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# THE INFLUENCE OF EQUILIBRIUM FLOWS ON TEARING MODES

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A B S T R A C T

Early investigations of resistive instability assumed that the plasma was at rest. However a recent paper investigated the influence of the natural diffusion velocity ($v \sim \eta/a$) on resistive tearing modes and showed it to be important even in the limit $\eta \to 0$. In the present paper we investigate the effect of a more general velocity, of the same order as the natural resistive diffusion velocity but otherwise arbitrary. It is found that as $\eta \to 0$ the effect on the stability threshold is finite and independent of the velocity except for its sign. Hence the threshold is discontinuous at $v = 0$. There is an additional effect of velocity on modes of finite growth rate which may be stabilising or destabilising according to the sign and magnitude of the velocity. The present calculations of these effects agree well with numerical simulations of tearing modes.

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## I. INTRODUCTION

The influence of resistivity on plasma stability was first systematically investigated by Furth, Killeen and Rosenbluth[1] (FKR) for the plane slab model and later by Coppi, Green and Johnson[2] (CGJ) for low-$\beta$ plasma in a circular cylinder. These authors showed that in the high conductivity limit ($\eta \to 0$) resistivity is important only in a thin layer, outside which the motion follows the ideal ($\eta = 0$) mhd equations with zero growth rate. The solution within the resistive layer determines $\Delta'(p)$, the change in logarithmic derivative across the layer, and matching this to the difference in logarithmic derivative of the ideal solutions in the outer regions ($\Delta'_{ext}$) determines the growth rate $p$ of the tearing modes. The stability criterion obtained in this way is found to be $\Delta'_{ext} \leqslant 0$ .

These and all subsequent calculations assumed a stationary equilibrium, ignoring the equilibrium diffusion velocity $v_o$ and the field diffusion ($\partial \underset{\sim}{B}/\partial t$) on the grounds that these occur on the resistive timescale $\tau_R \sim a^2/\eta$ whereas the tearing modes grow on the much faster timescale $p = \omega\tau_R \sim S^{2/5}$ (where $S = \tau_R/\tau_A$ and $\tau_A$ is the Alfven wave transit time ($\rho^{\frac{1}{2}}a/B$) so that $S \to \infty$ as $\eta \to 0$).

Recently, however, Dobrott, Prager and Taylor[3] (DPT) have pointed out that the neglect of the diffusion velocity is incorrect, since the length scale which determines its importance is the thickness of the resistive layer $\delta$ <u>not</u> a macroscopic dimension such as $a$ . For the tearing modes $\delta/a \sim S^{-2/5}$ so that diffusion across the resistive layer occurs on the <u>same</u> timescale as the growth of the tearing modes and is therefore significant even in the limit $\eta \to 0$ .

The relevant diffusion velocity is that of the fluid relative to the resistive layer, whose location is determined by $\underset{\sim}{k} . \underset{\sim}{B} = 0$ where

$\underset{\sim}{k}$ is the wave-number of the perturbation perpendicular to the density gradient, and which itself moves if $\partial \underset{\sim}{B}/\partial t \neq 0$ . This velocity $v_o$ is defined by

$$\frac{\partial \underset{\sim}{B}}{\partial t} + \nabla \times (\eta \nabla \times \underset{\sim}{B}) - \nabla \times (\underset{\sim}{v}_o \times \underset{\sim}{B}) = 0 \qquad (1)$$

and it was the influence of this "natural" diffusion velocity on tearing modes which was investigated by DPT. For the plane slab model they calculated $\Delta'(p)$ at the threshold of stability $(p = 0)$ and found that in the presence of this "natural" diffusion $\Delta'(0) > 0$ , indicating a stabilising influence of the diffusion velocity. A stabilising influence was also found when $p\delta/v_o$ was large.

In the present paper we extend the calculations of DPT in several directions. The first extension is to "decouple" the diffusion velocity from the equilibrium profile. That is we consider the effect of an arbitrary velocity which, although it is of the same order in $\eta$ as the natural diffusion velocity, does not necessarily satisfy Eq. (1). This extension allows one to investigate the stability of equilibria which do not satisfy the elementary Ohm's law which lies behind Eq. (1) (such as the force free paramagnetic model) and to make comparison with numerical calculations, such as those of Shestakov and Killeen[4] in which the equilibrium velocity is treated as an arbitrary parameter.

The second extension is to calculate $\Delta'(p)$ for values of $p$ in the vicinity of $p = 0$ as well as at the stability threshold itself. This allows one to investigate the effect of diffusion on weakly unstable tearing modes, in addition to the strongly unstable and marginal cases discussed by DPT. Finally, in Section III we extend the calculation to investigate the influence of diffusion velocity on tearing modes in a cylindrical plasma.

## II. BASIC EQUATIONS (DECOUPLED FLOW)

In the standard plane slab configuration equilibrium quantities depend only on $y$, the density is $\rho(y)$ and the magnetic field $\underset{\sim}{B} = \hat{\underset{\sim}{z}}B_{oz} + \hat{\underset{\sim}{x}}B_{ox}(y)$. The flow velocity is in the $y$ direction and, as in FKR and DPT, we assume that the plasma is incompressible and the resistivity uniform (to eliminate ripple modes). For modes of the form $\sim \exp[i(k_x x + k_z z) + \omega t]$ the linear perturbations are given by

$$\omega \underset{\sim}{B}_1 = \nabla \times (\underset{\sim}{v}_1 \times \underset{\sim}{B}_o) + \nabla \times (\underset{\sim}{v}_o \times \underset{\sim}{B}_1) - \eta_o \nabla \times (\nabla \times \underset{\sim}{B}_1) \qquad (2)$$

$$\rho_o \nabla \times (\omega \underset{\sim}{v}_1 + \underset{\sim}{v}_o . \nabla \underset{\sim}{v}_1) = \nabla \times (\underset{\sim}{B}_o . \nabla \underset{\sim}{B}_1) + \nabla \times (\underset{\sim}{B}_1 . \nabla \underset{\sim}{B}_o) \qquad (3)$$

$$\nabla . \underset{\sim}{v}_1 = 0 \quad \nabla . \underset{\sim}{B}_1 = 0 \, . \qquad (4)$$

