(\underline{r}, \underline{v}, t | \underline{r}_o, \underline{v}_o, t_o) \equiv \langle G(r, v, t | r_o, v_o, t_o) \rangle , \qquad (1)$$

where G satisfies the equations

$$\left(\frac{D}{Dt} + \frac{e}{m} \underline{E}(r, t) . \frac{\partial}{\partial \underline{v}} \right) G(\underline{r}, \underline{v}, t | \underline{r}_o, \underline{v}_o, t_o) = \delta(\underline{r} - \underline{r}_o)\delta(\underline{v} - \underline{v}_o)\delta(t - t_o) \quad (2)$$

and

$$G(\underline{r}, \underline{v}, t_o | \underline{r}_o, \underline{v}_o, t_o) = \delta(\underline{r} - \underline{r}_o)\delta(\underline{v} - \underline{v}_o) \qquad (3)$$

In equation (2) $\underline{E}(\underline{r}, t)$ is the turbulent electric field, assumed to be a zero-mean random function, and D/Dt is the derivative along the trajectory the particle would traverse in the absence of $\underline{E}$. For the most part, in this paper, it will not be necessary to specify D/Dt more precisely.

The derivation of an equation satisfied by Γ has been discussed by many authors (e.g. DUPREE, 1966, 1967; ORSZAG and KRAICHNAN, 1967; WEINSTOCK, 1969; BENFORD and THOMPSON, 1972; COOK and SANDERSON, 1974; MARCUVITZ, 1974; COOK, 1975; ROLLAND, 1976; DEWAR, 1977) using a variety of intuitive and formal methods. In this paper the problem is tackled in a new way. By assuming that $\underline{E}(\underline{r}, t)$ is a Gaussian random function and using a theorem about functionals of such functions it proves possible, without approximation, to cast the problem in a form which is closer to a closed equation for Γ and a more suitable starting point for approximations than is equation (2).

We begin by averaging equation (2), to obtain

$$\frac{D\Gamma}{Dt} + \frac{e}{m} \frac{\partial}{\partial v^{\alpha}} \langle E^{\alpha} G \rangle = \delta(\underline{r} - \underline{r}_o)\delta(\underline{v} - \underline{v}_o)\delta(t - t_o) \; \ldots \qquad (4)$$

We now suppose that $\underline{E}(r, t)$ is a Gaussian random function. There are grounds (O'NEIL, 1974) for supposing this to be approximately true for a system of weakly interacting waves and particles. The restriction can be removed at the cost of an increase in analytical complexity which we do not believe to be worthwhile at present. With this assumption the average in equation (4) can be written explicitly as a functional integral (MARTIN and

SEGAL, 1964) :-

$$\langle E^{\alpha} G\rangle = \eta \int \delta \underline{E}\; E^{\alpha}(\underline{r}, t)\, G(\underline{r}, \underline{v}, t \,|\, \underline{r}_o, \underline{v}_o, t_o \,|\, \{E\})$$

$$.\exp\left[-\frac{1}{2}\int d\underline{r}_1\, d\underline{r}_2\, dt_1\, dt_2\, E^{\lambda}(\underline{r}_1 t_1) E^{\nu}(\underline{r}_2 t_2)\; K^{\lambda\nu}(\underline{r}_1 - \underline{r}_2,\; t_1 - t_2)\right] \qquad (5)$$

In equation (5) η is the normalisation (which it is not necessary to calculate), the functional dependence of G on $\underline{E}$ is indicated by curly brackets and $\underline{r}$, $\underline{v}$, t, $\underline{r}_o$, $\underline{v}_o$ and t_o are merely parameters. $K^{\lambda\nu}$ is the inverse of the covariance $Q^{\alpha\beta}$:-

$$\int d\underline{r}' \int dt'\; Q^{\alpha\gamma}(\underline{r} - \underline{r}', t - t')\, K^{\gamma\beta}(\underline{r}' - \underline{r}_o,\; t' - t_o) = \delta(\underline{r} - \underline{r}_o)\; \delta(t - t_o)\, \delta^{\alpha\beta} \qquad (6)$$

and $Q^{\alpha\beta}$ is defined as

$$Q^{\alpha\beta}(\underline{r} - \underline{r}', t - t') \equiv \langle E^{\alpha}(\underline{r}, t)\, E^{\beta}(\underline{r}', t')\rangle , \qquad (7)$$

it being assumed for convenience that the turbulence is stationary and homogeneous.

Equation (5) appears as a matter of record only: we shall not need to evaluate it. (However we return to the subject of functional integrals in the appendix.) Rather than attempting to evaluate the functional integral we employ the Novikov-Furutsu theorem.

