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ON THE BIFURCATION OF MAGNETOHYDRODYNAMIC EQUILIBRIA IN A MAGNETIC QUADRUPOLE FIELD

by

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Abstract

We have investigated the equilibria of plasmas maintained by a quadrupole field in straight and toroidal geometries. Using asymptotic analysis we have studied the uniqueness of uniform-current straight systems for rail, point and rectangular limiters. Assuming a family of simple current profiles, numerical work has revealed the possibility of three types of bifurcation; our asymptotic analysis is fully confirmed. The value of b/a at bifurcation depends strongly on the current profile and poloidal beta, b and a being the maximum height and width of the plasma cross-section, respectively. For non-zero current density at the plasma boundary $(j_B \neq 0)$ the bifurcation b/a can approach unity. For a profile with $j_B = 0$, however, bifurcation disappears through the intervention of the separatrix; there is then a maximum value of b/a, which, for an aspect radio of 5, is approximately 2.0 irrespective of poloidal beta.

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1. INTRODUCTION

The present paper is concerned with the MHD equilibria of plasmas maintained by a magnetic quadrupole field. Numerical investigations of plasma configurations maintained by external conductors involve the determination of a free-boundary, and are, therefore, necessarily non-linear. Although many such investigations have been reported in the literature (see, eg. Refs. [1 - 4]), the essential non-linearity leads to considerable "trial-and-error". In general, the number of solutions that can arise is unknown, and further, if a trivial solution exists, then constraints must be applied if this solution is to be avoided.

In a previous paper [5], we have studied a very simple equilibrium namely, a plasma with uniform longitudinal current and confined by the field of a straight symmetric quadrupole. The simplicity of the model enabled us to investigate both analytically, and numerically, the possible solutions which can occur, their relationship to bifurcation, and the rôle of certain constraints in determining uniqueness. In particular, we obtained vertical ellipses and demonstrated the existence of an 'infinity' of bifurcation points; the first bifurcation point occurs at $b/a = 2.9$, where a and b denote the semi-axes [6].

Our objective in the present paper is to show, for both straight and toroidal plasmas, that the latter bifurcation point is strongly dependent on the form of current profile; we also investigate the effect of the choice of constraints upon bifurcation. The forms of current profile which we have adopted here, were suggested by work on TOSCA [7]. In running this experiment as a quadrupole, reasonable agreement between observation and computation [7] has been obtained assuming 'flat' or 'peaked' currents, as appropriate to the stage of the discharge. Thus we consider current forms intermediate to, and including, these two extreme cases; we assume toroidal current density profiles which are linearly dependent on the poloidal flux.

Apart from modelling experiments and generally improving our basic understanding of the application of numerical methods to equilibrium calculations, there is a more fundamental reason for interest in the present problem. It is well-known that the MHD stability of a configuration can be greatly influenced by the presence of ellipticity (and/

or triangularity) see, eg. [8]). Hence, it is important to know whether or not a specific set of conductors and given plasma current profile can set an intrinsic limit on the b/a that can be achieved. This might very well depend on the way the plasma is formed and/or constrained. In particular, such a limit could be set by the appearance of a bifurcation; the interpretation of bifurcation points as marginal stability points for ideal or dissipative axisymmetric modes was referred to in our previous work [5].

2. ANALYSIS

We begin by developing those features of our previous paper [5] which are relevant to the present work. Consider the straight configuration shown in Fig. 1. The longitudinal current density in the plasma, j , is assumed uniform; the four wires carry a current of magnitude, I , and have a half-separation distance d . Asymptotic analysis shows the existence of elliptic equilibria, and in particular, leads to the relation