Again, following previous authors, a pair of equations can be separated from this set which involve only $B_{1y}$ and $v_{1y}$. Then in the dimensionless variables

$$\psi = B_{1y}/B \qquad W = -\, ikv_{1y}\,\tau_R \qquad F = \underset{\sim}{k} \cdot \underset{\sim}{B}/kB$$

$$\alpha = ak \qquad p = \omega\tau_R \qquad C = v_o \tau_R/a$$

$$S = \tau_R/\tau_H \qquad y = a\mu \qquad \tau_R = a^2/\langle\eta\rangle$$

$$\tau_H = \frac{a\,\langle\rho\rangle^{\frac{1}{2}}}{B} \qquad \hat{\rho} = \rho_o/\langle\rho\rangle \qquad \hat{\eta} = \eta_o/\langle\eta\rangle$$

the equations governing the tearing mode in the presence of an equilibrium velocity $v_o$ are

$$\psi + \frac{FW}{p} + \frac{C\psi'}{p} = \frac{1}{p}(\psi'' - \alpha^2\psi) \tag{5}$$

$$\psi'' - \alpha^2\psi - \frac{F''}{F}\psi = \frac{1}{S^2\alpha^2 F}\{C(W''' - \alpha^2 W') + p(W'' - \alpha^2 W)\} \tag{6}$$

In the case discussed by DPT, when the velocity is that of natural diffusion given by Eq. (1), we have $C = F''/F'$ . Because the natural velocity is tied to the equilibrium profile the formulae of DPT do not describe the effect, on a particular profile, of changing the diffusion velocity. Also they cannot be applied to equilibria which are not themselves governed by the standard Ohm's law. We shall therefore consider $v_o$ to be a general velocity, of the same order in $\eta$ as the natural velocity but not tied to the equilibrium profile.

In the high conductivity limit, $S \to \infty$ , Eqs. (5,6) reduce to the zero frequency ideal mhd equations

$$\psi + \frac{FW}{p} = 0 \tag{7}$$

$$\psi'' - \alpha^2\psi - \frac{F''\psi}{F} = 0 \tag{8}$$

except in the neighbourhood of $F = 0$ . As discussed in FKR and DPT, solutions of these equations exhibit a discontinuity in logarithmic derivative

$$\Delta'_{ext} = (\psi_+'/\psi_+ - \psi_-'/\psi_-)$$

at $F = 0$ and the growth rate of resistive modes is determined by matching the solution of the full equations (5,6) within the resistive layer to the value of the discontinuity $\Delta'_{ext}$ .

As $\eta \to 0$ the resistive layer width $\delta \to 0$, the reduced growth rate $p \to \infty$ and $W/\psi \to \infty$. One therefore introduces appropriately scaled variables.[1,3]

$$p \sim \delta^{-1} \qquad \psi \sim 1 \qquad W \sim \delta^{-2} \qquad \alpha \sim 1 \qquad C \sim 1$$

$$F \sim \delta \qquad F' \sim 1 \qquad F'' \sim 1 \qquad S^2 \sim \delta^{-5} \quad \partial/\partial\mu \sim \delta^{-1}$$

when the resistive layer equations become

$$\psi_o'' = 0 \tag{9}$$

$$\psi_1'' = p(\psi_o + FW/p) \tag{10}$$

$$\frac{1}{S^2\alpha^2}(CW''' + pW'') - F^2W = pF\psi_o - F''\psi_o \tag{11}$$

The lowest order solution is $\psi_o = 1$ and the quantity we need, $\Delta'(p)$, is given by

$$\Delta_i' = \lim_{x \to \infty} \int_{-x}^{x} (p + FW)\, d\mu$$

After a further transformation $W \to p^{\frac{3}{4}}\left(\frac{\alpha^2 S^2}{\dot{F}^2}\right)^{\frac{1}{4}} h$, $\mu \to \delta\theta$, with $\delta = \frac{p^{\frac{1}{4}}}{(\alpha^2\dot{F}^2S^2)^{\frac{1}{4}}}$, the problem is reduced to:

$$\lambda_c h''' + h'' - \theta^2 h = \theta - \lambda_F \tag{12}$$

where

$$\lambda_c = \frac{C(\alpha\dot{F}S)^{\frac{1}{2}}}{p^{5/4}}, \quad \lambda_F = \frac{F''(\alpha\dot{F}S)^{\frac{1}{2}}}{\dot{F}\,p^{5/4}} \tag{13}$$

and

$$\Delta' = \frac{p^{5/4}}{(\alpha\dot{F}S)^{\frac{1}{2}}} \int_{-\infty}^{\infty} (1 + \theta h)\, d\theta \tag{14}$$

We now turn to the solution of Eqs. (12) and (14).

