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GRAVITATIONAL RESISTIVE INSTABILITY OF AN INCOMPRESSIBLE PLASMA IN A SHEARED MAGNETIC FIELD

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ABSTRACT

The gravitational resistive instability analysed by Furth, Killeen and Rosenbluth[1] and subsequent authors[2][3] is examined from a different point of view, which brings out the connection with ordinary Rayleigh-Taylor instability and thermal convection. In contrast to the modes found by these authors, which are either sharply localized in the vertical direction or require a boundary layer, it is shown that coherent motions of arbitrary vertical extent can occur. We are led to these new modes by first considering a simpler but related model in which resistivity is concentrated at the ends of a system of finite length. This shows that such systems may be unstable even if they satisfy the Newcomb[6] criterion. The new resistive modes do not have the usual periodic dependence $\exp(ik_z z)$ along the horizontal direction of the main field, but have finite length. They represent convective rolls[7] which are twisted so that they conform to the field lines. The relation of these new modes to the original periodic localized modes is examined and it is shown that there is a duality relation between them. The possibility of having two entirely different forms of normal mode arises from the near degeneracy of the original model.

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CONTENTS

I. INTRODUCTION

The analysis of resistive instabilities in a fluid supported by a sheared magnetic field, initiated by Furth, Killeen and Rosenbluth[1] (FKR), leads to a type of normal mode in which the influence of resistivity is concentrated in a thin region about the singular magnetic surface Σ at which $\underset{\sim}{k}.\underset{\sim}{B}_0 = 0$, (where $\underset{\sim}{k}$ is the component of the wave-vector normal to the direction of shear). This thin region plays a role similar to that of a boundary layer in hydrodynamics. In this paper we show that by adopting a different viewpoint one is led to consider a new class of unstable modes in which the influence of resistivity is not localized, and which do not have this 'boundary-layer' characteristic.

The model investigated by FKR was a plane slab of incompressible fluid, in which the destabilizing effect of field curvature was represented by a fictitious gravity, and this leads to three types of instability, called the rippling, tearing and gravitational modes. The gravitational or G-mode, which is the only one studied in this paper, was examined in more detail by Johnson, Greene and Coppi[2], and again by Coppi[3], who showed that in a limit which corresponds to $\beta \to 0$ a set of G-modes exists for which the perturbations effectively vanish outside the resistive layer. (In the case of zero resistivity, instabilities concentrated near the singular surface Σ had been found previously by Suydam[4] and Rosenbluth[5]).

Following FKR we will consider an equilibrium situation in which the magnetic field lies in the (y,z) plane, i.e. $\underset{\sim}{B} = (0, s\,x\,B_0, B_0)$, and there is a gravitational field g in the negative x-direction. To simplify the model we choose s, B_0 and the resistivity η to be uniform and set $s\,x \ll 1$. Rippling and tearing modes are therefore excluded from our analysis and the discussion is confined to gravitationally driven modes in a system with weak shear, such as the Stellarator. The low β approximation is also frequently used.

Because the equilibrium is independent of y and z, it has been customary in stability theory to look for normal modes of the form $f(x)\cdot\exp(i(k_y y + k_z z))$. The new approach introduced in this paper is to discard this assumption and to adopt a more general form $f(x,z)\cdot\exp(i\,k_y y)$; that is we do <u>not</u> Fourier analyse in z, the direction of the main field. With this changed viewpoint, modes are found which are neither localized near a particular horizontal surface Σ, nor have a 'boundary layer' character. Looked at in this way the localization of the instabilities found in sheared fields by Suydam[4], FKR[1] and others[2] [3] [5] seems to be a property, not so much of the physical

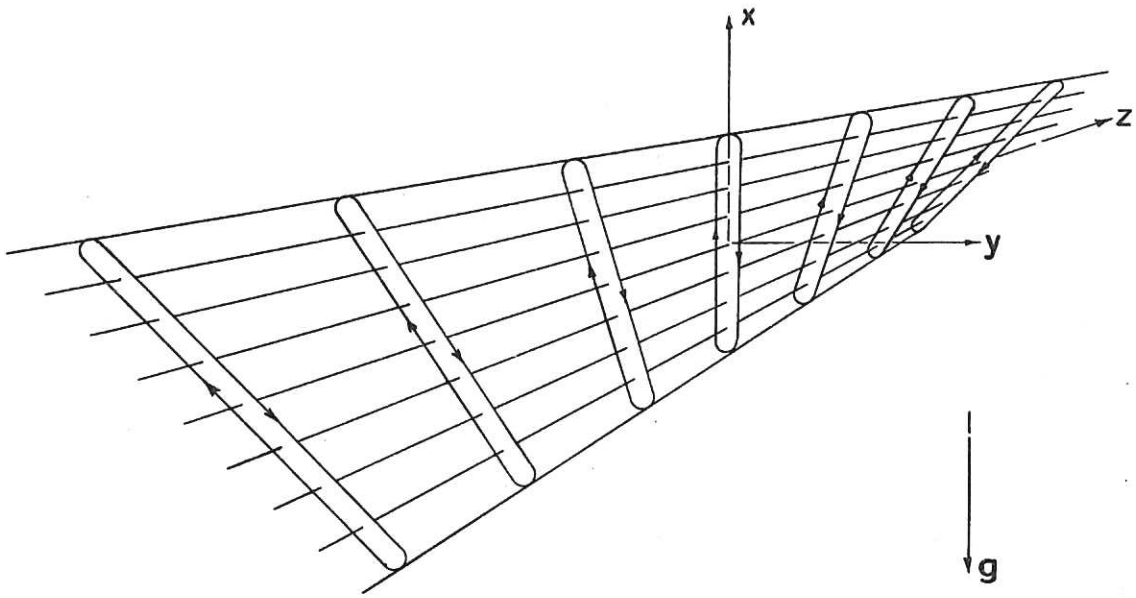
disturbances themselves, as of their Fourier transforms. An important advantage of our approach is that it immediately brings out the connection between resistive instabilities and convective cells in hydrodynamics. The mathematics is also simplified, there being no need to introduce boundary layers or to match internal and external solutions[1].