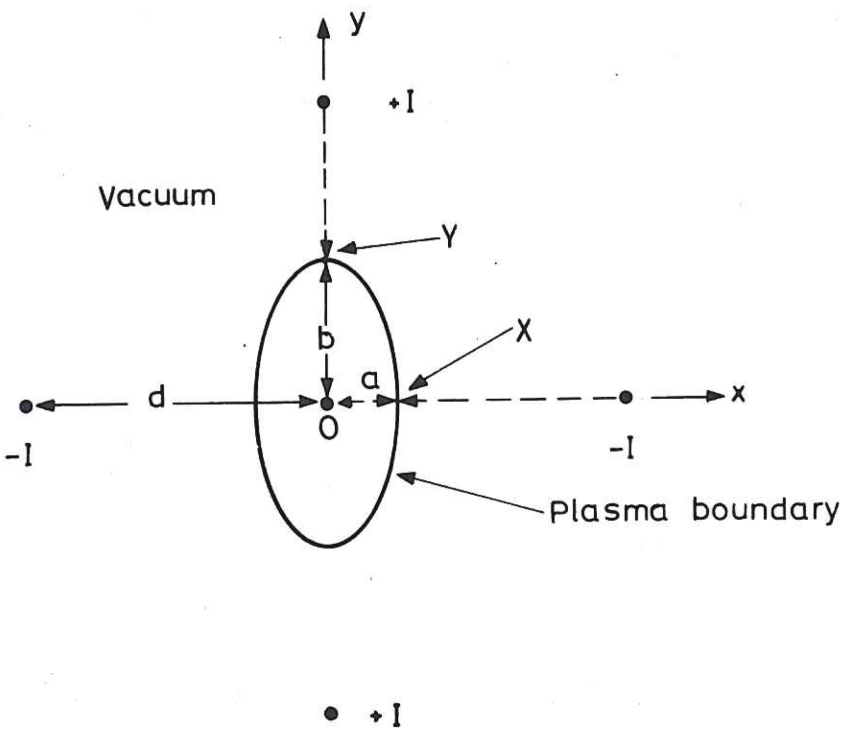

Fig.1 Elliptic equilibrium and coordinate system.

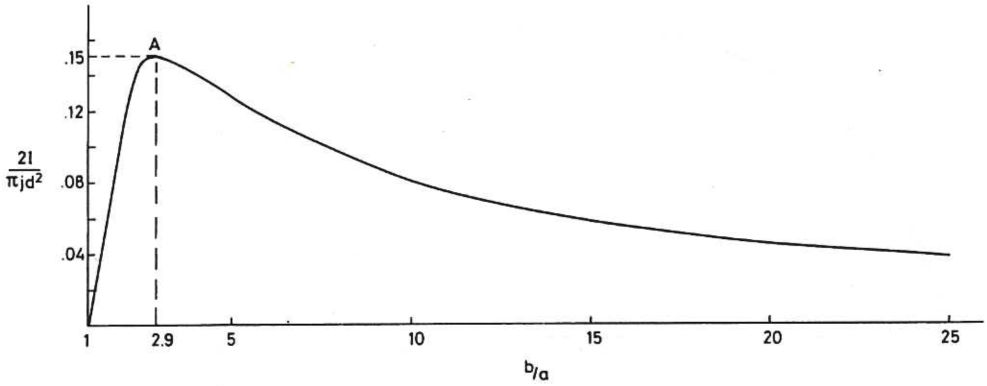

Fig.2 Plot of $\frac{2I}{\pi j d^2}$ against b/a for a straight uniform current plasma as obtained from asymptotic analysis.

$$\frac{2I}{\pi j d^2} = \frac{\frac{b}{a}\left(\frac{b}{a} - 1\right)}{\left(1 + \frac{b}{a}\right)\left(1 + \frac{b^2}{a^2}\right)} \quad . \tag{1}$$

Plotting the left-hand side of Eq. (1) against b/a, we obtain the curve shown in Fig. 2. Following Strauss [6], the point A is a bifurcation point (b/a = 2.9); for $j < \frac{4.24\, I}{d^2}$ there are no solutions, for $j = \frac{4.24\, I}{d^2}$ one solution, and for $j > \frac{4.24\, I}{d^2}$ two values for b/a are possible.