2. The Novikov-Furutsu Theorem

Applied to the present problem, this theorem (NOVIKOV, 1965, FURUTSU, 1963) states that

$$\langle E^{\alpha}(\underline{r}, t)\, G(\underline{r}, v, t \,|\, \underline{r}_o, v_o, t_o \,|\, \{\underline{E}\})\rangle = \iint d\underline{r}'dt' \langle E^{\alpha}(\underline{r},t) E^{\beta}(\underline{r}',t')\rangle \left\langle \frac{\delta}{E^{\beta}(\underline{r}'t')}\, G(\underline{r},v,t|\underline{r}_o, v_o, t_o|\{E\})\right\rangle \qquad (8)$$

Proofs of equation (8) may be found in the references cited above and will not be reproduced here. However, it may assist the reader if we point out that equation (8) is a generalisation of the following simple result for a function $g(\alpha)$ of a single random variable α . The equations analogous to equations (5), (6) and (7) are

$$\langle \alpha g(\alpha) \rangle = \int_{-\infty}^{\infty} d\alpha\, \alpha\, g(\alpha)\, \frac{e^{-\frac{\alpha^2}{2\sigma^2}}}{(2\pi)^{\frac{1}{2}}\, \sigma} , \qquad (9)$$

where

$$\sigma^2 \equiv \langle \alpha^2 \rangle . \qquad (10)$$

It is simple to show, by integrating by parts, that

$$\langle \alpha\, g(\alpha) \rangle = \sigma^2 \int_{-\infty}^{\infty} dx\, \frac{\partial g}{\partial \alpha}\, \frac{e^{-\frac{\alpha^2}{2\sigma^2}}}{(2\pi)^{\frac{1}{2}}\, \sigma} \qquad (11)$$

$$= \langle \alpha^2 \rangle \left\langle \frac{\partial g}{\partial \alpha} \right\rangle , \qquad (12)$$

which is analogous to equation (8).

We now substitute equations (7) and (8) into equation (4) to obtain

$$\frac{D\Gamma}{Dt} + \frac{e}{m} \frac{\partial}{\partial v^{\alpha}} \iint d\underline{r}'dt'\; Q^{\alpha\beta}(r - r', t - t') \left\langle \frac{\delta}{\delta E^{\beta}(r', t')}\; G(\underline{r}, \underline{v}, t \,|\, \underline{r}_o, \underline{v}_o, t_o) \right\rangle$$

$$= \delta(\underline{r} - \underline{r}_o)\, \delta(\underline{v} - \underline{v}_o)\, \delta(t - t_o) \qquad (13)$$

This result would be useless unless it was possible to express $\delta G/\delta E$ in terms of G. Rather surprisingly perhaps this can be done. We return to equation (2) and make small changes in $\underline{E}$ and G : $E \rightarrow E + \delta E$, $G \rightarrow G + \delta G$. The linearised equation satisfied by δG and δE is

$$\left(\frac{D}{Dt} + \frac{e}{m} \underline{E} . \frac{\partial}{\partial \underline{v}} \right) \delta G = - \frac{e}{m} \underline{\delta E} . \frac{\partial G}{\partial \underline{v}} \qquad (14)$$

Equation (14) can be formally integrated using G :-

$$\delta G(\underline{r}, \underline{v}, t \,|\, \underline{r}_o, \underline{v}_o, t_o) = - \frac{e}{m} \int_{-\infty}^{t} dt' \int d\underline{r}' \int d\underline{v}'\; G(\underline{r}, \underline{v}, t \,|\, \underline{r}', \underline{v}', t) \delta \underline{E}(r', t') . \frac{\partial G}{\partial \underline{v}'} (\underline{r}', \underline{v}', t' \,|\, \underline{r}_o, \underline{v}_o, t_o)$$

$$(15)$$

Thus the required functional derivative is

$$\frac{\delta G(\underline{r}, \underline{v}, t | r_o, v_o, t_o)}{\delta E^{\beta}(r', t')} = -\frac{e}{m}\int d\underline{v}' G(\underline{r}, \underline{v}, t | \underline{r}', \underline{v}', t') \frac{\partial G}{\partial v'_\beta} (\underline{r}', \underline{v}', t' | \underline{r}_o, \underline{v}_o, t_o) \tag{16}$$

If we substitute from equation (16) into equation (13) we obtain the desired equation :

$$\frac{D\Gamma}{Dt} - \frac{e^2}{m^2}\frac{\partial}{\partial v^\alpha}\iiint dr' dv' dt' Q^{\alpha\beta}(r - r', t - t')\left\langle G(\underline{r},\underline{v},t|\underline{r}'\underline{v}'t')\frac{\partial G}{\partial v'_\beta}(\underline{r}', \underline{v}', t' | \underline{r}_o, \underline{v}_o, t_o)\right\rangle$$

$$= \delta(r - r_o)\,\delta(v - v_o)\,\delta(t - t_o) \tag{17}$$

We emphasise that equation (17) is exact (for Gaussian $\underline{E}(\underline{r}, t)$). This equation is useful because, as we shall see below, it readily lends itself to the generation of approximate closed equations for Γ, untroubled by secularities, without the need to introduce the complicated apparatus of diagrammatic perturbation theory, projection operators, etc. In addition one gains a better feel for the circumstances in which the approximations are valid than can be obtained from those methods.