To justify and illustrate this more general type of mode we will approach the full problem of resistive instability in a sheared magnetic field through a series of simpler but related problems. In Sec. III we first consider the gravitational instability of a perfectly-conducting incompressible fluid in a sheared magnetic field, contained between conducting endplates which, however, are coated with a thin insulating layer so that the field lines are not tied. In the limit $B_0 \to \infty$, exact solutions are found which do not have the form $\exp(ik_z z)$ and which therefore do not fit within the framework of the stability analysis of Suydam[4] and Newcomb[6]. These modes may be unstable even if the system satisfies the Newcomb stability criterion which strictly applies only to an infinite system. These new modes represent twisted interchange motions in which fluid filaments or 'flux tubes' move as rigid bodies. As each filament rises or falls it rotates about a vertical axis to keep aligned with the local magnetic field at each height x and so avoid distortion of the field. These modes, which exist in finite systems but have no analogue in infinite systems, may be important in experiments but it is difficult to decide this as the real boundary conditions are much more complex than in our model. The importance of these modes in the present context is that they are due to resistive layers at the ends, and so provide a prototype for the true resistive instabilities which are examined in Secs. IV-VI.

We begin the discussion of resistive instabilities proper with the gravitational istability of a resistive fluid in a magnetic field without shear but with the field lines tied to conducting end plates (Sec. IV), and then examine what happens to these motions when a weak shear is imposed (Sec. V). When the shear is zero, but the field lines are tied at the ends, resistive gravitational modes occur which take the form of a 'slicing' motion, in which alternate vertical sheets of fluid move up and down. The sheets are parallel to the unperturbed magnetic field and the most dangerous modes have small longitudinal wave-number k_z (so that the field is only slightly distorted), small vertical wave-number k_x, and large transverse wave-number k_y. This corresponds to long thin convective cells ('slices') which extend the full height of the fluid. Such modes might occur in the unstable sectors of '$\int d\ell/B$ stable' devices[9]. They form a special case of convective rolls[7], modes which have been identified by Danielson with the penumbral

filaments observed in sunspots[10].

If now a weak shear is imposed on the magnetic field in this system, one finds very similar motions still to be possible, but with the convective cells twisted so that their surfaces remain everywhere approximately parallel to the field lines. This 'twisted slicing' motion is illustrated in Fig. 1. As a fluid filament rises or falls it must now



CLM - P 52 Fig. 1
Twisted slicing motion

rotate about a vertical axis, (just as in the model problem with resistance confined to layers at each end), in such a way that it always lies along the local direction of $\underset{\sim}{B}$, since this minimizes the field distortion. There is no reason for such a motion to be confined to a thin layer in the vertical direction, or to possess a boundary layer like the modes found by FKR. Once a filament has begun to rise it continues moving until it reaches the upper boundary, just as for a non-sheared field.

Finally we consider what happens as the length of the system is increased indefinitely. In the non-sheared case, tying at the ends becomes ineffective and the most dangerous modes have $k_z \to 0$, i.e. they represent interchange motion of infinitely long filaments, and the growth rate is independent of B and η. When shear is present the filaments cannot become infinitely long, since they are constrained by the field to rotate as they rise or

fall and the rotational kinetic energy would increase without limit if the mode length adjusted itself to the length of the system. We therefore expect the mode to settle down to a finite length. In fact the mode length ℓ automatically adjusts itself to give a balance between the rotational kinetic energy and the dissipative loss due to motion across the field. The growth rate is then the same, apart from a numerical factor of order unity, as that for a system with no shear and finite length $\Lambda = \ell$. It turns out that $\ell \sim \eta^{-1/3}$ so that the growth rate $p \sim \eta^{1/3}$ in agreement with FKR.

By the chain of argument outlined above we are led to discover unstable modes of a resistive fluid in a sheared magnetic field which are of quite a different character to the localized modes found hitherto. It is natural to ask how these 'twisted slicing' modes, with finite length ℓ and arbitrary height, are related to the G-modes of FKR and others, which have finite height h and unlimited length, and this question is examined in Sec. VI. It is found that the two types of mode have the same growth rate and there is a duality relation connecting the height h of the G-mode with the length ℓ of our mode. Each of our twisted slicing modes is in fact a linear superposition of the localized G-modes introduced by Coppi[5], one for every value of x. This superposition is possible because a system with weak shear is almost degenerate, so that localized G-modes centred at different heights have almost identical growth rates. If the degeneracy were exact there would be no unique normal mode, and any combination of degenerate modes would be a mode with identical growth rate. In the present case the growth rate of the G-modes centred at points x varies only as

$$p(x) = p(o)\ (1 + \frac{2}{3}\ (sx)^2) \tag{1.1}$$

where $(sx)^2_{max}$ may be extremely small, e.g. $\sim 10^{-3}$ for a typical Stellarator field. The individual components in any combination of these modes will therefore not increase at precisely the same rate - and for this reason we shall use the term 'quasi-mode' - but it will take many e-folding periods for a significant discrepancy to occur, and by this time the instability should be out of the linear phase.

Quasi-modes may be compared to unstable states in quantum mechanics, e.g. compound or radioactive nuclei, which do not have precise energy levels. The exact energy eigenstates of the problem are scattering or reaction states, but from a physical point of view it would be inconvenient if the concept of unstable levels had therefore not been introduced. In the same way it is not always convenient in stability theory to demand that a mode has a <u>precise</u> growth rate. It is sufficient if its growth rate is defined to within Δp

where $\Delta p \ll p$.

II. THE GRAVITATIONAL MODEL

The plane incompressible fluid model used in this paper is intended to describe pure gravitational or G-modes; we eliminate rippling and tearing modes[1] by assuming the resistivity η and the shear s to be uniform. A uniform gravitational field g is directed downwards, and the unperturbed density distribution is Rayleigh-Taylor unstable, increasing linearly with height x acording to

$$\frac{\partial \rho_o}{\partial x} = \alpha \rho_o \tag{2.1}$$

There is a uniform horizontal magnetic field $(0,0,B_o)$, together with a transverse field $B_y = sxB_o$ (where s = constant), so that the field direction changes with height. (We take $s = 0$ for the un-sheared model studied in Sec.IV). The fluid is contained between perfectly-conducting rigid walls at $x = \pm H$, where the boundary conditions for perturbed variables are $v_x = B_x = E_z = E_y = 0$, and is unbounded in the y-direction. The shear is assumed to be weak, so that $sH \ll 1$. In Secs. III and IV we impose boundary conditions at $z = \pm L$, which will be discussed later; elsewhere the system is assumed to be infinitely long.

