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STABILITY OF AN ANISOTROPIC HIGH-β TOKAMAK TO BALLOONING MODES

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Abstract

We have applied the Kruskal-Oberman energy principle to a simple model of an anisotropic tokamak in which the pressure varies round flux-surfaces. We show that the weighting of pressure towards regions of favourable curvature leads to a significant stabilisation of the high-n ballooning modes.

(Submitted for publication in Physical Review Letters)

April 1978

Following recent theoretical investigations of the MHD stability of scalar pressure tokamaks[1-6], it is now generally believed that the upper-limit to β is set by the ballooning mode. Apart from its use as an additional heat source, neutral injection has been proposed as a method for "pumping-up" β in the flux-conserving tokamak[7]; it is also fundamental to the counterstreaming ion concept[8]. These applications have led us to consider the MHD stability of an anisotropic model of tokamak to high-n ballooning, n being the toroidal mode-number.

Our analysis is based on the Kruskal-Oberman energy principle[9]; using the property of adiabatic invariance, Andreoletti[10] has shown their result to be independent of the form of distribution function. We assume that neutral injection is applied at an angle to the magnetic field such that hot ions are created only in the untrapped region of velocity space, so that the distribution function for the trapped particles is not significantly anisotropic. Then for small inverse aspect ratio, δ, the kinetic term in Kruskal-Oberman is $O(\delta^{7/2})$,[11] whereas the fluid terms are $O(\delta^2)$, when $\beta \sim \delta$. Thus, we drop the kinetic term, anticipating that our general analysis will be applied to a large aspect-ratio model. Writing the fluid terms in a form as closely analogous to that for scalar pressure[5] as possible, we obtain

$$\delta W = \int d\tau \left\{ (1-\sigma_-)\vec{Q}_\perp^{\,2} - (1-\sigma_-)\frac{\vec{J}.\vec{B}}{B^2}(\vec{\xi}\times\vec{B}.\vec{Q}) - 2(\vec{\xi}.\vec{\kappa})(\vec{\xi}.\nabla\bar{p}) \right.$$
$$\left. + B^2(1+\sigma_\perp)\left[\left(1+\frac{1-\sigma_-}{1+\sigma_\perp}\right)\vec{\xi}.\vec{\kappa} + \nabla.\vec{\xi}\right]^2 + B^2\left(\frac{1-\sigma_-}{1+\sigma_\perp}\right)(\sigma_\perp+\sigma_-)(\vec{\xi}.\vec{\kappa})^2 \right\} \quad (1)$$

where $\bar{p} = (p_\perp + p_\parallel)/2$, $\sigma_- = (p_\parallel - p_\perp)/B^2$, $\vec{Q} = \mathrm{curl}(\vec{\xi}\times\vec{B})$, $\xi_\parallel \equiv 0$,

and $\vec{\kappa}$ denotes the field-line curvature. In order to define $\sigma_\perp$, we introduce the pressure-like moment

$$C = \sum_j m_j \iint \frac{B d\mu d\epsilon}{v_{\parallel}} \frac{\partial f_j}{\partial \epsilon} (\mu B)^2 \qquad (2)$$

where the velocity space variables are $\epsilon = \frac{1}{2}v^2$ and $\mu = \frac{\frac{1}{2}v_{\perp}^2}{B}$, so that $v_{\parallel}^2 = 2(\epsilon - \mu B)$. Thus $\sigma_{\perp} = \frac{2p_{\perp} + C}{B^2}$. It follows from Eq.(2) and the definitions of $p_{\perp}$ and $p_{\parallel}$[10] that C and $\bar{p}$ are related by[13]

$$\vec{B} \cdot \nabla\bar{p} = \frac{2\bar{p} + C}{2B^2} \vec{B} \cdot \nabla(\tfrac{1}{2}B^2) \qquad (3)$$

In practice, the criteria for stability to the "firehose" and "mirror modes[13,14], namely, $1 - \sigma_{-} > 0$ and $1 + \sigma_{\perp} > 0$, will be satisfied.

Following Dobrott et al.[5] and expanding in $1/n$, we find that to lowest order the minimising displacements satisfy $\nabla \cdot \vec{\xi} = O(1)$ and $\vec{B} \cdot \nabla\vec{\xi} = O(1)$. The lowest-order contribution of the kink term (second in Eq. (1)) vanishes, and after minimising with respect to the first-order displacement, $\vec{\xi}^{(1)}$, the "field compression" term (fourth) also vanishes. In zeroth order, δW is then a functional of the zeroth-order $\vec{\xi}$ only. Employing the usual axisymmetric (ψ, χ, φ) coordinate system, we express $\vec{\xi}$ as a fourier mode $\vec{\xi} = \vec{X}(\psi, \chi)\, e^{in\varphi}$, and obtain