We note in passing that equation (17) is very reminiscent of a Green function description of a peculiar kind of many-body problem, with $Q^{\alpha\beta}(\underline{r},t)$ serving as the "interaction potential". In the appendix we make an attempt to capitalise on this idea.

3. Two simple approximations

In this section we show how equation (17) can be reduced, in appropriate circumstances, to some results which we shall need in later sections.

(a) Suppose that the fluctuating force is a function of time only, so that

$$Q^{\alpha\beta}(\underline{r}, t) = C^{\alpha\beta}(t), \text{ say}, \tag{18}$$

and that

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v}\cdot\frac{\partial}{\partial \underline{r}} . \tag{19}$$

In this case G is a function of $\underline{v} - \underline{v}'$ only, not of $\underline{v}$ and $\underline{v}'$ separately, so we can write equation (17) as

$$\frac{D\Gamma}{Dt}(\underline{r}, \underline{v}, t|\underline{r}_o, \underline{v}_o, t_o) - \frac{e^2}{m^2}\frac{\partial^2}{\partial v^\alpha \partial v^\beta}\int dt' C^{\alpha\beta}(t-t')\iint d\underline{r}' d\underline{v}'$$

$$\langle G(\underline{r}, \underline{v}, t|\underline{r}', \underline{v}', t')\, G(\underline{r}', \underline{v}', t'|\underline{r}_o, \underline{v}_o, t_o)\rangle = \delta(\underline{r}-\underline{r}_o)\,\delta(\underline{v}-\underline{v}_o)\,\delta(t-t_o) . \quad (20)$$

We now observe that because G propagates a Markov process (or by direct computation) it satisfies the Chapman-Kolmogorov equation

$$G(\underline{r}, \underline{v}, t|\underline{r}_o, \underline{v}_o, t_o) = \iint d\underline{r}' d\underline{v}'\, G(\underline{r}, \underline{v}, t|\underline{r}', \underline{v}', t')\, G(\underline{r}', \underline{v}', t'|\underline{r}_o, \underline{v}_o, t_o) , \quad (21)$$

for any t' satisfying $t > t' > t_o$.

Substituting the average of equation (21) into the right hand side of equation (20), we obtain

$$\frac{D\Gamma}{Dt} - \frac{e^2}{m^2}\int_{t_o}^{t} dt' C^{\alpha\beta}(t-t') . \frac{\partial^2\Gamma}{\partial v^\alpha \partial v^\beta} = \delta(\underline{r}-\underline{r}_o)\,\delta(\underline{v}-\underline{v}_o)\,\delta(t-t_o) \quad (22)$$

If the correlation time is very short (compared to the time-scale of Γ) $C^{\alpha\beta}(t)$ can be approximated as

$$C^{\alpha\beta}(t) = 2\, D^{\alpha\beta} . \delta(t) , \quad (23)$$

when equation (22) reduces to the following familiar equation

$$\frac{D\Gamma}{Dt} - \frac{e^2}{m^2} D^{\alpha\beta} \frac{\partial^2\Gamma}{\partial v^\alpha \partial v^\beta} = \delta(\underline{r}-\underline{r}_o)\,\delta(\underline{v}-\underline{v}_o)\,\delta(t-t_o) . \quad (24)$$

However we shall be more interested, in section 5, in the opposite case of long correlation times when $C^{\alpha\beta}$ can be approximated as a static average,

$$C^{\alpha\beta}(t) = \langle E^\alpha E^\beta\rangle , \quad (25)$$

and equation (22) becomes

$$\frac{D\Gamma}{Dt} - \frac{e^2}{m^2}(t-t_o)\,\langle E^\alpha E^\beta\rangle \frac{\partial^2\Gamma}{\partial v^\alpha \partial v^\beta} = \delta(\underline{r}-\underline{r}_o)\,\delta(\underline{v}-\underline{v}_o)\,\delta(t-t_o) . \quad (26)$$

(b) Suppose the fluctuating force has correlation length R_c and correlation time T_c . Then if the relative change in velocity experienced by the particle in a time T_c or in the time taken to move a distance R_c

(whichever is the less) is small it is clear that the first G on the right hand side of equation (17) may be replaced by the unperturbed orbit propagator Γ_o, which satisfies

$$\frac{D\Gamma_o}{Dt} = \delta(\underline{r} - \underline{r}')\,\delta(\underline{v} - \underline{v}')\,\delta(t - t') \tag{27}$$

The criterion for the validity of this replacement if, for example, D/dt is given by equation (19) and the random electric field is isotropically distributed is

$$\frac{e\,E_o\,T_c}{m\,V} \ll 1 \tag{28}$$

or

$$\frac{e\,E_o\,R_c}{m\,V^2} \ll 1, \tag{29}$$

(where $E_o^2 = \langle E^2 \rangle$). If the geometry is more complicated this criterion will also be more complicated.