The linearised equations are

$$\rho_o \frac{\partial \underset{\sim}{v}}{\partial t} = - \underset{\sim}{\nabla} P + \frac{1}{4\pi} (\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B}_o + \frac{1}{4\pi} (\underset{\sim}{\nabla} \times \underset{\sim}{B}_o) \times \underset{\sim}{B} + \rho \underset{\sim}{g} \tag{2.2}$$

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \underset{\sim}{\nabla} \times (\underset{\sim}{v} \times \underset{\sim}{B}_o) + \frac{\eta}{4\pi} \nabla^2 \underset{\sim}{B} = (\underset{\sim}{B}_o \cdot \underset{\sim}{\nabla}) \underset{\sim}{v} - (\underset{\sim}{v} \cdot \underset{\sim}{\nabla}) \underset{\sim}{B}_o + \frac{\eta}{4\pi} \nabla^2 \underset{\sim}{B}, \tag{2.3}$$

$$\frac{\partial \rho}{\partial t} = - (\underset{\sim}{v} \cdot \underset{\sim}{\nabla}) \rho_o = - v_x \alpha \rho_o , \tag{2.4}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0, \quad \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0 , \tag{2.5}$$

where $\underset{\sim}{B}_o = (0, sxB_o, B_o)$. In equilibrium the weight of the fluid and the force $\frac{1}{4\pi} (\underset{\sim}{\nabla} \times \underset{\sim}{B}_o) \times \underset{\sim}{B}_o$ are balanced by the fluid pressure P_o. We shall treat ρ_o as uniform in the inertial term of (2.2), (Boussinesq approximation) and also in (2.4).

We shall be interested in applying the gravitational model to plasma devices with low β and weak shear, such as the Stellarator. If R_o, r_o are the major and minor radii of the Stellarator; ρ, P the plasma density and pressure; v_s the sound speed;

and $\iota(r)$ the rotational transform with $\iota(r_o) = O(1)$, then we assume a correspondence

$$\mathcal{J} \approx \frac{v_s^2 r_o}{R_o}, \quad \beta = \frac{8\pi P}{B_z^2} \approx \frac{\rho v_s^2}{B_z^2},$$

$$P \approx \frac{B_\theta^2 R_o}{r_o}, \quad \alpha \approx r_o^{-1}, \tag{2.6}$$

$$sx = \frac{B_y}{B_z} \to \frac{B_\theta}{B_z} = \frac{\iota(r)}{2\pi} \frac{r}{R_o}.$$

In Secs. III and IV we deal with the full set of equations (2.2) - (2.5) while the principal approximation which we shall make in Secs. V and VI is to neglect the term $\partial \underset{\sim}{B}/\partial t$ in (2.3), which can be shown to be of order β (for the modes discussed in this paper), compared to the term $\eta\nabla^2\underset{\sim}{B}/4\pi$. Equations (2.2) - (2.5) can then be combined to give an equation for the vertical velocity v_x;

$$p^2\rho_o\eta\nabla^2 v_x + pB_o^2 \left(\frac{\partial}{\partial z} + sx\frac{\partial}{\partial y}\right)^2 v_x - \alpha g\rho_o\eta \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z}\right) v_x = 0, \tag{2.7}$$

where we have assumed a time dependence $\sim \exp(pt)$.

As previously remarked, we shall solve (2.7) without making a Fourier analysis in the z direction. Suppose however that one does assume a dependence

$$\underset{\sim}{v}_x \sim f(x) \exp(pt + ik_y y + ik_z z),$$

and sets

$$\tilde{k}^2 = k_y^2 + k_z^2; \quad k_z + sxk_y = sXk_y,$$

then (2.7) becomes

$$\left\{\frac{\partial^2}{\partial X^2} - \frac{B_o^2 s^2 k_y^2}{p\eta\rho_o} X^2 + \tilde{k}^2 \left(\frac{\alpha g}{p^2} - 1\right)\right\} v_x = 0, \tag{2.8}$$

which is Weber's equation in the vertical coordinate X; and, as shown in Sec. VI, leads to a set of G-modes localized near the point $X = 0$, in terms of which the quasi-modes of Sec. V may be expanded. These localized G-modes are <u>related</u> to the G-modes originally found by FKR - for example the growth rates and characteristic widths are the same - but they are not identical, in fact they are the modes investigated by Coppi[3]. The distinction may be explained as follows. Consider the full set of equations (2.2) - (2.5), without the approximation $\partial\underset{\sim}{B}/\partial t \ll \eta\nabla^2\underset{\sim}{B}$ and assume a dependence $\sim \exp(ik_z z)$. The equations are then of fourth order in $\partial/\partial x$ and symmetric about $x = 0$. Because of this

symmetry each eigenvalue is doubly degenerate, and so to each eigenvalue p_n of the growth rate p there is an eigenfunction S_n in which v is symmetric, and another A_n in which v is antisymmetric. The full set of eigenfunctions may then be grouped into two sequences in order of decreasing growth rate, sequence I containing $(S_0, A_1, S_2, A_3, \ldots)$, while II contains $(A_0, S_1, A_2, S_3, \ldots)$.

As $\beta \to 0$, the eigenfunctions of sequence I become localized[3] within the resistive layer near $X = 0$, and the external part of the solution vanishes; these are the eigenfunctions for which the approximation $\partial \underset{\sim}{B}/\partial t \approx 0$ is valid and which are given by (2.8). On the other hand FKR made the assumption $B_x \approx$ constant within the resistive layer, ($\psi \approx$ constant in their notation); this ruled out all S-modes (for which B_x is antisymmetric) and all localized modes of sequence I, since these have B_x varying rapidly over the layer. The sequence of modes which FKR found accordingly contained only $(A_0, A_2, A_4, \ldots)$, with growth rates corresponding to <u>alternate</u> modes of sequence I.

III. RAYLEIGH-TAYLOR INSTABILITY OF AN IDEAL FLUID IN A SHEARED FIELD

In this section we demonstrate, by means of a simple example, that interchange modes exist in a sheared system of finite length which are not of the form $\exp(ik_z z)$. Although the conductivity is assumed here to be perfect, this type of interchange may be regarded as a prototype of the twisted slicing mode in a resistive fluid (to be discussed in Sec. V), and it shows some analogies with more general types of instability in sheared systems with both finite and zero η.

