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A B S T R A C T

A procedure which reconciles long parallel wave-length, and short perpendicular wave-length, characteristic of plasma instabilities, with periodicity in a sheared toroidal magnetic field is described. Applied to the problem of high-n ballooning-modes in tokamaks it makes possible a full minimisation of δW and shows that previous calculations overestimated stability.

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In many investigations of plasma stability[1,2,3], in both fluid and kinetic theories, the principal difficulty is that of reconciling the characteristics of unstable oscillations - such as long parallel wave-length and short perpendicular wave-length - with the constraints imposed by periodicity in a sheared toroidal magnetic field. An important problem of current interest[1,2] in which the balance between these conflicting factors plays a crucial role is the calculation of the stability limit for ballooning modes - which in turn determines the maximum β attainable in a tokamak.

In our discussion we employ standard[4] orthogonal coordinates (ψ,χ,ζ) in which ψ labels the magnetic surfaces, χ is a poloidal angle-like variable and ζ is the toroidal angle. The magnetic field is $\underset{\sim}{B} = \nabla\psi \times \nabla\zeta + f(\psi)\nabla\zeta$ and the metric

$$(ds)^2 = (d\psi/RB_p)^2 + (JB_p d\chi)^2 + (Rd\zeta)^2$$

with J the Jacobian, R the radius and B_p the poloidal field $|\nabla\psi \times \nabla\zeta|$. We also define $\nu = fJ/R^2$ so that $\oint \nu d\chi = 2\pi q$ where q is the "safety factor".

The conventional representation of waves with short perpendicular, and long parallel, wave-length is in the eikonal form

$$\varphi(\psi,\chi) \exp \left(in[\zeta - \int^{\chi} \nu d\chi]\right) \tag{1}$$

with $n \gg 1$. (The phase is constant along $\underset{\sim}{B}$ but varies rapidly perpendicular to $\underset{\sim}{B}$.) However it is easily seen[5] that in a magnetic field with shear $(dq/d\psi \neq 0)$, this form is incompatible with periodicity in χ.

Recent work on ballooning modes[1,2] attempted to overcome this difficulty by imposing an artificial constraint that $\varphi = 0$ at the ends of the basic interval in χ. However, as we shall show, one does not then obtain the most unstable mode so that such calculations overestimate the stability of the system. Other authors[3] have attempted to circumvent the problem by introducing discontinuous jumps in φ at the ends of the basic interval in χ - but this is incompatible with the assumption that the amplitude $\varphi(\psi,\chi)$ varies slowly compared to the phase. An alternative approach[5,2] is to modify the eikonal by an arbitrary function G such that $\oint(\nu + G)d\chi = 2\pi m$ on all surfaces - but no satisfactory method for determining G has been given.

These difficulties can be overcome as follows[6]. In any axisymmetric system the determination of stability can be reduced to a two-dimensional eigenvalue problem

$$\mathcal{L}(\theta,x)\,\varphi(\theta,x) = \lambda\varphi(\theta,x) \tag{2}$$

where θ represents the poloidal angle and x the flux surface coordinate. The operator $\mathcal{L}$ is periodic in θ and $\varphi(\theta,x)$ must also be periodic in θ and bounded in x. If we express φ in the form

$$\varphi(\theta,x) = \sum_m e^{-im\theta}\int_{-\infty}^{\infty} e^{im\eta}\,\hat{\varphi}(\eta,x)\,d\eta \tag{3}$$

then periodicity of $\varphi(\theta,x)$ in θ is automatically ensured. Furthermore, φ will satisfy Eq. (2) if $\hat{\varphi}(\eta,x)$ itself satisfies the equation

$$\mathcal{L}(\eta,x)\,\hat{\varphi}(\eta,x) = \lambda\hat{\varphi}(\eta,x) \tag{4}$$

but now in the <u>infinite</u> domain $-\infty < \eta < \infty$. The function $\hat{\varphi}$ <u>need not</u> be periodic and can be calculated in the eikonal form (1). Of course, $\hat{\varphi}(\eta,x)$

is not the actual plasma perturbation, but the real, periodic, perturbation can be constructed from it.

We now apply this technique to the topical problem of high mode number ballooning modes[1,2] in tokamaks. Stability of these modes can be determined by minimising the potential energy functional $\delta W(\xi,\xi)$. The perturbation is decomposed into modes $\sim \exp in\zeta$ and, provided shear is non-vanishing, δW is minimised by displacements which are divergence free; then δW can be expressed in terms of the components of ξ perpendicular to $\underset{\sim}{B}$. For small values of n further minimisation of δW has been done numerically[7,8], but this fails for large n. However in this limit the minimisation can be performed analytically. When $n \gg 1$, δW will be positive unless the perpendicular gradients of the perturbation are of order n but the parallel gradients remain of order unity (i.e. unless the mode varies rapidly perpendicular to $\underset{\sim}{B}$ but slowly parallel to $\underset{\sim}{B}$). The divergence $(\nabla . \underset{\sim}{\xi}_\perp)$ must also be of order unity. A further minimisation can be carried out and δW then depends only on the normal displacement through $X \equiv RB_p \xi_\psi$. Thus, in an expansion in $1/n$, the dominant contribution to δW is