For $j > \frac{4.24\, I}{d^2}$, it can be shown from Eq. (1) and Fig. 2, that each value of b/a corresponds to an infinity of ellipses. Thus in order that an equilibrium calculation be determinate, further information must be prescribed. In our numerical work [5] this was achieved by constraining the plasma boundary to pass through the prescribed point Y shown in Fig. 1. The philosophy of the constraints used in the numerical method is evident from our asymptotic analysis. We can rewrite Eq. (1) in the forms

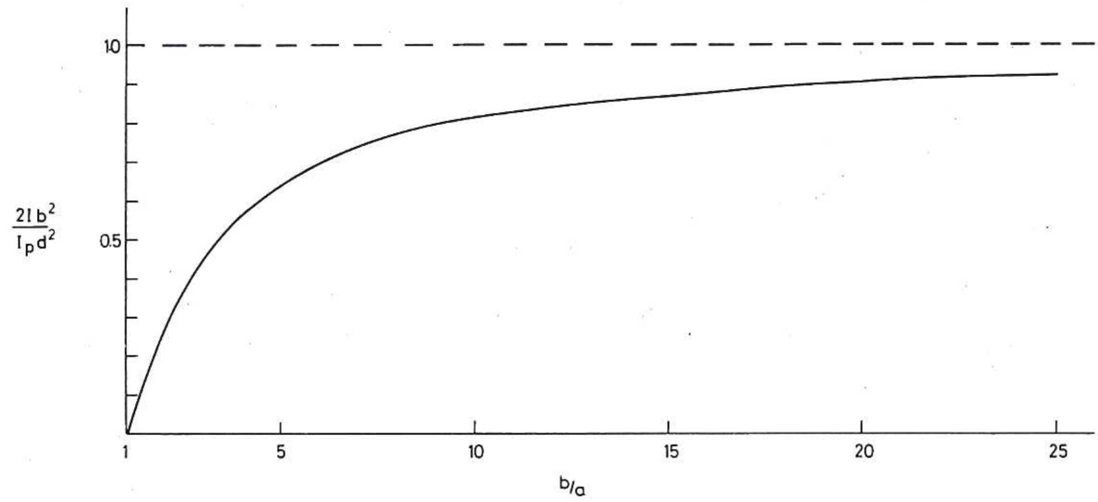

Fig.3 Plot of $\frac{2Ib^2}{I_p d^2}$ against b/a for a straight uniform current plasma as obtained from asymptotic analysis.

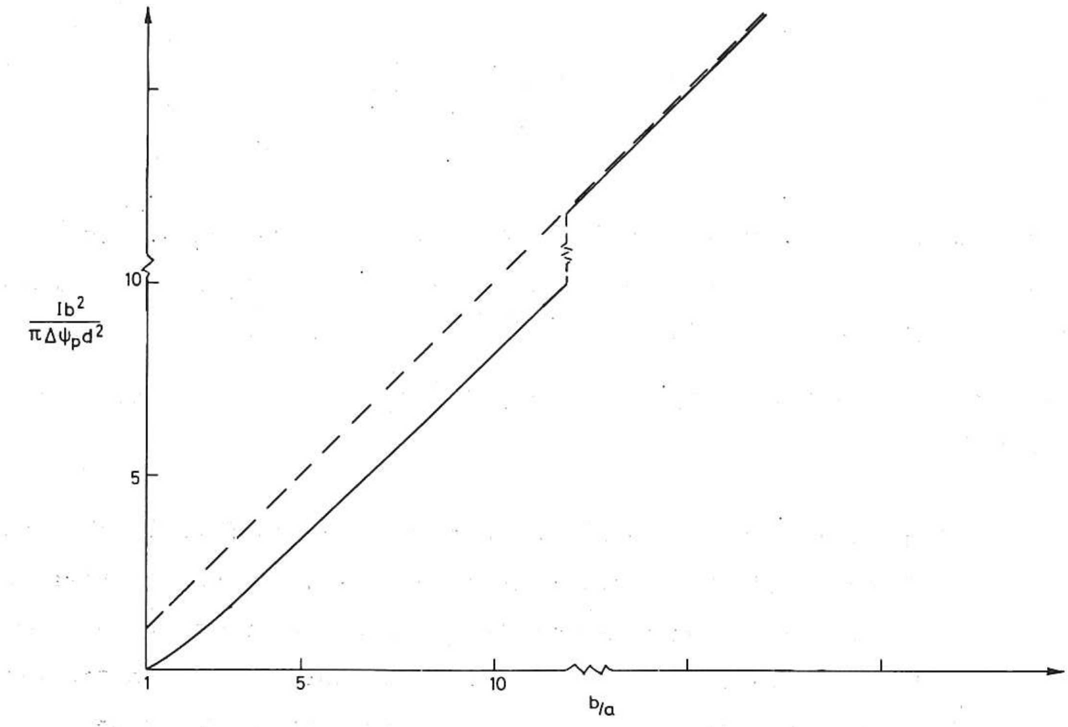

Fig.4 Plot of $\frac{Ib^2}{\pi\Delta\psi_p d^2}$ against b/a for a straight uniform current plasma as obtained from asymptotic analysis.