As a preliminary, consider a system without shear, that is to say one with uniform field, zero resistivity, and perfectly-conducting rigid endplates at $z = \pm L$. However the endplates are imagined to be coated with a thin perfectly-insulating layer, so that the lines of force are <u>not</u> tied and interchanges can occur. These motions are, in fact, resistive instabilities but the resistivity is here concentrated at the endplates instead of being uniformly distributed. The boundary conditions to be applied at $z = \pm L$ are

$$B_z = v_z = (\underset{\sim}{\nabla} \times \underset{\sim}{B})_z = 0. \tag{3.1}$$

We consider normal modes with $k_z = 0$, for which equations (2.2) - (2.5) show that all components of $\underset{\sim}{B}$, together with v_z, are everywhere zero and (3.1) is satisfied identically. The magnetic field has no effect on these motions, and arbitrary two-dimensional

interchanges can occur with

$$v_x\ (x,y,z) = v_x\ (x,y,o),$$

$$v_y\ (x,y,z) = v_y\ (x,y,o), \qquad (3.2)$$

$$v_z = 0, \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

The growth rate of a normal mode is

$$p = (\alpha g)^{\frac{1}{2}} \frac{|k_y|}{(k_x^2 + k_y^2)^{\frac{1}{2}}}$$

It may now be conjectured that the imposition of shear on this system, by introducing an extra transverse component $B_{yo}(x)$, would not prevent instability, since the volume of each flux tube $\int d\ell/B = \int dz/B_o$ remains unaltered. Interchange motions should still occur freely, but the flux tubes must <u>twist</u> during the interchange to follow the local field whose direction changes with height. In the limit $B_o \to \infty$ the motion again becomes two-dimensional, and may be defined by the values of v_x, v_y on the midplane. The gravitational energy released by any interchange depends only on v_x and is unaffected by the shear, but the kinetic energy increases with the length of the system L because of the transverse velocity due to twisting. Therefore we must expect p to be decreased by increased length, or increased shear, but we certainly should not expect shear to stabilise the system. In the remainder of this section we show that these conjectures are correct.

As in Sec. II we assume $B_{yo} = sxB_o$. The unperturbed current $\underset{\sim}{j}_o$ is in the z-direction and is uniform, the force $\underset{\sim}{j}_o \times \underset{\sim}{B}_o$ being balanced by pressure. Some further justification of the boundary conditions is needed, since $\underset{\sim}{j}_o$ has to pass to the conducting endplates across a layer which we have assumed to be an insulator. However it is consistent to assume a very large voltage drop across this layer in the unperturbed state, i.e. its conductivity may be made so small that it can be treated as an insulator so far as the perturbed variables are concerned, while still permitting the equilibrium $\underset{\sim}{j}_o$.

The component $u \equiv v_x$ satisfies a fourth-order equation derived from (2.2) - (2.5) namely;

$$p^2\rho_o\nabla^2 u = \frac{B_o^2}{4\pi}\left(\frac{\partial}{\partial z} + sx\frac{\partial}{\partial y}\right)\nabla^2\left(\frac{\partial}{\partial z} + sx\frac{\partial}{\partial y}\right)u + \alpha g\rho_o\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u \qquad (3.3)$$

This equation can be solved in the limit $B_o \to \infty$ by expanding u in powers of $1/B_o^2$.

That is we write $u = u_o + u_1 + \ldots\ldots$ Then

$$D_o\, u_o = 0, \quad D_o\, u_1 = D_1\, u_o, \ \ldots$$

etc, where

$$D_o \equiv B_o^2 \left(\frac{\partial}{\partial z} + sx\frac{\partial}{\partial y}\right) \nabla^2 \left(\frac{\partial}{\partial z} + sx\frac{\partial}{\partial y}\right), \tag{3.4}$$

and

$$D_1 \equiv \rho_o \left\{ p^2\nabla^2 - \alpha g \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right\}, \tag{3.5}$$

are self adjoint operators. We also introduce a <u>twisted coordinate</u> system adapted to the unperturbed magnetic field,

$$\xi = x, \ \chi = y - sxz, \ \zeta = z,$$

then

$$\frac{\partial}{\partial z} + sx\frac{\partial}{\partial y} = \frac{\partial}{\partial \zeta},$$

and ξ, χ are constant along the lines of force. Any function $f(\xi,\chi)$ which is independent of ζ is then a solution of $D_o\, u_o = 0$.

Now the next equation in the sequence, $D_o\, u_1 = D_1\, u_o$, can posses a solution u_1 only if $D_1\, u_o$ is orthogonal to all solutions of $D_o\, u_o = 0$. This imposes a further constraint on u_o which must therefore satisfy the equations

$$D_o u_o = 0, \qquad \int_{-L}^{L} D_1\, u_o\, d\zeta = 0. \tag{3.6}$$

Solutions of the pair of equations (3.6) exist of the form

$$u_o = u_o\,(\xi)\, \exp(ik\chi), \tag{3.7}$$

if

$$\frac{d^2u_o}{d\xi^2} + (A\xi^2 + B)\, u_o = 0, \tag{3.8}$$

where

$$A = k^2s^2 \left(\frac{\alpha g}{p^2} - 1\right),$$

$$B = k^2 \left(\frac{\alpha g}{p^2} - 1 - \frac{1}{3}\, s^2L^2\right). \tag{3.9}$$

In order to fit the boundary conditions $u_o = 0$ at $x = \pm H$, the quantity $\frac{1}{u_o} \frac{\partial^2 u_o}{\partial \xi^2}$ must be negative in some range, which means that

$$\left(\frac{\alpha g}{p^2} - 1\right) > \frac{\frac{1}{3} s^2 L^2}{1 + s^2 H^2} , \tag{3.10}$$

therefore $A > 0$ and by a <u>real</u> transformation (3.8) can be reduced to a form of Weber's equation,

$$\frac{d^2 u_o}{dw^2} + \left(\tfrac{1}{4} w^2 - a\right) u_o = 0 . \tag{3.11}$$

The solutions of this equation are tabulated[11] but it is unnecessary to solve it in detail; we simply remark that for any finite values of s, k, H, L it is possible to find a real positive value of $p^2/\alpha g$ such that the solution of (3.11) satisfies the boundary conditions. In other words the system is always unstable for $\alpha g > 0$, as was physically obvious from the argument with which we introduced this section. In the limit $B_o \to \infty$, the function u_o represents the complete solution (since $u_1 \to 0$), and the growth rate is independent of the magnitude of the field, depending only on its form. For finite B_o there would be a complicated correction due to bending of the field lines by the moving fluid.