Small $\lambda$ limit

When $\lambda_F, \lambda_c \ll 1$ we can expand $h$ in powers of $\lambda_F$ and $\lambda_c$ (which are treated as the same order in accordance with the basic supposition that the equilibriumvelocity, represented by $\lambda_c$, is of the same order as the natural velocity represented by $\lambda_F$). Then if $h = h_o + h_1 + h_2 ..$ one finds

$$\Delta' = \frac{p^{5/4}}{(\alpha F S)^{\frac{1}{2}}} \left[ \int_{-\infty}^{\infty} (1 + \theta h_o)\, d\theta + \int_{-\infty}^{\infty} \theta h_2 d\theta \right] \tag{15}$$

with

$$\begin{aligned} h_o'' - \theta^2 h_o &= \theta \\ h_1'' - \theta^2 h_1 &= -\lambda_F - \lambda_c h_o''' \\ h_2'' - \theta^2 h_2 &= -\lambda_c h_1''' \end{aligned} \tag{16}$$

Clearly $h_2$ is the sum of two parts, proportional to $(\lambda_c \lambda_F)$ and $\lambda_c^2$ respectively, so that

$$\Delta' = A + B_1 \lambda_c \lambda_F + B_2 \lambda_c^2$$

where $A, B_1$ and $B_2$ are constants. These coefficients have been determined both by numerical solution of Eqs. (16) and analytically by expanding $h_o, h_1, h_2$ in Hermite polynomials

$$h_{o,1,2}(\theta) = e^{-\theta^2/2} \sum_{n=0}^{\infty} \alpha_{o,1,2}^{(n)} H_n(\theta)$$

and using standard properties of $H_n(\theta)$ (see e.g. Ref. (5)) to evaluate the coefficients $\alpha_{o,1,2}^{(n)}$ and from them the quantity $\Delta'$. Then one finds

$$\Delta'(p) = \frac{2.12p^{5/4}}{|\alpha\dot{F}S|^{\frac{1}{2}}} + \frac{|\alpha\dot{F}S|^{\frac{1}{2}}}{p^{5/4}}(0.78CF''/F' + 0.49\,C^2) \quad . \tag{17}$$

The first term is that given by FKR, while if the diffusion velocity has the natural value $C = F''/F'$ the sum of all three terms reduces to the result obtained by DPT. We see that the natural velocity always exerts a stabilising effect, as does any velocity in the same direction. For velocities in the reverse direction the effect is at first destabilising but may become stabilising again when the velocity is sufficiently large.

Large $\lambda$

Following DPT, the solution of Eqs. (12,14) for $\lambda \to \infty$ determines a stability threshold ($p \to 0$) for tearing modes. However in addition to extending the calculation of DPT to the case of uncoupled flow we shall also extend it to obtain $\Delta'$ for small $p$, so that the behaviour of weakly unstable modes as well as the marginal mode can be discussed.

Returning to Eq. (12) we write $\theta \to \lambda_c^{1/5}x$, $h \to \lambda_F\lambda_c^{-2/5}g$, (for the present we consider $\lambda_c$ positive so that $x$ is a real variable) then in the new variables

$$g''' - x^2g + 1 = \frac{1}{\lambda_c^{4/5}}\left\{\frac{\lambda_c}{\lambda_F}x - g''\right\} \quad . \tag{18}$$

For large $\lambda_F,\lambda_c$ the appropriate expansion is in powers of $\lambda_c^{-4/5}$, then

$$g_o''' - x^2g_o = -1$$

$$g_1''' - x^2g_1 = \frac{1}{\lambda_c^{4/5}}\left\{\frac{\lambda_c}{\lambda_F}x - g_o''\right\}$$

and

$$\Delta' = \frac{F''}{F'}\int_{-\infty}^{\infty}\left(x g_o + x g_1 + \frac{\lambda_c^{1/5}}{\lambda_F}\right) dx \quad . \tag{19}$$

Again following DPT, Eqs. (19) may be solved by Fourier transform

$$g_{o,1} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_{o,1}(k)\ e^{-ikx}\ dk$$

where

$$G_o'' + ik^3 G_o = -\ 2\pi\delta(k)$$

$$G_1'' + ik^3 G_1 = \frac{1}{\lambda_c^{4/5}}\left\{k^2 G_o - 2\pi i\frac{\lambda_c}{\lambda_F}\delta'(k)\right\}$$

and $G_{o,1} \to 0$ as $|k| \to \infty$ . The function $G_o$ is continuous but its derivative is discontinuous at $k = 0$ and the derivative of $G_1$ is ill defined. However if we introduce $Y = G_1 - i\frac{\lambda_c^{1/5}}{\lambda_F} G_o'$ the problem is reduced to

$$G_o'' + ik^3 G_o = -\ 2\pi\delta(k)\ , \quad |G_o| \to 0 \quad |k| \to \infty \tag{20}$$

and

$$Y'' + ik^3 Y = \frac{1}{\lambda_c^{4/5}}\ k^2 G_o\left\{1 - 3\frac{\lambda_c}{\lambda_F}\right\}, \quad |Y| \to 0 \quad |k| \to \infty \tag{21}$$

where the derivative of $Y$ is continuous at $k = 0$ . Then the required quantity $\Delta'$ is given by

$$\Delta' = -\frac{i}{2}\frac{F''}{\dot{F}}\left\{\left(\frac{dG_o}{dk}\right)_{o_-} + \left(\frac{dG_o}{dk}\right)_{o_+}\right\} - \frac{iF''}{\dot{F}}\left(\frac{dY}{dk}\right)_{k=0} \ . \tag{22}$$