With this replacement equation (17) becomes

$$\frac{D\Gamma_1}{Dt} - \frac{e^2}{m^2}\frac{\partial}{\partial v^\alpha}\iiint d\underline{r}'\,d\underline{v}'\,dt'\,Q^{\alpha\beta}(r - r', t - t')\;\Gamma_o(\underline{r}, \underline{v}, t\,|\,\underline{r}', \underline{v}', t')\,\frac{\partial\Gamma_1}{\partial v'_\beta}\,(\underline{r}', \underline{v}', t'\,|\,\underline{r}_o, \underline{v}_o, t_o)$$

$$= \delta(\underline{r} - \underline{r}_o)\,\delta(\underline{v} - \underline{v}_o)\,\delta(t - t_o) \tag{30}$$

The notation has been changed from Γ to Γ_1 in anticipation of the derivation in the following sections of a sequence, Γ_n, of approximations to Γ.

Equation (30) may, by further approximation be reduced to the Fokker-Planck equation (STURROCK, 1966) which forms the basis of quasi-linear theory (DRUMMOND, 1965, BERNSTEIN and ENGELMANN, 1966). In this paper, however, we shall concentrate on deriving improvements to equation (30).

4. A sequence of higher order approximations

We now return to equation (13) :

$$\frac{D\Gamma}{Dt} + \frac{e}{m}\frac{\partial}{\partial v^\alpha}\iint d\underline{r}'\,d\underline{t}'\,Q^{\alpha\beta}(\underline{r} - \underline{r}', t - t')\left\langle \frac{\delta}{\delta E^\beta(\underline{r}', t')}\;G(\underline{r}, \underline{v}, t\,|\,\underline{r}_o, \underline{v}_o, t_o)\right\rangle$$

$$= \delta(\underline{r} - \underline{r}_o)\,\delta(\underline{v} - \underline{v}_o)\,\delta(t - t_o) \tag{13}$$

We propose now to leave this exact equation unapproximated and use the Novikov-Furutsu method to obtain an equation satisfied by $\langle \delta G/\delta E \rangle$. This second equation is then approximated, leading to a kinetic equation for Γ. This process may be carried further and the approximation indefinitely postponed, with results which we shall see below.

To obtain an equation satisfied by $\delta G/\delta E$ we functionally differentiate equation (2), for $t' < t$, to obtain :

$$\frac{D}{Dt}\frac{\delta G(\underline{r}, \underline{v}, t|\underline{r}_o, \underline{v}_o, t_o)}{\delta E^{\beta}(r', t')} + \frac{e}{m} E^{\alpha}(\underline{r}, t)\frac{\partial}{\partial v^{\alpha}}\frac{\delta}{\delta E^{\beta}(r', t')} G(\underline{r}, \underline{v}, t|r_o, v_o, t_o) = 0 . \tag{31}$$

Averaging equation (31) :

$$\frac{D}{Dt}\left\langle\frac{\delta}{\delta E^{\beta}(r', t')} G(\underline{r}, \underline{v}, t|\underline{r}_o, \underline{v}_o, t_o)\right\rangle = -\frac{e}{m}\frac{\partial}{\partial v^{\alpha}}\left\langle E^{\alpha}(\underline{r}, t)\frac{\delta}{\delta E^{\beta}(r',t')} G(r, v, t|r_o, v_o, t_o)\right\rangle \tag{32}$$

$$= -\frac{e}{m}\frac{\partial}{\partial v^{\alpha}}\iint d\underline{r}''dt'' \langle E^{\alpha}(\underline{r}, t)\, E^{\gamma}(\underline{r}'', t'')\rangle\left\langle\frac{\delta^2}{\delta E^{\gamma}(r'', t'')\;\delta E^{\beta}(r', t')} G(\underline{r}, \underline{v}, t|\underline{r}_o, \underline{v}_o, t_o)\right\rangle , \tag{33}$$

where the Novikov-Furutsu theorem has been used.

The second order functional derivative on the right hand side of equation (33) may be calculated by the techniques used in section 3, with the result that equation (33) becomes

$$\frac{D}{Dt}\left\langle\frac{\delta}{\delta E^{\beta}(\underline{r}', t)} G(\underline{r}, \underline{v}, t|\underline{r}_o, \underline{v}_o, t_o)\right\rangle = \frac{e^2}{m^2}\frac{\partial}{\partial v^{\alpha}}\iiint d\underline{r}''d\underline{v}''Q^{\alpha\gamma}(\underline{r} - \underline{r}'', t - t'')$$

$$\left\langle G(\underline{r}, \underline{v}, t|\underline{r}'', \underline{v}'', t'')\frac{\partial}{\partial v_{\gamma}''}\frac{\delta}{\delta E^{\beta}(r', t')} G(r'', v'', t''|r_o, v_o, t_o)\right\rangle \tag{34}$$