We observe, then, that in a system of finite length with perfect conductivity but in which lines of force are not tied at the ends, there are instabilities even when the shear is sufficient to stabilize the corresponding infinitely-long system, i.e. even when the Newcomb criterion[6] is satisfied. The effect of field shear on these modes is to introduce a constraint which determines the shape of the fluid motion. The growth rate is lowered because this constraint induces rotational kinetic energy, but the stability <u>criterion</u> is unaltered and there is no tendency for the convective motions to be localized vertically. These modes, which exist in finite systems but have no analogue in infinite systems, may be of experimental importance but the boundary conditions are in reality much more complicated than those used in this section. Their importance to this paper is that they can be regarded as due to the presence of resistive layers at the ends, and one can therefore expect similar motions to be possible when the resistivity is spread throughout the system. We specifically look for such motions in resistive fluids in Sec. V, but first in Sec. IV we obtain further guidance by examining the motions possible in a <u>resistive</u> fluid in which the lines of force <u>are</u> tied to endplates.

IV. RESISTIVE INSTABILITY IN UNIFORM FIELD

In the perfect conductivity example discussed in Sec. III the introduction of shear did not alter the fundamental character of the Rayleigh-Taylor interchange modes; it simply twisted them to conform to the field lines and so slowed down their growth rate because of the rotational kinetic energy induced by this 'twisting'. We expect shear to produce a qualitatively similar behaviour when the resistivity is distributed uniformly throughout the system instead of being concentrated into thin layers at the ends. Accordingly we first consider interchange-like motions in a resistive system with zero shear, but with perfectly-conducting endplates at $z = \pm L$. Some care must be taken with the boundary conditions, since the equations are of fourth order in d/dz and will therefore not have sinusoidal solutions in a bounded region in general. We choose $B_z = v_x = v_y = 0$ at $z = \pm L$ but place no restriction on v_z. These conditions are equivalent to tying the tubes of force at the ends, and also preventing any transverse fluid displacement there.

It can be shown that v_z is any case very small at $z = \pm L$, so that the precise choice of boundary conditions should have little effect. Solutions then have the form

$$v_x \sim \text{Cos}\,\frac{m\pi z}{2L}\,\text{Cos}\,\frac{\ell\pi x}{2H}\left\{\begin{matrix}\text{Cos}\,k_y y\\ \text{Sin}\,k_y y\end{matrix}\right\}, \tag{4.1}$$

and correspondingly for other components, but we shall assume a dependence $\exp(pt + ik_x + ik_y y + ik_z z)$ and represent the influence of the ends by the requirement $k_z \geqslant \pi/2L$. In particular the case $k_z = 0$ is now forbidden. The finite height is represented by $k_x \geqslant \pi/2H$, and we set $k^2 = k_x^2 + k_y^2 + k_z^2$, $\tilde{k}^2 = k_y^2 + k_z^2$. Equations (2.2) - (2.5) then yield;

$$p^3 + \frac{\eta k^2 p^2}{4\pi} + \left(\frac{B_o^2 k_z^2}{4\pi\rho_o} - \alpha g\,\frac{\tilde{k}^2}{k^2}\right) p - \frac{\alpha g \eta \tilde{k}^2}{4\pi} = 0. \tag{4.2}$$

It can be proved that no overstable modes exist, so we need only consider instabilities with real p. Also since the fluid is to be stable for $\eta = 0$ we must assume $B_o^2 k_z^2/4\pi\rho_o > \alpha g$, which for a real plasma is a condition on the ratio β of particle to magnetic pressure. We shall suppose $\beta \ll 1$ and therefore drop the second term in the bracket. Finally, the first term of (4.2) must be negligible if $p < c_A k_z$, where $c_A = B_o/(4\pi\rho_o)^{\frac{1}{2}}$ is the Alfvén speed, i.e. if the growth rate is less than the frequency of an Alfvén wave of wavelength $\approx L$. Then the approximate dispersion relation is

$$p^2 + \frac{B_o^2 k_z^2 p}{\eta k^2 \rho_o} - \frac{\alpha g \tilde{k}^2}{k^2} = 0 \tag{4.3}$$

When p is small one root of this equation is

$$p \sim \left\{ \frac{\alpha g \rho_o}{B_o^2 k_z^2} \right\} \eta \tilde{k}^2 \sim \left\{ \frac{\alpha g \rho_o L^2}{B_o^2} \right\} \eta \tilde{k}^2 \quad , \tag{4.4}$$

and to apply this to a real plasma we identify g in terms of pressure P_o and radius of curvature R_c by $g \sim {}^{2P_o}/_{\rho_o R_c}$ so that

$$p \sim \lambda \left\{ \frac{\beta \eta \tilde{k}^2}{4\pi} \right\} , \tag{4.5}$$

where $\lambda = 8\alpha L^2/\pi^2 R_c$ and so is a purely geometric factor of order unity. We can assume $k_x, k_z \ll k_y$, then the growth rate is almost independent of k_x and is proportional to k_y^2. The pattern of this instability is that of a 'slicing' motion in which alternate thin vertical layers, parallel to the magnetic field, are moving up and down. The growth rate p increases as the transverse thickness of the slices becomes smaller, but eventually it is necessary to include the first term of (4.3), and $p \to (\alpha g)^{\frac{1}{2}}$ as $k_y \to \infty$.

The physical reason why fluid can move across the field in this way is the following. Imagine two vertical layers $\simeq k_y^{-1}$ apart, with the fluid moving up in one and down in the other, and suppose that the density perturbation has reached an amplitude $A \cos \pi z/2L$, so that the vertical fluid displacement $\varepsilon \simeq A/\alpha$. At each stage equilibrium would be maintained if the field lines were displaced by a distance

$$\delta\varepsilon \simeq \frac{A g \rho_o}{B_o^2 k_z^2} ,$$

so that the weight of the fluid would be balanced by tension in the field. This generates a transverse field $B_x \sim \sin \pi z/2L$ with opposite sign in the two layers. Within a time $\simeq (\eta k_y^2)^{-1}$ this B_x field disappears by transverse resistive diffusion so that the motion can proceed; the growth rate is therefore

$$\frac{1}{\varepsilon} \frac{d\varepsilon}{dt} \simeq \frac{\alpha g \rho_o}{B_o^2 k_z^2} \eta k_y^2 ,$$

in agreement with (4.4).

It is worth noting that while the fluid motion is vertical, the field diffusion is

horizontal and can be made arbitrarily fast by choosing a large k_y (i.e. thin 'slices'). This mode could be particularly dangerous because there is no relation between the large scale length (k_x^{-1}) of the fluid motion and the small scale length (k_y^{-1}) over which the field diffusion occurs. One can thus retain large-scale eddies in the x direction while allowing $k_y \to \infty$; in this limit the field has no influence on the motion at all. (In practice the rapid short-wavelength slicing motion would be limited by viscosity or some non-fluid effect such as finite Larmor radius).