$$\delta W^{(o)} = \int d\tau \Bigg\{ (1 - \sigma_{-}) \left[\frac{|\vec{B} \cdot \nabla \xi_{\psi}|^2}{|\nabla\psi|^2} + \frac{|\nabla\psi|^2}{B^2} \left| \frac{1}{n} \frac{\partial}{\partial\psi} (\vec{B} \cdot \nabla \xi_{\psi}) \right|^2 \right]$$
$$- \left(\left\{ \xi_{\psi}\kappa_{\psi} + \xi_s \kappa_s \right\} \left\{ \overset{*}{\xi}_{\psi} \bar{p}_{\psi} + \frac{|\nabla\psi|}{B} \overset{*}{\xi}_s \bar{p}_s \right\} + \text{c.c.} \right) + \frac{B^2(1 - \sigma_{-})(\sigma_{\perp} + \sigma_{-})}{(1 + \sigma_{\perp})} \left| \xi_{\psi}\kappa_{\psi} + \xi_s \kappa_s \right|^2 \Bigg\} \qquad (4)$$

where

$\vec{\xi} = \frac{\vec{\nabla}\psi}{|\nabla\psi|^2} \xi_{\psi} + \frac{\vec{B} \times \vec{\nabla}\psi}{B^2} \xi_s$, with $\xi_s = \frac{1}{in} \frac{\partial \xi_{\psi}}{\partial \psi}$; $\vec{\kappa} = \vec{\nabla}\psi\, \kappa_{\psi} + \frac{\vec{B} \times \nabla\psi}{|\nabla\psi|^2} \kappa_s$;

$\bar{p}_{\psi} = \frac{\partial\bar{p}}{\partial\psi}$, and $\bar{p}_s = \frac{\vec{B} \times \nabla\psi \cdot \nabla\bar{p}}{B|\nabla\psi|}$. Minimising Eq. (4) with respect to

ξ_ψ we obtain an Euler equation containing partial derivatives which act on the rapid ψ- variation of ξ_ψ, as well as derivatives with respect to χ; following Connor et al.[6], this equation is reduced to an ordinary differential equation. Thus, we define the transformation $\xi_\psi(\psi, \chi) \rightarrow F(\psi, y)$ by

$$\xi_\psi = \sum_m e^{im\chi} \int_{-\infty}^{\infty} F(\psi, y) \; e^{-i(my + n\int^y \nu d\chi')} \, dy$$

where $\nu = \dfrac{B_\varphi}{|\nabla\psi||\nabla\chi|}$, and all the rapid ψ-variation of ξ_ψ is contained within the phase-factor $e^{-in\int^y \nu d\chi'}$. Defining $G(\psi, y) = \int_0^y \nu d\chi$, then in transform space the Euler equation becomes

$$\frac{1}{J}\frac{\partial}{\partial y}\left[\frac{(1-\sigma_-)}{|\nabla\psi|^2}\left\{1 + \frac{|\nabla\psi|^4}{B^2}\left(\frac{\partial G}{\partial\psi}\right)^2\right\}\frac{1}{J}\frac{\partial F}{\partial y}\right] + 2\left(\bar{p}_\psi - \frac{\kappa_\psi}{\kappa_s}\frac{|\nabla\psi|}{B}\bar{p}_s\right)\left(\kappa_\psi - \kappa_s\frac{\partial G}{\partial\psi}\right)F = 0 \tag{5}$$

From the mode radial structure defined above, we deduce the physical boundary condition $|y|^{\frac{1}{2}}F \rightarrow 0$ as $|y| \rightarrow \infty$. Asymptotic analysis of Eq. (5) leads to the localised interchange criterion[12], as was noted in the scalar pressure case[6]. In general, anisotropic equilibria are of the form $\bar{p} = \bar{p}(\psi, \chi)$; this suggests that if equilibria can be produced such that the pressure surfaces are displaced inwards relative to the flux-surfaces, then the "loading" of pressure into regions of favourable curvature could lead to stability at higher β. We now demonstrate this to be the case.

Using simple forms for $\bar{p}$ and the toroidal current density, we expand the equilibrium equations[14] in δ and develop a circular boundary (radius = a), 'flat'-current, $\beta \sim \delta$ model of tokamak. Introducing co-ordinates (r, θ) based on the plasma centre, and with the major radius $R = R_o\left(1 + \frac{r}{R_o}\cos\theta\right)$, we obtain

$$\psi(r,\theta) = \frac{-B_o a^2}{\bar{q}} \left(\frac{1}{2\left(1 + \frac{\alpha k}{4}\right)} \right) (1 - r^2/a^2) \left\{ 1 + kr/a \cos\theta + \frac{\alpha k}{6} \left[\frac{3}{4} (1 + r^2/a^2) + r^2/a^2 \cos 2\theta \right] \right\}$$