This exact equation is the analogue of equation (17). We now make the approximation of replacing G by Γ_o in the right hand side of equation (34), which becomes

$$\frac{D}{Dt}\left\langle\frac{\delta}{\delta E^{\beta}(\underline{r}', t')} G(\underline{r}, \underline{v}, t|\underline{r}_o, \underline{v}_o, t_o)\right\rangle = \frac{e^2}{m^2}\frac{\partial}{\partial v^{\alpha}}\iiint d\underline{r}''d\underline{v}''dt''\; Q^{\alpha\gamma}(\underline{r} - \underline{r}'', t - t'')$$

$$\Gamma_o(\underline{r}, \underline{v}, t|\underline{r}'', \underline{v}'', t'')\frac{\partial}{\partial v_{\gamma}''}\left\langle\frac{\delta}{\delta E^{\beta}(r', t')} G(\underline{r}'', \underline{v}'', t''|\underline{r}_o, \underline{v}_o, t_o)\right\rangle \tag{35}$$

If equation (35) is compared with equation (30) it will be seen that $\langle \delta G/\delta E \rangle$ and Γ_1 satisfy the same equation apart from the presence of the δ-functions on the right hand side of equation (30). In fact, in this approximation, Γ_1 is the propagator of $\langle \delta G/\delta E \rangle$ and the solution of equation (35) may be expressed as

$$\left\langle \frac{\delta}{\delta E^{\beta}(r', t')} G(\underline{r}, \underline{v}, t | \underline{r}_o, \underline{v}_o, t_o) \right\rangle = \iint d\underline{y}\, d\underline{u}\, \Gamma_1(\underline{r}, \underline{v}, t | \underline{y}, \underline{u}, \tau) \left\langle \frac{\delta}{\delta E^{\beta}(\underline{r}', t')} G(\underline{y}, \underline{u}, \tau | \underline{r}_o, \underline{v}_o, t_o) \right\rangle , \tag{36}$$

where τ is any time satisfying $t > \tau > t_o$. In particular one may choose $\tau = t'$. The value of $\delta G/\delta E$ for $\tau = t'$ (the "initial" value) may be obtained from equation (16) :

$$\frac{\delta}{\delta E^{\beta}(r', t')} G(\underline{y}, \underline{u}, t' | \underline{r}_o, \underline{v}_o, t_o) = -\frac{e}{m} \int dv'' G(\underline{y}, \underline{u}, t' | \underline{r}', \underline{v}', t') \frac{\partial G}{\partial v_{\beta}''} (\underline{r}', \underline{v}'', t' | \underline{r}_o, \underline{v}_o, t_o \tag{37}$$

Using the fact that

$$G(\underline{y}, \underline{u}, t' | \underline{r}', \underline{v}', t') = \delta(\underline{y} - \underline{r}')\, \delta(\underline{u} - \underline{v}') , \tag{38}$$

equation (37) may be reduced to

$$\frac{\delta}{\delta E^{\beta}(r', t')} G(\underline{y}, \underline{u}, t' | \underline{r}_o, \underline{v}_o, t_o) = -\frac{e}{m} \delta(\underline{y} - \underline{r}') \frac{\partial G}{\partial u^{\beta}} (\underline{y}, \underline{u}, t' | \underline{r}_o, \underline{v}_o, t_o) . \tag{39}$$

Finally, substituting equation (39) into equation (36) (with $\tau = t'$) we obtain

$$\left\langle \frac{\delta}{\delta E^{\beta}(r', t')} G(\underline{r}, \underline{v}, t | r_o, v_o, t_o) \right\rangle = -\frac{e}{m} \int d\underline{u}\, \Gamma_1(\underline{r}, \underline{v}, t | \underline{r}', \underline{u}, t') \frac{\partial \Gamma}{\partial u^{\beta}} (\underline{r}', \underline{u}, t' | \underline{r}_o, \underline{v}_o, t_o) \tag{40}$$

Now we substitute this approximation for $\langle \delta G/\delta E \rangle$ into equation (13) to obtain the following equation for Γ (renamed Γ_2) ;

$$\frac{D}{Dt}\,\Gamma_2(\underline{r},\underline{v},t|\underline{r}_o,\underline{v}_o,t_o) - \frac{e^2}{m^2}\frac{\partial}{\partial v^{\alpha}}\iiint d\underline{r}'\,d\underline{v}'\,dt'\;Q^{\alpha\beta}(\underline{r}-\underline{r}',t-t')$$

$$\Gamma_1(\underline{r},\underline{v},t|\underline{r}',\underline{v}',t')\,\frac{\partial\Gamma_2}{\partial v'_{\beta}}(\underline{r}',\underline{v}',t'|\underline{r}_o,\underline{v}_o,t_o) = \delta(\underline{r}-\underline{r}_o)\,\delta(\underline{v}-\underline{v}_o)\,\delta(t-t_o)\,. \tag{41}$$