$$\delta W_o = \pi \int J d\chi d\psi \left\{ \frac{B^2}{R^2 B_p^2} \left| k_{\parallel} X \right|^2 + R^2 B_p^2 \left| \frac{1}{n} \frac{\partial}{\partial \psi} k_{\parallel} X \right|^2 - 2 \frac{dp}{d\psi} \left(\frac{\kappa_n}{R B_p} |X|^2 - \frac{i f B_p}{B^2} \kappa_s \frac{X}{n} \frac{\partial X^*}{\partial \psi} \right) \right\} \tag{5}$$

where κ_n and κ_s are the normal and geodesic components of the curvature $\underset{\sim}{\kappa} = - \underset{\sim}{B} \times [\underset{\sim}{B} \times \nabla(p + B^2/2)] B^{-4}$ and

$$ik_{\parallel} \equiv \frac{1}{JB} \left(\frac{\partial}{\partial \chi} + in\nu \right) \quad .$$

Eq. (5) represents the starting point for the investigation of high-n ballooning modes.

The Euler equation, obtained by minimising (5) over all functions $X(\psi,\chi)$ which are periodic in χ, is

$$Bk_{\parallel}\left\{ \frac{B}{R^2B_p^{\,2}}\left[1-\left(\frac{R^2B_p^{\,2}}{nB}\right)^2\frac{\partial^2}{\partial\psi^2}\right]k_{\parallel}X\right\} + 2\,\frac{dp}{d\psi}\left\{\frac{\kappa_n X}{RB_p} - \frac{i\kappa_s fB_p}{B^2 n}\frac{\partial X}{\partial\psi}\right\} = 0 \qquad (6)$$

We now introduce the transformation discussed earlier, namely,

$$X(\psi,\chi) = \sum_m \exp\left(\frac{-2\pi i m\chi}{\oint d\chi}\right)\int_{-\infty}^{+\infty}\exp\left(\frac{2\pi i m y}{\oint d\chi}\right)\hat{X}(\psi,y)\,dy \qquad (7)$$

then $X(\psi,\chi)$ will be periodic in χ and will satisfy the Euler Eq. (6) provided $\hat{X}$ satisfies the same equation in the infinite domain $-\infty<y<\infty$. The solution of Eq. (6) in the infinite domain may be obtained by writing $\hat{X}$ in the form of a (non-periodic) "quasi-mode"[9]

$$\hat{X}(\psi,x) = F(\psi,y)\exp\left(-in\int^{y}\nu\,dy\right) \qquad (8)$$

where the exponential factor contains all the rapid cross field variation and where F satisfies the ordinary differential equation

$$\frac{1}{J}\frac{d}{dy}\left\{\frac{1}{JR^2B_p^{\,2}}\left[1+\left(\frac{R^2B_p^{\,2}}{B}\int^{y}\frac{\partial\nu}{\partial\psi}\,dy\right)^2\right]\frac{dF}{dy}\right\}$$

$$+\frac{2}{RB_p}\frac{dp}{d\psi}\left\{\kappa_n - \frac{fRB_p^{\,2}}{B^2}\kappa_s\int^{y}\frac{\partial\nu}{\partial\psi}\,dy\right\}F = 0 \qquad (9)$$

in which ψ appears only as a parameter. Eq. (9) can readily be solved for any prescribed equilibrium and determines its stability against high-n ballooning modes. (In this lowest order calculation the slow ψ variation of F is not determined; it can be obtained from higher orders of the $1/n$ expansion[6].)

We see, therefore, that the simultaneous application of the transformation (7) and the quasi-mode form (8) decouples the stability analysis from surface to surface and provides a complete minimisation of δW at large n. If the quasi-mode form were introduced directly, as in Ref. (1), one obtains Eq. (9) but an additional constraint, such as $F = 0$ at $\chi = \pm \frac{1}{2} \oint d\chi$, must be introduced to make the solution periodic. One does not then obtain the full minimisation of δW .

As a specific example we have considered a model problem representing a large aspect ratio tokamak with circular flux surfaces. In this model the magnetic field is uniform over the magnetic surface but the shear is non-uniform. Eq. (9) becomes

$$\frac{d}{d\eta}\, [1 + (s\eta - \alpha \sin \eta)^2]\, \frac{dF}{d\eta} + \alpha[\cos \eta + \sin \eta(s\eta - \alpha \sin \eta)]F = 0 \quad (10)$$

where

$$s \equiv \frac{d(\ell n\, q)}{d(\ell n\, r)} \quad \text{and} \quad \alpha \equiv \frac{-\,2Rq^2}{B^2}\frac{dp}{dr}$$

are measures of the mean shear and pressure gradient respectively.