In this simple model <u>no instabilities have been found in which p depends on fractional powers of η</u>, as in the modes obtained by FKR and others[2] [3]. Nevertheless we assert that G-modes with $p \sim \eta^{\frac{1}{3}}$ are generically the same as the slicing mode discussed in this section, which has $p \sim \eta$. The reason for this, as will be shown in Sec. V, is that in a sheared field the slicing mode becomes twisted and as a result it automatically takes up a length which is proportional to $\eta^{-\frac{1}{3}}$. It can be seen from (4.4) that if we put $L \sim \eta^{-\frac{1}{3}}$ then we do indeed get a growth rate $p \sim \eta^{\frac{1}{3}}$ as found by FKR and others.

V. TWISTED SLICING MODES

In this section we examine the final model, that is a plasma slab of infinite extent in the z direction, with finite resistivity and finite shear. Guided by the simpler situations discussed in the previous sections we are led to look for modes which have the form of 'twisted slicing motions', that is modes which are similar to those of Sec. IV but twisted to follow field lines like those of Sec. III. We first give a simple direct derivation of such modes and later show how they can be obtained from the work of FKR[1] and Coppi[3].

Since the presence of shear will not increase the growth rate, $p \ll \eta k^2$ as in Sec. IV, and the $\partial \underset{\sim}{B}/\partial t$ term can be omitted. We can therefore start from equation (2.7). To investigate modes which are similar to those in Sec. IV but twisted to follow the field lines, it is natural to introduce the twisted coordinate system used in Sec. III

$$\xi = x, \ \chi = y - sxz, \ \zeta = z \ .$$

then

$$\frac{\partial}{\partial z} + sx \frac{\partial}{\partial y} = \frac{\partial}{\partial \zeta} \ , \tag{5.1}$$

and the coordinates ξ, χ are constant along a field line. We transform equation (2.7)

into these new coordinates and look for solutions of the form

$$v_x = v(\zeta)\ \exp(ik_x\xi + ik_y\chi)\ , \tag{5.2}$$

then equation (2.7) gives

$$(1 + \varepsilon^2 q)\cdot \frac{1}{k_y^2}\frac{\partial^2 v}{\partial\zeta^2} - 2\varepsilon^2 qis\xi\cdot \frac{1}{k_y}\frac{\partial v}{\partial\zeta} - \varepsilon^2\left[q\ (1 + s^2\xi^2) + \frac{p^2}{\alpha g}\ s^2\zeta^2\right] v$$

$$- \frac{\varepsilon^2 p^2}{\alpha g}\left(\frac{k_x^2}{k_y^2} - 2s\zeta\frac{k_x}{k_y}\right) v = 0, \tag{5.3}$$

where

$$\varepsilon^2 = \frac{\alpha g\rho_o\eta}{pB_o^2}\ ,\quad q = \frac{p^2}{\alpha g} - 1 \tag{5.4}$$

Now the results of the earlier sections, particularly Sec. IV equation (4.4), lead us to expect a 'twisted slicing' mode for which

$$p \simeq \frac{\alpha g\rho_o\eta}{B_o^2}\cdot k_y^2\ L^2\ , \tag{5.5}$$

where L is the length of the mode in the z, or ζ, direction. For such a mode

$$\varepsilon^2 \simeq \frac{1}{k_y^2 L^2}\ ,$$

and in the limit $k_y L \gg 1$ and $k_x/k_y \ll 1$ equation (5.3) reduces to

$$\frac{d^2 v}{d\zeta^2} - \frac{p\rho_o\eta}{B_o^2}\ (sk_y)^2\zeta^2 v + \frac{p\rho_o\eta k_y^2}{B_o^2}\left(\frac{\alpha g}{p^2} - 1\right) v = 0\ , \tag{5.6}$$

which can again be transformed into Weber's equation

$$\left(\frac{d^2}{dw^2} - \frac{w^2}{4} + a\right) v = 0\ , \tag{5.7}$$

(but note that this is now an equation for the dependence <u>along</u> the field), with

$$\frac{w^2}{4} = \frac{\zeta^2}{2\Delta^2}\ ,\quad \Delta = \left(\frac{B_o}{sk_y}\right)^{1/2}\ \frac{1}{(p\rho_o\eta)^{1/4}}\ , \tag{5.8}$$

and

$$2a = \frac{(p\rho_o\eta)^{1/2}}{sB_o}\ k_y\left(\frac{\alpha g}{p^2} - 1\right) \tag{5.9}$$

In a long system equation (5.7) has eigenvalues and eigenfunctions

$$a = n + \tfrac{1}{2} , \tag{5.10}$$

$$D_n (w) = H_n (w) \exp(- w^2/4) , \tag{5.11}$$

where H_n are the Hermite polynomials

$$H_n (w) = (- 1)^n \exp(w^2/2) \frac{d^n}{dw^n} \exp(- w^2/2). \tag{5.12}$$

The corresponding solutions of (5.6) are

$$v = v_n (\zeta) = \exp(- \zeta^2/2\Delta^2) H_n (\zeta \sqrt{2}/\Delta) , \tag{5.13}$$

and the growth rate of this mode is (for $\alpha g \gg p^2$ as in Sec. III)

$$p_n = \left(\frac{\eta k_y^2}{4\pi}\right)^{1/3} (\alpha g)^{2/3} \left(\frac{4\pi\rho_o}{B_o^2 s^2}\right)^{1/3} (2n + 1)^{-2/3} , \tag{5.14}$$

which is the same as that found by FKR for G-modes. (As $k_y \to \infty$ we also find that $p \to (\alpha g)^{\frac{1}{2}}$ as before). The half width of the disturbance in ζ or z, i.e. the length of the mode along the field, is

$$\Delta_n = \left(\frac{4\pi}{\eta k_y^2}\right)^{1/3} \frac{1}{(\alpha g)^{1/6}} \left(\frac{B_o^2}{4\pi\rho_o}\right)^{1/3} \frac{1}{s^{2/3}} (2n + 1)^{2/3} . \tag{5.15}$$

The vertical wave number k_x has no effect on the growth rate or the length Δ of the mode, provided only that $k_x \ll k_y$. Hence we can replace the x dependence of our modes by any arbitrary dependence $g(x)$ so long as this is slowly varying compared to the width of the slices (k_y^{-1}). We thus obtain modes of the form

$$v_x(x,y,z) = g(x) v_n(z) \exp [ik_y(y - sxz)] , \tag{5.16}$$

where $v_n(z)$ is the appropriate Hermite function (5.13) with growth rate given by (5.14).