Eqn. (20) may be solved in terms of Hankel functions

$$G_o = Ak^{\frac{1}{2}} H^{(1)}_{1/5} \left( \frac{2}{5} e^{i\pi/4} k^{5/2} \right)$$

and the Green's function for Eqn. (20) can also be constructed from Hankel functions so that both $G_o$ and $Y$ can be evaluated analytically. The detailed calculations are given in Appendix A and the final expression for $\Delta'$ is

$$\Delta' = \pi \tan(\pi/10) \frac{F''}{\dot{F}} \left[ 1 - \frac{4\pi p}{3|\alpha\dot{F}S|^{2/5}} \frac{5^{3/5}}{C^{4/5}} \left( 1 - \frac{3CF'}{F''} \right) \frac{\cot \pi/5}{(\Gamma(1/5)^2)} \right] . \qquad (23)$$

This expression has been derived only for $\lambda_c > 0$ . However the symmetry of the original Eqns. (12,14) ensures that $\Delta'(\lambda_F, -\lambda_c, p) = \Delta'(-\lambda_F, \lambda_c, p)$ so that the appropriate expression for negative $\lambda_c$ can easily be obtained. Taking this symmetry into account and inserting numerical values for the $\Gamma$ functions the final expression can be written;

$$\Delta' = 1.021\ \sigma \left| \frac{F''}{\dot{F}} \right| - \frac{0.735p}{(\alpha\dot{F}S)^{2/5} |C|^{4/5}} \left( \sigma \left| \frac{F''}{\dot{F}} \right| - 3|C| \right) \qquad (24)$$

where $\sigma = \text{sign}\ (CF''/\dot{F})$.

The first term in this expression is the shift in the stability threshold due to diffusion, found by DPT. It can now be seen that this shift occurs for <u>any</u> non-zero diffusion velocity, not just the natural velocity, and indeed is independent of the velocity except for its sign.

It will also be seen from Eq. (24) that in addition to the shift in the stability threshold, which depends only on the sign of $v_o$ , the velocity has an additional influence on modes with small but finite growth. This small additional effect depends on the magnitude and direction of the diffusion velocity and may be stabilising or destabilising. For the "natural" velocity it is always an additional stabilising term.

## III. CYLINDRICAL PLASMAS WITH DIFFUSION

So far we have considered only the plane slab model. In this section we derive the resistive layer equations for tearing modes in a cylindrical plasma with equilibrium diffusive flow. The notation and the ordering follows closely that used by CGJ.

We assume a stationary equilibrium magnetic field

$$\underset{\sim}{B} = B_\theta(r)\ \hat{\underset{\sim}{\theta}} + B_z(r)\ \hat{\underset{\sim}{z}}$$

in the presence of a radial flow $v_o$. The basic equations governing the evolution of the system are

$$\nabla \times (\eta \nabla \times \underset{\sim}{B} - \underset{\sim}{v} \times \underset{\sim}{B}) = - \frac{\partial \underset{\sim}{B}}{\partial t} \tag{25}$$

$$\frac{d}{dt}\ (p\rho^{-\gamma}) = 0 \tag{26}$$

$$- \nabla p + (\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B} = \rho\ \frac{d\underset{\sim}{v}}{dt} \tag{27}$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla . \underset{\sim}{v} + \underset{\sim}{v} . \underset{\sim}{\nabla} \rho = 0 \tag{28}$$

where $v_o$ is calculated from $\nabla \times (\underset{\sim}{v}_o \times \underline{B}) = - \eta \nabla^2 \underset{\sim}{B}$ . Linearising Eqs. (25-28) and applying perturbations of the form

$$f_1(\underset{\sim}{r}) = f_1(r)\ \exp\ (im\theta - ikz + qt) \quad .$$

We obtain the equations for small displacements $\underset{\sim}{\xi}$

$$\rho_o q (\underset{\sim}{v}_o \cdot\nabla + q)\underset{\sim}{\xi} + \rho_o q\, \underset{\sim}{\xi}\cdot\nabla \underset{\sim}{v}_o = (\nabla\times \underset{\sim}{b})\times \underset{\sim}{B} + (\nabla\times\underset{\sim}{B})\times\underset{\sim}{b} - \nabla p_1 \quad (29)$$

$$q\underset{\sim}{\xi}\cdot\nabla(p_o\rho_o^{-\gamma}) + \underset{\sim}{v}_o\cdot\nabla\,(p_1\rho_o^{-\gamma} - \gamma\rho_1 p_o\rho_o^{-(1+\gamma)}) = -q(p_1\rho_o^{-\gamma} - \gamma\rho_1 p_o \rho_o^{-(1+\gamma)}) \quad (30)$$

$$q\rho_1 + q\rho_o\nabla\cdot\underset{\sim}{\xi} + q\underset{\sim}{\xi}\cdot\nabla\rho_o = -\underset{\sim}{v}_o\cdot\nabla\rho_1 - \rho_1\nabla\cdot\underset{\sim}{v}_o \quad (31)$$

$$\underset{\sim}{b} - \frac{\eta}{q}\nabla^2\underset{\sim}{b} - \nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}) = \frac{1}{q}\nabla\times(\underset{\sim}{v}_o\times\underset{\sim}{b}) \quad (32)$$

where uniform resistivity has been assumed.

We proceed to the limit $\eta \to 0$ by applying the ordering procedure used by CGJ (extended to include the scaling of $v_o$)

$$\eta \equiv \epsilon^5\eta^* \qquad \beta \equiv \epsilon^2\beta^* \qquad m = m^* \qquad n = n^*$$

$$q \equiv \epsilon^3 q^* \qquad r - a \equiv \epsilon^2 x^* \qquad v_o \equiv \epsilon^5 v_o^* \quad .$$

The scaling on $v_o$, which is fixed if the velocity $v_o$ has the same order in $\eta$ as the natural diffusion velocity, implies that flow terms must be present in the resistive layer equations as they were for the plane slab. Consequently resistive diffusion will affect the stability of tearing modes in cylindrical plasmas, even in the limit $\eta \to 0$, just as it did in the plane slab.