Equation (41) is our second order approximation. We note that it can be obtained from the first order approximation (equation (30)) by the replacements $(\Gamma_0 \to \Gamma_1\,,\ \Gamma_1 \to \Gamma\ )$. To carry the discussion further it is convenient to adopt a much more concise and symbolic notation. Thus we say that the 1st order approximation, Γ_1, satisfies

$$(D - Q\,\Gamma_o)\,\Gamma_1 = \delta\,. \tag{30a}$$

The second order approximation, Γ_2, satisfies

$$(D - Q\,\Gamma_1)\,\Gamma_2 = \delta\,. \tag{41a}$$

We have verified by direct calculation that the next stage is

$$(D - Q\,\Gamma_2)\,\Gamma_3 = \delta\,, \tag{42}$$

and clearly if the approximation procedure is deferred to the n th order equation the result will be a chain of equations terminating in

$$(D - Q\,\Gamma_{n-1})\,\Gamma_n = \delta\,. \tag{43}$$

If this process is converging, that is if $\hat{\Gamma}$, defined as

$$\hat{\Gamma} \equiv \lim_{n\to\infty} \Gamma_n\,, \tag{44}$$

exists, it will satisfy the equation

$$(D - Q\,\hat{\Gamma})\,\hat{\Gamma} = \delta\,. \tag{45a}$$

Written out in full this is

$$\frac{D}{Dt}\,\hat{\Gamma}(\underline{r}, \underline{v}, t|\underline{r}_o, \underline{v}_o, t_o) - \frac{e^2}{m^2}\frac{\partial}{\partial v^{\alpha}}\iiint d\underline{r}'\,d\underline{v}'\,dt'\,Q^{\alpha\beta}(\underline{r}-\underline{r}', t-t')$$

$$\hat{\Gamma}(\underline{r}, \underline{v}, t|\underline{r}', \underline{v}', t')\frac{\partial}{\partial v'_{\beta}}\hat{\Gamma}(\underline{r}', \underline{v}', t'|\underline{r}_o, \underline{v}_o, t_o) = \delta(\underline{r}-\underline{r}_o)\delta(\underline{v}-\underline{v}_o)\delta(t-t_o). \quad (45)$$

5. Discussion

Equation (45) was first proposed by ORSZAG and KRAICHNAN (1967), who discuss how it is related to Dupree's theory. The method used by these authors gives little clue as to the domain of validity of the equation. At first sight it appears that a succinct and reasonable derivation of equation (45) can be based on the analysis in section 2 of this paper, and that the long argument in section 4 is unnecessary. Returning to the exact equation (17) we see that it contains an average of two Green functions. Writing

$$G \equiv \Gamma + \tilde{G}, \quad (46)$$

where
$$\langle\tilde{G}\rangle \equiv 0, \quad (47)$$

we can write the average of two G's as

$$\langle GG'\rangle \equiv \Gamma\Gamma' + \langle\tilde{G}\tilde{G}'\rangle \quad (48)$$

It is tempting to suppose, as a reasonable first approximation, that the fluctuations in G are small compared to its mean, so

$$\langle GG'\rangle \approx \Gamma\Gamma' \quad (49)$$

Put into equation (17), this produces equation (45) immediately.

It may be noted that if this type of approximation is applied directly to the governing equation (2) it gives rise to quasi-linear theory, but when applied to the exact reformulation, equation (17), it gives rise to an equation which is usually thought of as a consequence of a "renormalised" or "consolidated" perturbation theory.

However tempting it may be, the above argument is not valid. The average $\langle G(\underline{r}, \underline{v}, t|\underline{r}', \underline{v}', t')\, G(\underline{r}', \underline{v}', t'|r_o, v_o, t_o)\rangle$ only factorises if the motion of the particle in the time interval $t' \to t$ is statistically independent of its motion in the earlier interval $t_o \to t'$. It is easy to think of examples where this is not even approximately true. For example if the random field $\underline{E}$ is independent of $\underline{r}$ and t it is clear that the motion before and after t' is perfectly correlated. Indeed, if in addition D/Dt is given by equation (19) one can readily show that

$$\langle G(\underline{r}, \underline{v}, t|\underline{r}', \underline{v}', t')\, G(\underline{r}', \underline{v}', t'|\underline{r}_o, \underline{v}_o, t_o)\rangle = \delta\Big(\underline{r} - \underline{r}' - \underline{v}'(t-t') -$$

$$\frac{(t-t')^2}{(t'-t_o)^2}\{r' - r_o - v_o(t'-t_o)\}\Big)\delta\Big(\underline{v} - \underline{v}' - \frac{(t-t')}{(t'-t_o)}(\underline{v}' - \underline{v}_o)\Big)\Gamma(\underline{r}', \underline{v}', t'|\underline{r}_o, \underline{v}_o, t_o)\,. \tag{50}$$

Factorisation of $\langle GG\rangle$ is just nonsense in this case. The correlation time and distance are infinite in this example, but the conclusion still follows if they are merely large compared to the other times and distances in the problem and we are led to the view that factorisation of $\langle GG\rangle$ is a good approximation only when T_c or R_c are small in some sense. The analysis in sections 3(b) and 4 may be regarded as a more precise and much expanded version of this argument.