$$\frac{2I}{I_p}\frac{b^2}{d^2} = \frac{\left(\frac{b}{a}\right)^2\left(\frac{b}{a} - 1\right)}{\left(1 + \frac{b}{a}\right)\left(1 + \frac{b^2}{a^2}\right)} \tag{2}$$

and

$$\frac{I}{\pi\Delta\psi_p}\frac{b^2}{d^2} = \frac{\frac{b}{a}\left(\frac{b}{a} - 1\right)}{\left(1 + \frac{b}{a}\right)} , \tag{3}$$

where I_p and $\Delta\psi_p$ are the total current and poloidal-flux within the plasma, respectively. Plotting the left-hand sides of Eqs. (2) and (3) against b/a we obtain the curves shown in Figs. 3 and 4. Thus we can immediately draw the following conclusion:

For fixed $\frac{I}{d^2}$, equilibria corresponding to points on the curve in Fig. 2 are uniquely determined by specifying Y (geometrical constraint in Fig. 1), and either the total plasma current, or the poloidal-flux within the plasma; no a priori knowledge of j is required.

The point Y can, of course, be interpreted as a rail-limiter. We further note, that within the ordering, there is no upper limit on the value of b/a that can be obtained. For a fixed I and d, and prescribed I_p, there is an upper limit to b; this occurs when $b/a \to \infty$. Thus from Eq.(2), $b \to d\sqrt{\frac{I_p}{2I}}$. We note, however, that for a fixed I and d, and prescribed $\Delta\psi_p$, the quantity b is unlimited (again within the ordering).

From the above discussion, it might be supposed that uniqueness could be ensured by specifying I_p or $\Delta\psi_p$, together with any geometrical constraint. That this is not the case can be shown very simply. Thus suppose we take the point X in Fig.1 to be the geometrical constraint (a point limiter). We now write Eqs.(2) and (3) in the alternative forms:

$$\left.\begin{aligned} \frac{2Ia^2}{I_p d^2} &= \frac{\frac{b}{a} - 1}{\left(1 + \frac{b}{a}\right)\left(1 + \frac{b^2}{a^2}\right)} \\ &\text{and} \\ \frac{Ia^2}{\pi\Delta\psi_p d^2} &= \frac{\frac{a}{b}\left(\frac{b}{a} - 1\right)}{1 + \frac{b}{a}} \end{aligned}\right\} \tag{4}$$

Plotting the left-hand sides of Eqs.(4) against b/a we obtain the curves shown in Figs. 5 and 6. It is evident that prescribing I_p or $\Delta\psi_p$ together with the point X, does not lead to a unique value of b/a. We now find new bifurcations at values of b/a somewhat lower than 2.9, namely b/a = 1.84 or 2.41. Furthermore, for a given I_p or $\Delta\psi_p$ there is a bound on the positioning of the point-limiter; from Figs. 5 and 6 we observe that there are upper limits on 'a' given by $a = 0.18\, d\sqrt{\frac{I_p}{I}}$ and $a = 0.73\, d\sqrt{\frac{\Delta\psi_p}{I}}$, respectively.

Since the above discussions of the point and rail-limiters are based on the same equations, it is obvious that the two approaches are intimately connected. Indeed, if we take the results of a rail-limiter calculation where a and b are both known and plot either $\frac{2Ia^2}{I_p d^2}$ or $\frac{Ia^2}{\pi\Delta\psi_p d^2}$ against b/a, then we are again led to the curves shown in Figs.5 and 6. From our earlier argument, however, only one point on each curve is compatible with the known a and b, and in this case the points B are not to be interpreted as bifurcation points; the rail limiter must also set the same upper-limits on 'a' as observed for the bounds on the positioning of the point-limiter.