$$\bar{p}(r,\theta) = - \frac{\delta B_o}{a^2 \bar{q}} \frac{2k}{\left(1 + \frac{1}{4}\alpha k\right)} \psi(1 + \alpha\, r/a \cos\theta) , \qquad (6)$$

where $I_\varphi = \frac{2\pi a^2 B_o}{R_o \bar{q}}$ is the toroidal current, B_o characterises the magnetic field, and k and α are free parameters. As a consequence of the ordering, $p_\parallel = p_\parallel(\psi)$ and $p_\perp = p_\perp(\psi, r\cos\theta)$, and hence from Eq.(6), $p_\perp/p_\parallel = A + (1 + A)\alpha r\cos\theta$, where A is arbitrary. We have three cases: (a) scalar $p(\alpha = 0, A = 1)$, (b) $\bar{p} = \bar{p}(\psi)(\alpha = 0, A \neq 1)$ and (c) $\bar{p} = \bar{p}(\psi, r\cos\theta)(\alpha \neq 0, A \neq 1)$. It is the general case (c) which is of interest here; α is a measure of the extent to which $\bar{p} \neq \bar{p}(\psi)$. For given α, k is defined through $\beta_p (= 8\pi I_\varphi^{-2} \int \bar{p}\, r dr d\theta)$ which becomes $\beta_p = \delta^{-1} k (1 + \frac{1}{3}\alpha k)(1 + \frac{1}{4}\alpha k)^{-2}$. Flux and pressure surfaces for a typical case are shown in Fig.1. We note that for $p_\parallel \gg p_\perp$ and $\alpha = 0$ the model reverts to that of Cordey and Haas[15].

We have taken the large aspect ratio form of Eq.(5), and solved it numerically for the above equilibrium. Except for the immediate vicinity of the magnetic axis, the localised interchange criterion is always satisfied when $\alpha \leq 0$. On a surface of given shape and magnetic field, with a prescribed amount of shear, there are, in general, two marginally-stable pressure gradients which bound a range of unstable values. When $\alpha = 0$, the equilibrium value of $\partial\bar{p}/\partial\psi$ is found always to lie in the unstable range, but rather close to the higher marginal point. As α falls below zero, the unstable range narrows. Plotting β_p versus α, our results are presented as a marginal stability line in Fig. 2. We also indicate the equilibrium limit, and for completeness, the current reversal limit. We observe that for very modest values of $\alpha(-0.1)$, our model is ballooning stable right up to the equilibrium limit; as β_p approaches this limit, the value of α corresponding to marginal stability begins to decrease. This effect is thought to result from a stabilisation associated with large values of the major radius displacement function Δ' close to the boundary, where $\Delta' \sim k$. We note that for $\alpha = -0.1$ and $A = 1$, $0.8 \leqslant p_\perp/p_\parallel \leqslant 1.2$. In obtaining the marginal curve, we have excluded the magnetic axis and a small surrounding region (less than 1% of plasma volume). Any instability in the excluded region will thus be strongly localised round the axis and is therefore disregarded.

In the case of scalar pressure ($\alpha = 0$) as $k \to 0$ the shear at the boundary vanishes like k^2, and as a result our equilibrium is unstable even in the limit of small pressure. However, with a current profile producing finite shear at the boundary we expect stability up to a finite limit in β_p. When the additional shear is small this value is in the vicinity of point A in Fig.2; the latter point is obtained by applying the result of Connor et al[6] to our circular boundary.

As a consequence of large aspect ratio and Eq.(3), $\bar{p}$ can only have the correct lowest-order variation round a flux-surface if C is O(1). Calculations of $p_\perp$, $p_\parallel$ and C for a distribution function which models neutral injection[16], show that substantial variation of $\bar{p}$ is to be expected only for near-perpendicular injection, with $v_\parallel/v \sim \delta^{\frac{1}{2}}$ for the injected ions. Perpendicular injection is known to have less favourable microstability properties than parallel injection, although the non-linear consequences are uncertain[16,17].

For the same class of current profiles, it is clear from Fig. 2 that by a modest inward weighting of pressure, a significant improvement in β can be obtained over the scalar pressure value. Although the weighting modifies the shear, this effect is small at the values of $\alpha(-0.1)$

necessary to ensure stability up to the equilibrium limit. Naturally, we expect this class of profiles to be kink unstable; stabilisation of this mode requires shaping of the current profile[18]. We conjecture that the effect to which attention has been drawn in this letter, may also give rise to improved ballooning stability for equilibria possessing more realistic current profiles.

We are grateful to Mr R.J. Hastie and Drs J.W. Connor, J.B. Taylor and J.A. Wesson for helpful discussions.