If this argument is correct, equation (45) should be regarded as an improved version of equation (30), valid in essentially the same circumstances. Of course, it is likely to be free from certain difficulties, arising more from an inappropriate mathematical method than the physics of the situation, such as accidentally vanishing denominators, secularities, etc. This may be a considerable advantage in practice. (This assessment is at variance with the hope expressed by COOK and SANDERSON (1974) that equation (45), because of its non-Markovian character, might be adequate for the treatment of strong plasma turbulence.)

We now turn to the consideration of what can be done with equation (45). We begin by observing that we only need to know $\hat{\Gamma}(\underline{r}, \underline{v}, t|\underline{r}', \underline{v}', t')$ for time intervals $(t-t')$ shorter than or comparable with the correlation time of the fluctuating force experienced by the particle. Let us call an approximation to $\hat{\Gamma}$ in these circumstances $\hat{\Gamma}_{SH}$ (for "short time"). If $\hat{\Gamma}_{SH}$ can be independently calculated, we can write a linear equation for $\hat{\Gamma}$:-

$$\frac{D}{Dt}\hat{\Gamma}(\underline{r}, \underline{v}, t|\underline{r}_o, \underline{v}_o, t_o)$$

$$-\frac{e^2}{m^2}\frac{\partial}{\partial v^\alpha}\iiint d\underline{r}'\, d\underline{v}'\, dt'\, Q^{\alpha\beta}(\underline{r}-\underline{r}', t-t')\,\hat{\Gamma}_{SH}(\underline{r}, \underline{v}, t|\underline{r}', \underline{v}', t')\frac{\partial\hat{\Gamma}}{\partial v_\beta}(\underline{r}', \underline{v}', t'|\underline{r}_o, \underline{v}_o, t_o)$$

$$= \delta(\underline{r}-\underline{r}_o)\,\delta(\underline{v}-\underline{v}_o)\,\delta(t-t_o)\,. \qquad (51)$$

In fact the independent determination of $\hat{\Gamma}_{SH}$ is fairly straightforward. As we are only concerned with times shorter than the correlation time we can ignore the spatio-temporal dependence of $\underline{E}(\underline{r}, t)$ and take it to be a random constant vector. Thus (at least when equation (19) holds, and with analogues in more complicated geometry) $\hat{\Gamma}_{SH}$ satisfies equation (26) :

$$\frac{D}{Dt}\hat{\Gamma}_{SH} - \frac{e^2}{m^2}(t-t')\langle E^\alpha E^\beta\rangle\frac{\partial^2}{\partial v^\alpha \partial v^\beta}\hat{\Gamma}_{SH} = \delta(\underline{r}-\underline{r}')\delta(\underline{v}-\underline{v}')\delta(t-t')\,. \qquad (52)$$

This seems a reasonable point at which to stop. The approximate solution of equations (51) and (52) and applications to wave growth will be treated in a subsequent paper. Obviously the theory will be rather like that of Dupree but with short-time, rather than long-time, mean propagators in the resonance functions.

Appendix

We had originally intended the material in this appendix to occupy a more honourable position, but growing doubts as to the validity of the argument have led to a loss of nerve. However we believe it to be worth presenting, partly because of its considerable intrinsic interest, and partly because of a belief that really the argument is sound (or could be made sound) and might be made to yield more.

We recall an earlier remark that equation (17) looks like the average of a hypothetical Green function formulation of a strange sort of many-body problem, with Q serving as the "interaction potential" (which depends on temporal as well as spatial separations). We also arrived at this idea in another way : by expressing Γ as a path-integral, but we will spare the reader this argument. In fact, when you think of it, it is unavoidable that the exact equation for $\langle G \rangle$ should have a "many-body" character. After the random field $\underline{E}(\underline{r}, t)$ has been averaged out of the problem, what else could remain but the interaction of G with itself, via Q? As thus formulated, the problem shows affinities to certain problems in the statistical mechanics of polymers (EDWARDS and FREED, 1970), self-avoiding random walks, critical-point dynamics (MA, 1976), amorphous materials (ECONOMOU et al., 1971), etc.

Might it be possible to replace the many-body problem with the problem of a single body moving in a self-consistent field (rather like the Thomas-Fermi atom)?

With the foregoing as motivation we return to a more conventional formulation of our problem. The mean Green function may be defined as a functional integral, analogous to equation (5) :

$$\Gamma = \eta \int \delta\underline{E}\; G(\underline{r}, \underline{v}, t | \underline{r}_o, \underline{v}_o, t_o | \{\underline{E}\}).\exp\left[-\tfrac{1}{2}\int d\underline{r}_1\, d\underline{r}_2\, dt_1\, dt_2 E^{\lambda}(\underline{r}_1, t_1) E^{\nu}(\underline{r}_2, t_2) K^{\lambda\nu}(\underline{r}_1 - \underline{r}_2, t_1 - t_2)\right] \tag{A1}$$