Resolving perturbations $\underset{\sim}{\xi}$ and $\underset{\sim}{b}$ along $\hat{\underset{\sim}{r}}$, $\underset{\sim}{B}$, $\hat{\underset{\sim}{r}}\times\underset{\sim}{B}$, we then rescale the leading order terms in Eqs. (29-32) precisely as done by CGJ. (See Appendix B.) The resistive layer equations are then

$$\frac{d\psi_o}{dX} = 0 \quad (33)$$

$$\frac{d^2\psi_2}{dX^2} = Q\,\psi_o + XW \quad (34)$$

$$C\frac{d^3W}{dX^3} + Q\frac{d^2W}{dX^2} = -\Gamma + X^2W + QX\psi_o - J_p\psi_o \tag{35}$$

$$\frac{d^2\Gamma}{dX^2} = W\left(S - \frac{2D}{\gamma\beta}\right) + \frac{2\ell^6Q}{\gamma\beta}\left(\Gamma + \frac{C}{Q}\frac{d\Gamma}{dX}\right) - DXQ\Lambda - \frac{2\ell^6QJ\psi_o}{\gamma\beta} \tag{36}$$

$$Q\Lambda + C\frac{d\Lambda}{dX} = -\frac{X\Gamma}{DQ} - \frac{\psi_o}{Q\ell^6} + \frac{X\psi_o J}{DQ} \quad . \tag{37}$$

In these equations $\psi$ is proportional to the radial magnetic field perturbation $b_r$, $\Gamma$ to the field perturbation along $\underset{\sim}{B}$, $W$ to the radial displacement $\xi_r$ and $\Lambda$ to the displacement along $\underset{\sim}{B}$. Other definitions are contained in Appendix (B).

We note here the quantities

$$J_p = \frac{2\pi}{kB_z^2 i'(a)}\frac{d}{dr}(\underset{\sim}{J}\cdot\underset{\sim}{B}) \quad , \quad J = \frac{4\pi B_\theta^2}{kaB_z^2 i'(a)}\frac{\underset{\sim}{J}\cdot\underset{\sim}{B}}{B^2}$$

which describe the parallel current in the resistive layer. In the plane slab limit only $J_p$ persists and it then reduces to $F''/F'$. Eqs. (33-37) reduce to those of CGJ when $C = 0$ and to those of DPT in the plane slab limit. The main cylindrical modification to the plane slab problem is the presence of the term $-\Gamma$ in Eq. (35).

The problem now involves solving Eqs. (33-37) for $W, \Lambda, \Gamma, \psi_2$ so that the solutions match smoothly on to those obtained in the outer hydromagnetic region. The matching condition again leads to an eigenvalue equation for $Q$

$$\Delta'(Q) = \Delta'_{ext}$$

$$\Delta'(Q) = \lim_{x\to\infty}\int_{-x}^{x}(Q\psi_o + XW)\,dX \quad .$$

Here, we shall analyse the layer equations only for equilibria with vanishing $\beta$ but finite shear $F'$ (such as force-free equilibria). Using the normalisation $\psi_o = 1$ Eq. (36) is then just

$$\Gamma + \frac{C}{Q}\frac{d\Gamma}{dX} = J \tag{38}$$

so that Eqs. (34,35,38) form a closed system for $W, \psi_2$ and $\Gamma$ only. By making the following transformations

$$W \to Q^{3/4}H \qquad \Gamma \to Q^{5/4}G \qquad C \to Q^{5/4}\lambda$$

$$X \to Q^{1/4}\theta \qquad J_p \to Q^{5/4}J_p^* \qquad J \to Q^{5/4}J^*$$

we can reduce Eqs. (34,35,38) to

$$\lambda H''' + H'' - \theta^2 H = \theta - G - J_p^* \tag{39}$$

$$\lambda G' + G = J^* \tag{40}$$

$$\Delta' = \frac{C}{\lambda}\int_{-\infty}^{\infty} (1 + \theta H)\, d\theta \tag{41}$$

which closely resemble Eqs. (12,14) for the plane slab problem. The stability threshold for tearing modes can again be determined from the large $\lambda$ limit of Eqs. (39-41). After a further transformation $\theta \to \lambda^{1/5}z$ , $H \to \lambda^{3/5} F$ , the problem becomes

$$F''' - z^2F = -\frac{1}{C}(J + J_p)$$

$$\Delta' = C\int_{-\infty}^{\infty} zF(z)\, dz + O(\lambda^{-4/5}) \tag{42}$$

These can again be solved by Fourier transform, with the result that the critical $\Delta'$ at the stability boundary is

$$\Delta'_{c,cyl} = \sigma|J + J_p|\,\pi \tan(\pi/10) \quad . \tag{43}$$

As expected, this is similar to the result obtained for the plane slab with the quantity $F''/F'$ replaced by $(J + J_p)$.

CONCLUSIONS

As noted by DPT, a velocity of order $\eta/a$ influences the stability of resistive tearing modes even in the limit $\eta \to 0$. They calculated the effect of the natural diffusion velocity. In the present paper we have extended these calculations in several directions. First, we have considered velocities which, although of the same order as the natural resistive diffusion velocity, are otherwise arbitrary. Second, we have considered weakly unstable modes as well as the strongly unstable and marginal stability situations discussed by DPT.