Having considered a plasma with uniform current and maintained by a straight quadrupole, it is now of interest to study non-uniform current profiles, and subsequently, to take account of toroidicity. Using the asymptotic analysis as a guide, we shall use numerical methods to re-investigate the phenomena of uniqueness and bifurcation.

3. NUMERICAL WORK

We now write down the equations appropriate to the straight and toroidal calculations. Denoting the poloidal flux by ψ, we define the plasma-vacuum boundary to be $\psi = 0$. For the straight system, we choose a model such that the flux in the plasma $(\psi < 0)$ satisfies the equation

$$\nabla^2\psi = -\, j[\psi_o - \psi] \qquad (5)$$

where j and ψ_o are constants. Since $\mu_o j_z(\psi) = BB_z'(\psi) + \mu_o p'(\psi)$, the model describes all p and B_z profiles consistent with the right-hand side of Eq.(5). It follows that our calculations for the straight system are for arbitrary total beta (or poloidal-beta). For the vacuum region $(\psi > 0)$

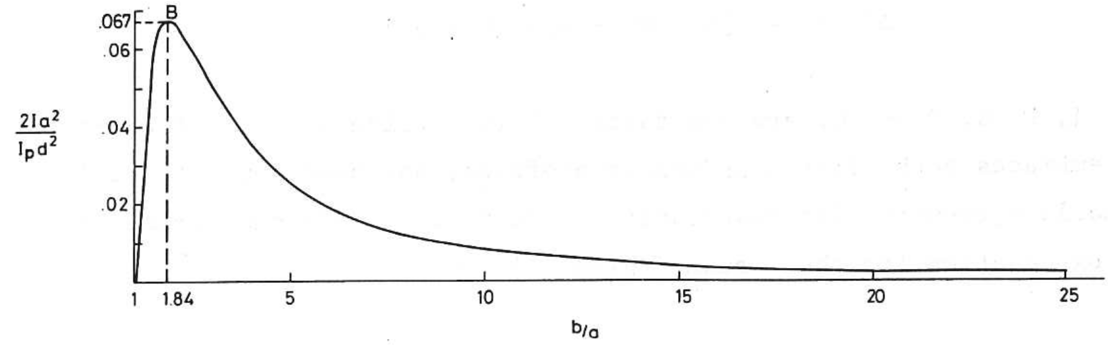

Fig.5 Plot of $\frac{2Ia^2}{I_p d^2}$ against b/a for a straight uniform current plasma as obtained from asymptotic analysis.

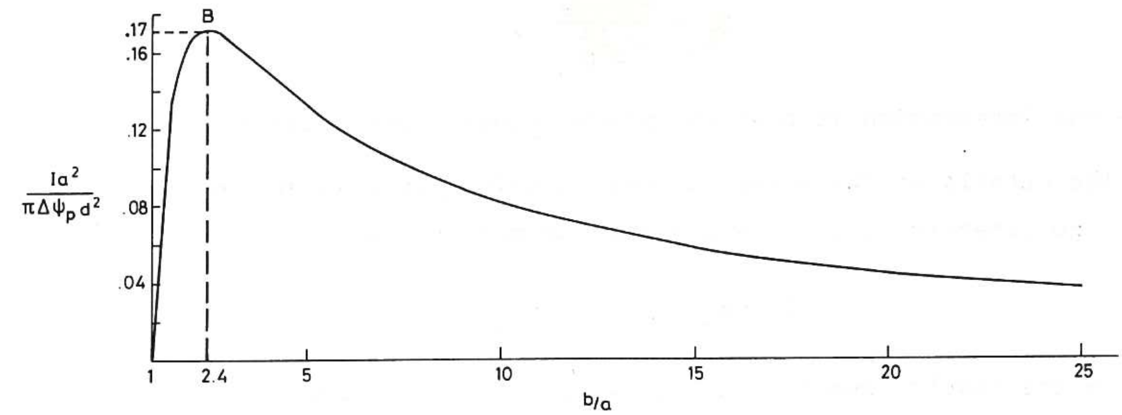

Fig.6 Plot of $\frac{Ia^2}{\pi \Delta \psi_p d^2}$ against b/a for a straight uniform current plasma as obtained from asymptotic analysis.