Let us denote the sought-after (non-random) self-consistent field by $\hat{E}(r, t)$ and define $\hat{G}$ by

$$\hat{G}(\underline{r}, \underline{v}, t | \underline{r}_o, \underline{v}_o, t_o) \equiv G(\underline{r}, \underline{v}, t | \underline{r}_o, \underline{v}_o, t_o | \{\hat{\underline{E}}\}) . \tag{A2}$$

$\hat{G}$ satisfies the equation

$$\frac{D}{D\tau} \hat{G}(\underline{y}, \underline{v}, \tau | \underline{r}_o, \underline{v}_o, t_o) + \frac{e}{m} \frac{\partial}{\partial \underline{v}} \{\hat{E}(y, \tau) \hat{G}(\underline{y}, \underline{v}, \tau | \underline{r}_o, \underline{v}_o, t_o)\}$$

$$= \delta(\underline{y} - \underline{r}_o)\delta(\underline{v} - \underline{v}_o)\delta(\tau - t_o) . \qquad \text{(A3)}$$

The linearity of this equation is rather misleading, for $\hat{E}(\underline{y}, \tau)$ will be a functional of the "answer", $\Gamma(\underline{r}, \underline{v}, t | \underline{r}_o, \underline{v}_o, t_o)$. We will eventually put

$$\hat{G}(\underline{r}, \underline{v}, t | \underline{r}_o, \underline{v}_o, t_o) \triangleq \Gamma(\underline{r}, \underline{v}, t | \underline{r}_o, \underline{v}_o, t_o) . \qquad \text{(A4)}$$

The distinction between $(\underline{y}, \tau)$ and $(\underline{r}, t)$, notwithstanding that eventually $\underline{y} \to \underline{r}$ and $\tau \to t$, has been introduced in an attempt to avoid confusion!

Comparing equations (A1) and (A3), we see that $\hat{\underline{E}}$ (if it exists) must be that field $\underline{E}(\underline{r}, t)$ (if it exists) in equation (A1) which dominates the average. (From this point on the argument is similar to that used by ECONOMOU et al. (1971).)

If we define B by writing equation (A1) as

$$\Gamma \equiv \eta \int \delta \underline{E} \; e^{B} , \qquad \text{(A5)}$$

then

$$B = \ln G - \frac{1}{2} \int d\underline{r}_1 d\underline{r}_2 dt_1 dt_2 \; E^{\lambda}(\underline{r}_1, t_1) E^{\nu}(\underline{r}_2, t_2) K^{\lambda\nu}(\underline{r}_1 - \underline{r}_2, t_1 - t_2) . \qquad \text{(A6)}$$

The dominant field is given by

$$\left. \frac{\delta B}{\delta E^{\lambda}(\underline{r}', t')} \right|_{\underline{E} = \hat{\underline{E}}} = 0 \qquad \text{(A7)}$$

(This is like the method of steepest descents.) Substituting equation (A6) into equation (A7), we obtain

$$\frac{1}{G}\frac{\delta}{\delta E^{\lambda}(\underline{r}', t')} G(\underline{r}, \underline{v}, t|\underline{r}_o, \underline{v}_o, t_o) - \iint d\underline{r}_2 dt_2\, E^{\nu}(\underline{r}_2, t_2) K^{\lambda\nu}(\underline{r}' - \underline{r}_2, t' - t_2) = 0\,. \tag{A8}$$

Now K is the inverse of Q, so multiplying equation (A8) by $Q^{\lambda\alpha}(\underline{y} - \underline{r}', \tau - t')$, integrating over r' and t' and using equation (6), we find

$$\hat{E}^{\alpha}(\underline{y}, \tau)\, G(\underline{r}, \underline{v}, t|\underline{r}_o, \underline{v}_o, t_o|\{\hat{E}\})$$

$$= \iint d\underline{r}' dt' Q^{\lambda\alpha}(\underline{y} - \underline{r}', \tau - t') \frac{\delta}{\delta E^{\lambda}(r', t')} G(\underline{r}, \underline{v}, t|\underline{r}_o, \underline{v}_o, t_o|\{E\})\Big|_{\underline{E} = \hat{\underline{E}}}\,. \tag{A9}$$

We have already evaluated $\delta G/\delta E$ (equation (16)). Using this result in equation (A9), substituting the result in equation (A3) and putting $\underline{y} \to \underline{r}$, $\tau \to t$, we finally obtain

$$\frac{D}{Dt}\hat{G} - \frac{e^2}{m^2}\frac{\partial}{\partial v^{\alpha}} \iiint d\underline{r}' d\underline{v}' dt' Q^{\alpha\beta}(\underline{r} - \underline{r}', t - t')\, \hat{G}(\underline{r}, \underline{v}, t|\underline{r}', \underline{v}', t') \frac{\partial}{\partial v'_{\beta}} \hat{G}(\underline{r}', \underline{v}', t'|\underline{r}_o, \underline{v}_o, t_o)$$

$$= \delta(\underline{r} - \underline{r}_o)\delta(\underline{v} - \underline{v}_o)\delta(t - t_o)\,. \tag{A10}$$

But this is just equation (45). All roads lead to Rome.

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