In the strongly unstable case, $p \gg C^{4/5}(\alpha\dot{F}S)^{2/5}$ the effect of velocity on tearing modes is represented by

$$\Delta'(p) = \frac{2.12p^{5/4}}{|\alpha\dot{F}S|^{\frac{1}{2}}} + \frac{|\alpha\dot{F}S|^{\frac{1}{2}}}{p^{5/4}}(0.78CF''/F' + 0.49\,C^2) \qquad (17a)$$

where $C$ is a measure of the velocity. The first term is the classic result of FKR and the other terms constitute the extension to arbitrary velocity of the calculation of DPT. One now sees that <u>any</u> velocity in the direction of the natural diffusion velocity exerts a stabilising influence but a velocity in the reverse direction may be stabilising or destabilising according to its magnitude. This expression has recently been compared by Killeen and Shestakov[4] with their extensive numerical simulations of resistive instabilities in a plane slab. In the appropriate limit $S \to \infty$ they find good agreement with Eq. (17a).

In the weakly unstable case, when $p \ll C^{4/5}(\alpha\dot{F}S)^{2/5}$, the effect of an equilibrium velocity on resistive tearing modes has also been calculated and is given by

$$\Delta' = 1.021\ \sigma \left|\frac{\overset{\shortparallel}{F}}{\dot{F}}\right| - \frac{0.735p}{(\alpha\dot{F}S)^{2/5}|C|^{4/5}} \left( \sigma\left|\frac{\overset{\shortparallel}{F}}{\dot{F}}\right| - 3|C| \right) \qquad (24a)$$

which also determines a stability threshold ($p = 0$). This formula extends that of DPT in two respects. It applies to any velocity of the same order as the natural velocity and it applies to modes of small growth rate as well as at the stability threshold. This result, Eq. (24a), is remarkable in that it shows that there is a shift in the stability threshold for any non-zero velocity and that this shift is independent of the velocity except for its sign. This means that $v_o = 0$ is a singular point of $\Delta'$ and that the stability threshold is discontinuous in the flow velocity, jumping from $\Delta'$ = zero when $v_o = 0$ to $\Delta' = \Delta'_{DPT}$ when $v_o \neq 0$ . This peculiar behaviour is less surprising when one recalls the basic equation

$$\lambda_c h''' + h'' - \theta^2 h = \theta - \lambda_F \quad .$$

For if we put $\lambda_c = 0$ not only does this reduce the order of the basic equation but it increases the symmetry of the problem so that although $\lambda_F$ may be non-zero it makes no contribution to $\Delta$!

Eq. (24a) shows that, in addition to the shift in the instability threshold when $v_o \neq 0$ , there is a further influence of velocity on modes of small but finite growth rate. This additional effect depends on the sign and magnitude of the velocity but for the natural diffusion velocity it is again stabilising.

The final extension we have made to the DPT theory is to cylindrical plasmas, where the first point to note is that the influence of velocities of order $\eta$ on tearing modes again persists in the limit $\eta \to 0$ . In the cylindrical case it is not possible, in general, to isolate a single pair of equations and the problem is governed by the full set of Eqs. (33-37). However in the limit $\beta \to 0$ this set does reduce to equations similar to those for the plane slab and $\Delta'$ can be evaluated. There is again a shift in the stability threshold which is independent of $v_o$ except for its sign and given by

$$\Delta'_{c,cyl} = \sigma|J_p + J|\pi \tan(\pi/10) \quad . \tag{43a}$$

This involves two contributions to the parallel current, the first of these is analogous to the corresponding term $F''/F'$ in the plane slab but the second has no analogue in the plane slab model.

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APPENDIX A

## Solution of Equations for $\Delta'$ in large $\lambda$ limit

Consider functions $Y(k)$, $G(k)$ satisfying

$$G'' + ik^3G = -2\pi\delta(k) \quad , \quad |G| \to 0 \quad \text{as} \quad |k| \to \infty \tag{A1}$$

$$Y'' + ik^3Y = \beta k^2G \quad , \quad |Y| \to 0 \quad \text{as} \quad |k| \to \infty \tag{A2}$$

Denote by $K(k, k')$ the Green's functions for the operators on the left side of (A1) and (A2) corresponding to decaying solutions as $|k| \to \infty$ so that

$$\frac{\partial^2 K}{\partial k^2}(k, k') + ik^3K(k, k') = \delta(k - k') \quad .$$

Then the function $Y$ can be expressed as

$$Y(k) = \beta \int_{-\infty}^{\infty} K(k, k')k'^2G(k')dk' \quad .$$

Using the symmetry properties

$$K(k, k') = K^*(-k, -k') \qquad G(k) = G^*(-k)$$

this can be written

$$Y(k) = \beta \int_0^{\infty} (K(k, k')k'^2G(k') + K^*(-k, k')k'^2G^*(k'))dk' \quad .$$

Hence, the quantity $(dY/dk)_o$ which is required for the determination of $\Delta'$ can be written

$$\left(\frac{dY}{dk}\right)_{k=0} = 2i\beta \,\mathrm{Im}\left(\int_0^{\infty} \frac{\partial K}{\partial k}(0, k')k'^2G(k')dk'\right) \quad . \tag{A3}$$

The Green's function $K(k, k')$ can be constructed from the linearly independent solutions $H^+(k)$, $H^-(k)$ of the homogeneous part of (A1) which decay at $+\infty$ and $-\infty$ respectively. These are

$$H^+(k) = k^{\frac{1}{2}} H^{(1)}_{1/5}\left(\frac{2}{5} e^{i\pi/4} k^{5/2}\right)$$

$$H^-(k) = k^{\frac{1}{2}} H^{(2)}_{1/5}\left(\frac{2}{5} e^{-i\pi/4} k^{5/2}\right)$$

where $H^{(1)}$ and $H^{(2)}$ are the Hankel functions.