These modes represent a 'twisted slicing' motion (Fig. 1) of the expected form. The motion is specified at one plane z = const. by the function $g(x)$ (which is arbitrary provided it is slowly varying and vanishes at $x = \pm H$). The motion in this plane takes the form of 'convective rolls' as in ordinary hydrodynamics, and its form at any other value of z is determined by the fact that the flow pattern is almost constant along any

field line ($y - sxz$ = constant). The rolls thus get twisted as one moves along z but at the same time the flow velocity also decays slowly away in z because of the term $v_n(z)$. For the fastest growing mode, $v_o(z)$ is a simple gaussian curve with characteristic width Δ. We see from (5.15) that $\Delta \sim \eta^{-1/3}$ so that the length (in z) of the twisted slices increases indefinitely as $\eta \to 0$. Higher modes have an oscillatory z-dependence.

The relation of this twisted slicing mode to those found in Sec. IV is now apparent. In Sec. IV the length L of the slices was set by the position of the endplates; in an infinitely long resistive plasma such as we consider in this section the length is set by the resistivity according to (5.15) and if we identify the length L of Sec. IV with the 'natural' length Δ then we do indeed find that the growth rate of the two types of slicing modes are in agreement.

Physically the natural length Δ is set by a compromise between (a) rate of release of gravitational potential energy, (b) rate of resistive dissipation and (c) rate of increase of kinetic energy. A feature of the twisted slicing motion is that in order to reduce (b) the fluid motion must follow the field lines which means that the fluid elements must rotate about a vertical axis as they rise or fall; most of the kinetic energy (c) is in this rotation and it is in order to keep this energy finite that the modes must have finite length in z. They achieve this finite length at the expense of some increase in (b), hence the length increases as $\eta \to 0$.

VI. DECOMPOSITION INTO SPATIALLY PERIODIC NORMAL MODES

We have established in the previous sections the existence of twisted slicing modes in a resistive fluid supported by a sheared magnetic field. These modes are of quite a different character to the G-modes found by FKR[1] and Coppi[3] for the same problem and one naturally asks what is the relation of our twisted slicing modes to the G-modes.

To determine this we first re-examine the G-modes. These are spatially periodic in z (unlike our modes which have a definite length Δ) and are of the form

$$v_x = v_g(x) \exp\,(pt + ik_y y + ik_z z) \ ,$$

where $v_g(x)$ satisfies equation (2.8) i.e.

$$\left\{\frac{\partial^2}{\partial^2} - \frac{B_o^2 s^2 k_y^2}{p\eta\rho_o} X^2 + \tilde{k}^2 \left(\frac{\alpha g}{p^2} - 1\right)\right\} v_g(X) = 0 \ , \qquad (6.1)$$

with

$$\tilde{k}^2 = k_y^2 + k_z^2, \qquad sXk_y = sxk_y + k_z.$$

If we put

$$\frac{w^2}{4} = \frac{X^2}{2\delta^2}, \quad \delta = \frac{(p\eta\rho_o)^{1/4}}{(B_o sk_y)^{1/2}}, \tag{6.2}$$

this can be again reduced to Weber's equation and leads to eigenfunctions

$$v_g(X) = u_n(X) = \exp(-X^2/2\delta^2)\, H_n(X\sqrt{2}/\delta), \tag{6.3}$$

and eigenvalues p_n satisfying

$$\tilde{k}^2 \left(\frac{\alpha g}{p_n^2} - 1\right) = \frac{B_o sk_y}{(p_n \eta\rho_o)^{1/2}} (2n+1). \tag{6.4}$$

These are G-modes which are highly localized around $X = 0$ i.e. around $x = -k_z/sk_y$; they have a half width in the x direction of order δ, and $\delta \to 0$ as $\eta \to 0$. Thus these G-modes shrink in x as the twisted slices grow in z. As we pointed out in Sec. II these particular G-modes form a sub-set of the complete set of modes for the problem, and are related to, but not identical with, the original modes of FKR[1].

Although these G-modes are so completely different in character to our twisted slicing modes, their growth rates p are almost exactly the same; the only difference is that in (6.4) the term $\tilde{k}^2 = k_y^2 + k_z^2$ replaces the terms k_y^2 in (5.14). This means that G-modes which have the same k_y, but are localized at different heights x by reason of having different k_z, will also have different growth rates. However if the shear is small this difference in growth rate is also small; in fact two G-modes of the same k_y, but with their k_z chosen so that they are localized at heights x_o apart, have growth rates which differ only by

$$\frac{\delta p}{p} = \frac{2}{3}\frac{k_z^2}{k_y^2} = \frac{2}{3}(sx_o)^2 \tag{6.5}$$

For a Stellarator $(sx_o)^2$ is to be identified with $(\frac{\iota}{2\pi}\frac{r_o}{R_o})^2$ where ι is the rotational transform and r_o, R_o are the minor and major radii; this is typically of order 10^{-3}. This means that all G-modes with the same k_y but varying x_o have almost identical growth

rates. (It also follows from (6.4) that

$$\frac{p\delta^2}{\eta} = \frac{\alpha g \rho_o}{B_o^2 s^2} (2n + 1)^{-1} \approx \beta(2n + 1)^{-1} , \tag{6.6}$$

which justifies our neglect of the term $\partial \underset{\sim}{B}/\partial t$ if we assume $\nabla^2 \sim \delta^{-2}$).

Now let us consider a combination of periodic G-modes, all having the same value of k_y and $n = 0$, but centred at different heights x_o. This can be achieved by taking a spread of values of k_z and leads to

$$u(x,y,z,t) = \exp(ik_y y) \int f(k_z)dk_z \exp[ik_z z - (x - x_o)^2/2\delta_o^2] \exp(pt), \tag{6.7}$$

where $f(k_z)$ is the weight function which we take to be slowly varying. The centre of each constituent mode x_o is related to k_z by $x_o = -k_z/sk_y$, and p also depends on k_z but is again slowly varying. To evaluate (6.7) we note that the integrand is of the form of a highly localized function $\exp(-x - x_o)^2/2\delta_o^2)$, multiplied by slowly varying functions which can therefore be replaced by their values at $x = x_o$. Then if $f(k_z)dz = - g(x_o)dx_o$,

$$u(x,y,z,t) = \delta\sqrt{2\pi}\, g(x) \exp[ik_y(y - sxz) - (sk_y z\delta_o)^2/2 + p(o) (1 + \tfrac{2}{3}(sx)^2)t], \tag{6.8}$$

which has the form of the twisted slicing mode studied in Sec. V, except for the weak dependence of growth rate on x. The arbitrary function $g(x)$ may be chosen to fit the boundary conditions.