Eq. (10) has been integrated numerically with the boundary condition $F \to 0$ as $|\eta| \to \infty$. The boundary between stability and instability is shown in Fig. 1, which indicates that, for this model, the critical pressure gradient for ballooning modes is rather insensitive to shear. Over most of the range it is roughly $(dp/dr) \sim 0.25\ B_o^2/Rq^2$. Also shown in Fig. 1 (dotted line) is the stability boundary obtained by imposing the boundary condition $F(\pm \pi) = 0$ used by Dobrott et al[1]. This overestimates the stability of the system and produces a higher threshold value of (dp/dr). This overestimate becomes more marked at low shear because, as shown in Fig. 2, the eigenfunction $F(\eta)$ then extends considerably beyond $\pm \pi$.

In conclusion, we have shown that a complete minimisation of δW in the limit $n \to \infty$ can be obtained by the transformation (7) together with the quasi-mode form (8). This reduces the problem to an ordinary differential equation which can readily be solved to determine the stability of any prescribed equilibrium. Incidentally analysis of the asymptotic behaviour of this equation yields a necessary criterion for stability - none other than the well-known Mercier[4] criterion.

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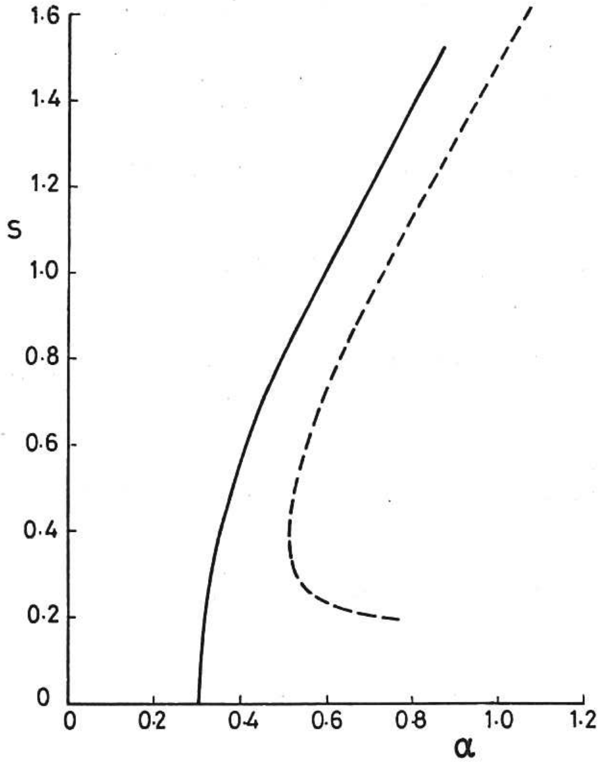

Fig.1 Maximum stable pressure α as a function of shear s.

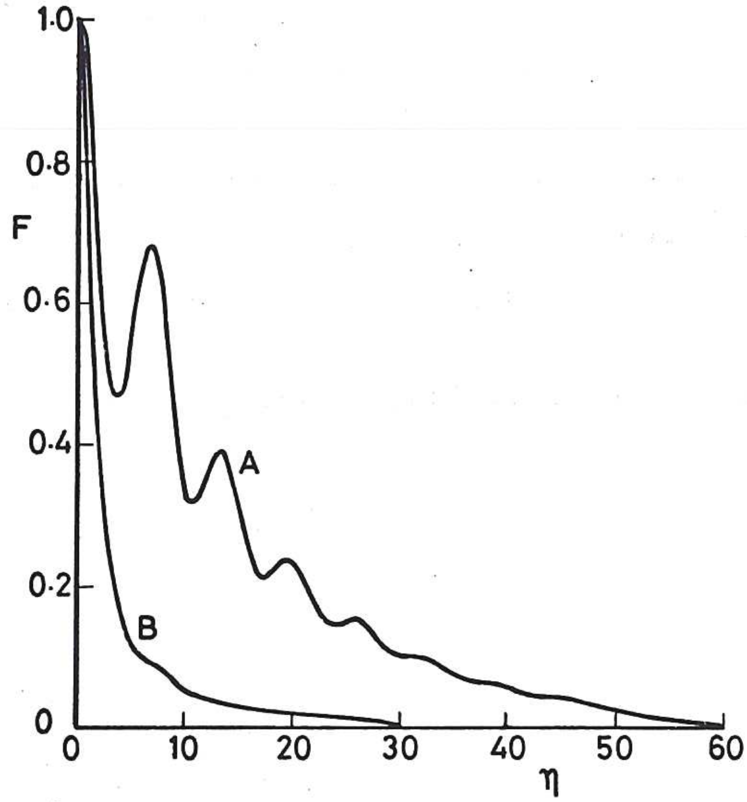

Fig.2 Marginally stable Eigenfunctions. A, low shear s = 0.1; B, high shear s = 0.7.