Then

$$K(k, k') = \begin{cases} \dfrac{H^+(k)H^-(k')}{W} & k > k' \\ \\ \dfrac{H^-(k)H^+(k')}{W} & k < k' \end{cases}$$

and $W$ is the Wronskian

$$W = \dot{H}^+(k)H^-(k) - \dot{H}^-(k)H^+(k) = -\frac{5}{\pi} \operatorname{cosec}(\pi/10) \quad . \qquad \text{(A4)}$$

Then equation (A3) becomes

$$\left(\frac{dY}{dk}\right)_{k=0} = 2i\beta \ \mathrm{Im}\left(\frac{\dot{H}^-(0)}{W} \int_0^\infty k^2 G(k) H^+(k) dk\right) \quad . \qquad \text{(A5)}$$

Similarly, the solution of equation (A1) can be written, for $k > 0$

$$G(k) = Ak^{\frac{1}{2}} H^{(1)}_{1/5}\left(\frac{2}{5} e^{i\pi/4} k^{5/2}\right)$$

where

$$A = 2i\pi \sin\left(\frac{\pi}{10}\right) \Gamma(6/5) e^{i\pi/20} (1/5)^{-1/5} \quad . \qquad \text{(A6)}$$

The quantity we required $\left(\frac{dY}{dK}\right)_{k=0}$ can therefore be expressed in terms of the integral

$$I = \int_0^\infty k^3 H^{(1)}_{1/5}\left(\frac{2}{5}\, e^{i\pi/4} k^{5/2}\right) H^{(1)}_{1/5}\left(\frac{2}{5}\, e^{i\pi/4} k^{5/2}\right) dk$$

which is performed along the real axis. The function $H^{(1)}(z)$ vanishes as $|z| \to \infty$ everywhere in the upper half plane. We can therefore substitute

$$\theta = \frac{2}{5} e^{i\pi/4}\, k^{5/2}$$

and reduce $I$ to the form

$$I = \left(\frac{5}{2}\right)^{3/5} e^{-2i\pi/5} \int_0^\infty \theta^{3/5}\, H^{(1)}_{1/5}(\theta) H^{(1)}_{1/5}(\theta)\, d\theta \quad .$$

This can be evaluated as follows. Consider for a moment the integral

$$M(\lambda) = \int_0^\infty \theta^{-\lambda} H^{(1)}_{1/5}(\theta) H^{(1)}_{1/5}(\theta)\, d\theta \quad .$$

Then recalling that the Hankel functions can be written

$$H^{(1)}_{1/5}(z) = i\,\mathrm{cosec}(\pi/5)\{e^{-i\pi/5} J_{1/5}(z) - J_{-1/5}(z)\} \quad .$$

$M(\lambda)$ is proportional to the integral

$$\int_0^\infty z^{-\lambda}\left(e^{-2i\pi/5} J_{1/5}(z) J_{1/5}(z) - 2e^{-i\pi/5} J_{-1/5}(z) J_{1/5}(z) + J_{-1/5}(z)\, J_{-1/5}(z)\right) dz \quad \text{(A7)}$$

which can be evaluated from the formula[6]

$$\int_0^\infty t^{-\lambda} J_\mu(t) J_\nu(t)\, dt = \frac{\Gamma(\lambda)\Gamma\left(\frac{\nu + \mu - \lambda + 1}{2}\right)}{2^\lambda \Gamma\left(\frac{\mu + \nu + \lambda + 1}{2}\right) \Gamma\left(\frac{\nu - \mu + \lambda + 1}{2}\right) \Gamma\left(\frac{-\nu + \mu + \lambda + 1}{2}\right)} .$$

(A8)

Unfortunately, this formula is valid only in $\mathrm{Re}\,\lambda > 0$ because the integral (A8) converges only in this domain, and is singular when $\lambda = 0$. However Eq. (A7) involves the sum of three such integrals and this sum converges for $\mathrm{Re}\,\lambda > -1$. Hence we may evaluate $M(\lambda)$ for $\mathrm{Re}\,\lambda > 0$ using (A7) and then analytically continue the result (which will be well-behaved at $\lambda = 0$) to $\lambda = -3/5$. In this way, one finds

$$I = -e^{-4i\pi/5}\, 5^{3/5} \operatorname{cosec}^2(\pi/5) \frac{\Gamma(-3/5)}{\Gamma(2/5)[\Gamma(1/5)]^2} . \quad \text{(A9)}$$

Collecting together (A4), (A5), (A6) and (A9) gives

$$\left(\frac{dY}{dk}\right)_{k=0} = -5^{3/5} \cdot \frac{4i\beta}{3} \pi^2 \frac{\tan \pi/10}{\tan \pi/5} \frac{1}{[\Gamma(1/5)]^2} .$$

Combining this with the other contribution to $\Delta'$ (calculated by DPT) gives the result quoted as Eq. (23) in the text.

APPENDIX B

The Equations for a Cylindrical Plasma

In the usual notation, we obtain for cylindrical geometry

$$\underset{\sim}{B} \cdot \nabla = \frac{imkB_z i'(a)[r - a]}{2\pi} \quad , \quad i(a) = \frac{2\pi}{m} \quad , \quad \beta = \frac{2p(a)}{B^2} \quad , \quad B_z \equiv B_z(a)$$

$$F' = \frac{kaB_z i'(a)}{2\pi B} \quad , \quad D = \frac{-8\pi^2 p'(a)}{am^2 B_z^2 [i'(a)]^2} \quad , \quad S = \left(\frac{ki^2(a)}{\pi i'(a)}\right)^2 .$$