The expression (6.8) does not therefore represent an <u>exact</u> normal mode, because it has no precise time dependence. Nevertheless, because sx is small it will behave like a normal mode for all practical purposes and we shall call it a <u>quasi-mode</u>. With $\Delta p/p \sim 10^{-3}$ as in a Stellarator, the different components would keep in step for 1000 e-folding times, an enormously long period. Probably even $\Delta p/p \sim 10^{-1}$ would allow components to hold together until the disturbance was out of the linear regime. As we have indicated earlier there is a slight indeterminacy of the growth rate of each quasi-mode, but this should not detract from its physical significance, any more than the slight indeterminacy of the energy of a compound nucleus detracts from the usefulness of that concept. The essential point of our argument is that the localized G-modes are <u>almost degenerate</u> so that any combination of them is itself 'almost' a true mode, and for weakly sheared systems such as the Stellarator the distinction between a true mode and a quasi-mode is imperceptible for many hundreds of e-folding periods.

There is a duality relation between the length of quasi-modes and the width of the G-modes, which is expressed by

$$\delta\Delta = \frac{1}{sk_y} \tag{6.9}$$

where δ is the width of a G-mode and Δ is the length of a twisted slicing quasi-mode. It should be noticed that this duality relation does not refer to the shape of the quasi-mode itself in the x-direction; that is, there is no relation between the length Δ of a quasi-mode and its localization $g(x)$, indeed $g(x)$ is assumed to vary by a negligible amount over a distance δ. The duality is between the length of a quasi-mode in z and the height of a G-mode in x.

A quasi-mode may be centered at any arbitrary z-plane $z = z_o$ by using the weight function

$$f(k_z) \exp(-ik_z z_o) ,$$

instead of the function $f(k_z)$. Slicing modes may also be constructed from periodic modes with $n > 0$, by setting

$$v_x(x,y,z,t) = \exp(ik_y y) \int f(k_z) dk_z \exp(ik_z z - X^2/2\delta^2) H_n\left(\frac{X\sqrt{2}}{\delta}\right) \exp(pt), \tag{6.10}$$

which leads to

$$v_x(x,y,z,t) = \delta\sqrt{2\pi}\, i^n g(x) \exp[ik_y (y - sxz) - (sk_y z\delta_n)^2/2 + p_n(o) (1 + \tfrac{2}{3}(sx)^2)t] H_n(sk_y z\delta_n\sqrt{2}) , \tag{6.11}$$

and corresponds to slicing modes of higher n-value (5.13).

VII. CONCLUSIONS

The problem of the resistive instability of a conducting fluid supported by a sheared magnetic field has been examined from a new point of view.

First we considered the simpler problem of a perfectly conducting fluid with resistive layers at its ends and so discovered some modes which are present in a finite length system but have no analogue in one of infinite length, thus lying outside the conventional stability analysis of Newcomb and Suydam. These modes may actually exist, but whether this is so or not, they form the prototype of instabilities in a resistive fluid supported by

sheared field and in Sec. V we did indeed find similar modes in this case. These take the form of twisted slicing motions or twisted convective cells and as such are clearly related to the well known instability occurring in ordinary convection. These modified convective cells have finite length in the direction of the field but may extend arbitrarily in the vertical direction. As fluid rises and falls the fluid filaments or 'flux tubes' rotate about a vertical axis and the finite length is a result of compromise between the rotational kinetic energy so induced and the resistive dissipation. The length therefore increases without limit as $\eta \to 0$.

Such modes as these are of a totally different character to the modes found by FKR and others which are periodic in the field direction and localized in the vertical direction. However, we have also shown that our new modes can be constructed from a sub-set of the totality of spatially periodic modes. Looked at in this way the localization of the resistive modes found by FKR and others is a feature not of any physical motion but only of its Fourier transform. In a real system a convective roll type of motion is far more likely to occur than a motion localized in one direction and periodic in the other.

The possibility of having two such dissimilar types of normal mode is a result of the near degeneracy of a system with weak shear. The relation between the 'twisted slicing' quasi-modes and the periodic modes is analogous to the relation between compound nucleus states and scattering states, or between a wave-packet in a slightly dispersive medium and infinite periodic waves. Although the scattering states or the periodic waves may be mathematically a more exact description than the compound nucleus or the wave-packet, the latter may be more useful in practice. In the case of instabilities such as we are discussing it is especially unrealistic to ask for precise normal modes; any mode which preserves its form for many growth periods is permissible since after many periods the system is in any case out of the linear phase. In this respect the fact that the concept of convective cells retains its usefulness in the non-linear phase of ordinary convective instability may suggest a similar utility of the twisted convective cells in the present problem.

Whatever equations might be used to describe the plasma, this near degeneracy of localized modes will be a feature of any calculation based on a sheared-field model. For suppose that the model were changed slightly, so that the magnetic field was of constant strength but changed its direction uniformly with height at a rate $A(x-x_0)$, then the field would be invariant under a 'helical' transformation in which the axes are translated

a distance d along the x-axis and rotated through an angle Ad about the x-axis. Any mode, centred at x_0 and aligned in a direction θ, is therefore almost equivalent to any other mode centred at x_1 and aligned in a direction $[\theta + A(x_1 - x_0)]$, so that all such centred modes are almost degenerate with all others in this model.

The importance of 'quasi-modes' for plasma loss can be seen from the following. The growth rate of conventional G-modes suggests that they would have a significant effect on plasma containment in devices which reply on shear for stabilization; however the localized nature of these modes makes it difficult to understand exactly how they contribute to plasma loss. For low β the growth rate $p \sim \beta\eta/h^2$, where h is a measure of the mode height, and if we assume that each G-mode corresponds to an eddy of height h and velocity hp, the eddy diffusion coefficient is $\sim \beta\eta$ which is of the same order as the classical diffusion. However the present paper shows that the G-modes may be combined coherently into extended quasi-modes which might considerably enhance their effect.

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