$$\nabla^2\psi = 0 \; . \qquad (6)$$

For our toroidal model, the equation for the plasma $(\psi < 0)$ is

$$\Delta^*\psi = - j[M + N\psi + \mu_o R^2(P + Q\psi)] \; , \qquad (7)$$

where j, M, N, P and Q, are constants. This particular form of current, which embraces both 'flat' and 'peaked' profiles, has been found to yield reasonable agreement with observations made in the TOSCA experiment [7][1]. The flux-equation for the vacuum region $(\psi > 0)$ is

$$\Delta^*\psi = 0 \; . \qquad (8)$$

Equations (7) and (8) are solved in the usual toroidal coordinates (R, φ, Z) based on the axis of symmetry; the operator Δ^* is defined by

$$\Delta^* = R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial z^2} \; .$$

For the toroidal problem, calculations depend on poloidal-beta, β_I, which we define as

$$\beta_I = \frac{8\pi\int p ds}{\mu_o I_p^2} \; ,$$

where the integration is over the plasma minor cross-section.

The details of the numerical method being given in an appendix, we now go directly to the results for the two systems.

3.1 Straight Quadrupole

In the results reported here, I and d are fixed, and b/a is to be interpreted as the ratio of maximum height to maximum width of the plasma cross-section. Following the asymptotic analysis for the uniform current, we give our results in terms of the quantities discussed in Section 2. Since the prescription of $\Delta\psi_p$ is difficult to implement

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[1] We define a profile to be 'flat' if the current density j_φ or j_z is independent of ψ. A 'peaked'-profile is defined such that j_φ or j_z is zero at the boundary, whilst a 'mixed'-profile is ψ-dependent and non-zero at the boundary.

in an experiment, we only present results for the case of prescribed I_p. Numerically, however, prescribing $\Delta\psi_p$ is expected to yield results completely analogous to those given below. We shall only quote results for the rail-limiter for reasons to be discussed later.

Fixing I_p and striding through a succession of prescribed b values for different current profiles, our results lead to the curves shown in Fig.7. The curve for the uniform current case agrees very well with that deduced from the analysis (see Fig.2). Each point on a given curve corresponds to a different value for b. We observe a decrease in the value of b/a for bifurcation as the current density at the plasma boundary, j_B, decreases. For the peaked profile, $j_B = 0$, the advancing separatrix has removed the possibility of bifurcation.

Somewhat surprisingly, if we fix b and stride through a succession of prescribed values for I_p, for the same current profiles as before, our results take on a different character. Plotting $2I/jd^2$ against b/a we now obtain the curves shown in Fig.8. Each point on a given curve corresponds to a particular value for I_p. As before, the b/a for bifurcation decreases with decreasing current density at the boundary. Now, however, b/a can approach unity, and finally, in the limit $j_B = 0$ the bifurcation point disappears. The flat-current curves in Figs.7 and 8 are identical.

We now give a simple intuitive argument which suggests that the different limiting behaviour of the peaked current in Figs. 7 and 8 is indeed correct. Thus consider a plasma with the equation $\nabla^2\psi = -j\psi$. Suppose we fix the quadrupole current, I, and increase the plasma current, I_p. For sufficiently large I_p the plasma cross-section tends towards a circle, and $j \to \left(\frac{Z_o}{b}\right)^2$, where Z_o is the first zero of $J_o(j^{\frac{1}{2}}r)$ and b is the radius at the limiter; for infinite I_p the plasma cross-section remains unchanged and hence j is unchanged. Thus in the limit b/a = 1, the asymptotic value of $2I/jd^2$ is finite. Alternatively, suppose we fix I_p and reduce the quadrupole current, I. For sufficiently small I the plasma cross-section becomes circular, and again, j approaches $\left(\frac{Z_o}{b}\right)^2$. In this case, however, as $b/a \to 1$ the quantity $\frac{2I}{jd^2} \to 0$. Although the latter procedure was not actually followed in the calculations, the arguments are very suggestive. Furthermore, for the flat-current, both these processes lead to the limit $\frac{2I}{jd^2} \to 0$.