Expanding $\underset{\sim}{\xi}$ and $\underset{\sim}{b}$ in powers of $\epsilon$ (effectively $\eta^{1/5}$)

$$\xi = \sum_{n=0}^{\infty} \epsilon^n \underset{\sim}{\xi} \quad , \quad \underline{b} = \sum_{n=0}^{\infty} \epsilon^n \underset{\sim}{b}^{(n)}$$

$$\underset{\sim}{\xi} = \hat{\underset{\sim}{r}}(\xi_r^{(0)} + \epsilon\xi_r^{(1)} + \ldots) + \underset{\sim}{B}(\xi_B^{(0)} + \epsilon\xi_B^{(1)} + \ldots) + \hat{\underset{\sim}{r}} \times \underset{\sim}{B}(\xi_\perp^{(0)} + \epsilon\xi_\perp^{(1)} + \ldots)$$

We extract the leading order terms of Eqs.(29-32) as $\epsilon \to 0$. Taking, respectively, components of the momentum equation (29) along $\underset{\sim}{B}, \hat{\underset{\sim}{r}} \times \underset{\sim}{B}$ and operating with the annihilator $\nabla \cdot \left(\frac{\underset{\sim}{B} \times}{B^2}\right)$ gives

$$-p_o' b_r^{(4)} - \underset{\sim}{B} \cdot \nabla p_1^{(4)} = \rho_o q^2 B^2 \xi_B^{(0)} + \rho_o q v_o B^2 \frac{d\xi_B^{(0)}}{dr} \qquad (B1)$$

$$(\underset{\sim}{J} \cdot \underset{\sim}{B}) b_r^{(4)} + \frac{i}{a}(mB_z + akB_\theta) p_1^{(4)} = -\frac{iB}{a}(mB_z + akB_\theta) b_B^{(4)} \qquad (B2)$$

$$-v_o \xi_r^{(2)'''} + q\xi_r^{(2)''} = \frac{(mB_z + akB_\theta)^2}{\rho_o a q B^3} \frac{dB^2}{dr} b_B^{(4)}$$

$$+ \frac{B^2}{\rho_o a q} \underset{\sim}{B}_o \cdot \nabla b_r^{(6)''} - \frac{i(mB_z + akB_\theta)}{\rho_o a q} \frac{d}{dr}\left(\frac{\underset{\sim}{J} \cdot \underset{\sim}{B}}{B^2}\right) b_r^{(4)} . \qquad (B3)$$

The radial and parallel components of the induction equation (32) yield

$$b_r^{(4)} - \frac{\eta}{q} b_r^{(6)''} = \underset{\sim}{B} \cdot \nabla \xi_r^{(2)} \tag{B4}$$

$$\frac{\eta}{q} b_B^{(4)''} - \nabla \cdot \underset{\sim}{\xi}^{(2)} = \frac{1}{B^2}(p_o + B^2)' \xi_r^{(2)} - \underset{\sim}{B} \cdot \nabla \xi_B^{(o)} \; . \tag{B5}$$

Eqs.(30) and (31) become

$$(1 - \gamma) p_o q \frac{\rho_o'}{\rho_o} \xi_r^{(2)} + q\left(p_1^{(4)} - \gamma \frac{p_o \rho_1^{(2)}}{\rho_o}\right) = \frac{\gamma p_o v_o}{\rho_o} \frac{d\rho_1^{(2)}}{dr} - v_o \frac{dp_1^{(4)}}{dr} \tag{B6}$$

$$v_o \frac{d\rho_1^{(2)}}{dr} + q\rho_1^{(2)} + q\rho_o \nabla \cdot \underset{\sim}{\xi}^{(2)} + q\xi_r^{(2)} \frac{d\rho_o}{dr} = 0 \; . \tag{B7}$$

Next, the leading order terms of $\nabla \cdot \underset{\sim}{b} = 0$ and $\nabla \cdot \underset{\sim}{\xi}^{(o)} = 0$ are

$$\frac{db_r^{(4)}}{dr} = 0 \tag{B8}$$

$$\frac{db_r^{(6)}}{dr} = \frac{i}{a}(mB_z + akB_\theta) b_\perp^{(4)} \tag{B9}$$

$$\frac{d\xi_r^{(2)}}{dr} = \frac{i}{a}(mB_z + kaB_\theta) \xi_\perp^{(o)} \; . \tag{B10}$$

These Eqs.(B1-B10) can then be reduced to 5 involving only $b_r^{(4)}$, $b_r^{(6)}$ $\xi_B^{(o)}$, $\xi_r^{(2)}$, $b_B^{(4)}$. Finally, introducing the following rescaled variables yields the resistive layer Eqs.(33-37) of the main paper

$$L_R^6 = \frac{\rho \eta^2 a^2}{m^2 \dot{F}^2 B^2} \; , \qquad Q_R^3 = \frac{m^2 \eta \dot{F}^2 B^2}{\rho a^2} \; , \qquad X = \frac{x a^{1/5}}{L_R^{6/5}}$$

$$Q = \frac{q a^{4/5}}{L_R^{4/5} Q_R} \; , \qquad \Xi = \xi_r^{(2)} \; , \qquad \Gamma = \frac{D B^2 b_B^{(4)} a^{6/5}}{p' L_R^{6/5}}$$

$$\psi_o = \frac{ia^{6/5} b_r^{(4)}}{m L_R^{6/5} \dot{F}_B} \quad , \quad \psi_2 = \frac{ia^{12/5} b_r^{(6)}}{m \dot{F} B L_R^{12/5}} \quad , \quad W = Q\,\Xi$$

$$C = \frac{v_o}{aQ_R} \quad , \quad \Lambda = \frac{iB^2 L_R^{16/5}}{\eta a^{1/5}} \left(\frac{i'(a)}{i(a)}\right) k B_z Q_R \xi_B^{(o)} \quad .$$

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