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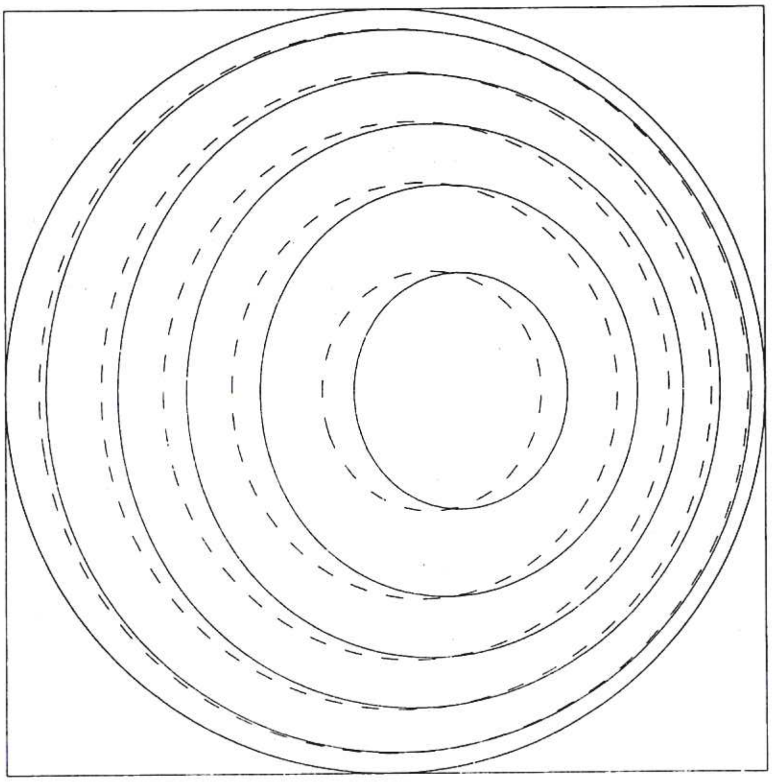

Fig.1 Flux (solid lines) and constant $\overline{p}$ surfaces (dashed lines) for equilibrium with k = 0·5 and $\alpha = -0{\cdot}2$. The major axis lies to the left.

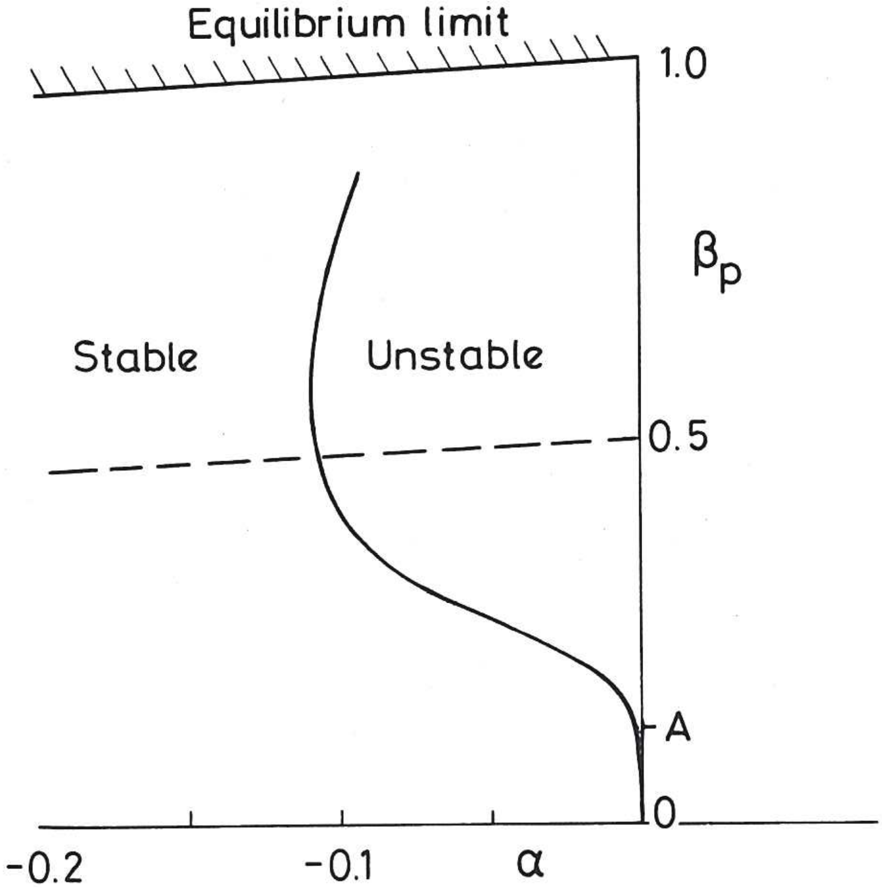

Fig.2 Variation of the marginally-stable poloidal β, measured in units of δ^{-1}, is plotted versus α. For β_p above the dashed line, the toroidal current reverses on the inside.