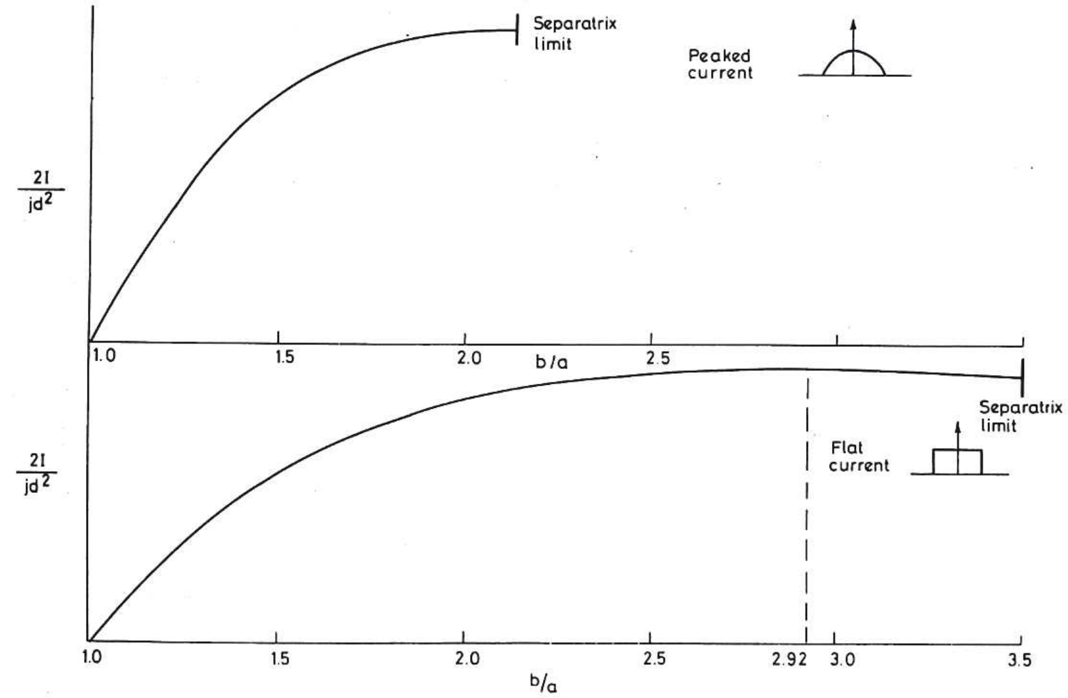

Fig.7 Plots of $\frac{2I}{jd^2}$ against b/a for straight uniform and peaked current profiles. In the computations I_p is held fixed and b varied.

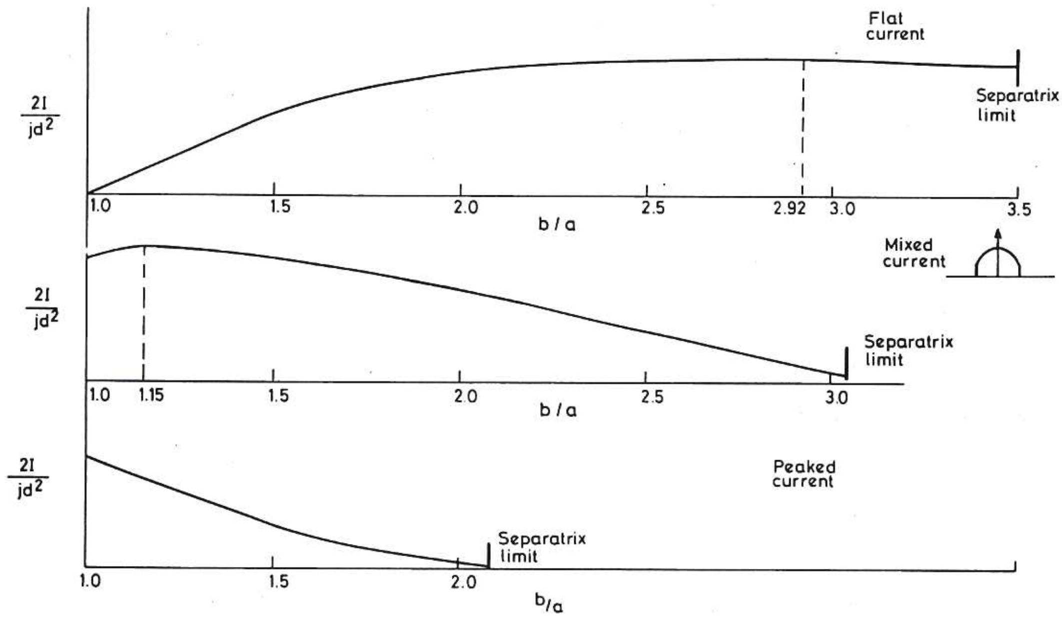

Fig.8 Plots of $\frac{2I}{jd^2}$ against b/a for straight uniform and peaked current profiles. In the computations b is held fixed and I_p varied.

By analogy with Fig.3, we now take the results from Figs.7 and 8 and plot $2Ib^2/I_p d^2$ against b/a for the various profiles (see Fig.9). The curves indicate that fixing b and I_p leads to a unique solution for current profiles described by our model; this result strongly suggests that fixing b and I_p can ensure uniqueness for any profile.

Our analytic work suggests that we should take the above results (Figs.7 and 8) and plot them in yet another form. Thus plotting $2Ia^2/I_p d^2$ versus b/a we obtain the curves shown in Fig.10. As described in Section 2, the figure is interpretable in two ways. Using a rail-limiter, the points B are not to be interpreted as bifurcation points. With a point-limiter, however, the points B denote a second type of bifurcation. As for the first-type, the b/a values for bifurcation are profile-dependent. We note that for the peaked profile ($j_B = 0$) the bifurcation point has vanished due to the intervention of the separatrix; the maximum attainable value of b/a for this profile is 2.1. Given I_p, we see that the point B for each profile implies a corresponding upper-bound on the value of 'a' attainable. In the case of a point-limiter, this sets a bound on the position of the limiter.

3.2 Toroidal Quadrupole

In order to maintain a plasma in toroidal equilibrium, it is necessary to supplement the quadrupole with additional current carrying conductors; adjustment of the current, I_w, in these windings, allows the plasma to be centralised with respect to the quadrupole. Details of the complete configuration of conductors, which models that used in the TOSCA experiment, are given in the appendix along with an outline of the numerical procedure. As for the straight case, calculations have been performed using the rail-limiter only; results for the point-limiter are inferred.

For a given I and d, and fixing I_p and I_w, we stride through a succession of prescribed b values for both flat and peaked current profiles; assuming an aspect-ratio of 5 and setting $\beta_I = 0$, our results are shown in Fig.11. Each point on a given curve corresponds to a different value of b. The results suggest that as the current at the edge of the plasma, j_B, is decreased, the value of b/a for bifurcation

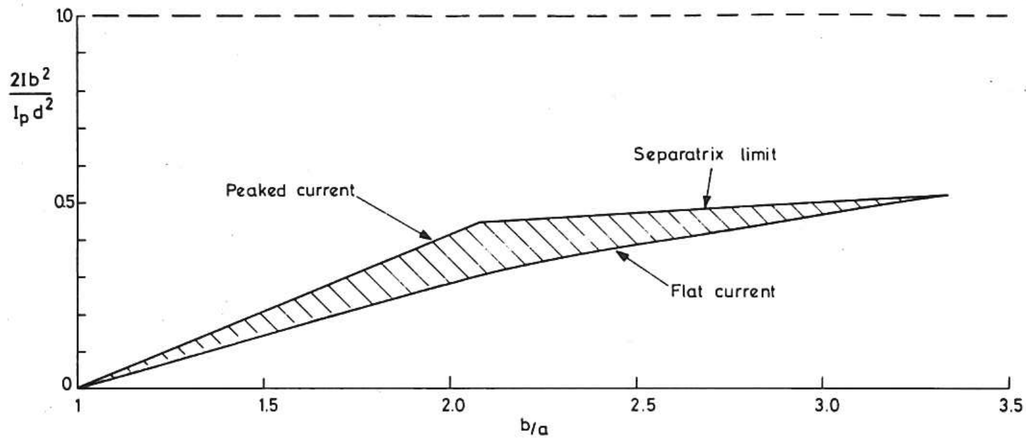

Fig.9 Plots of $\frac{2Ib^2}{I_p d^2}$ against b/a obtained from Figs.7 and 8. All results for mixed current profiles lie in the hatched region.

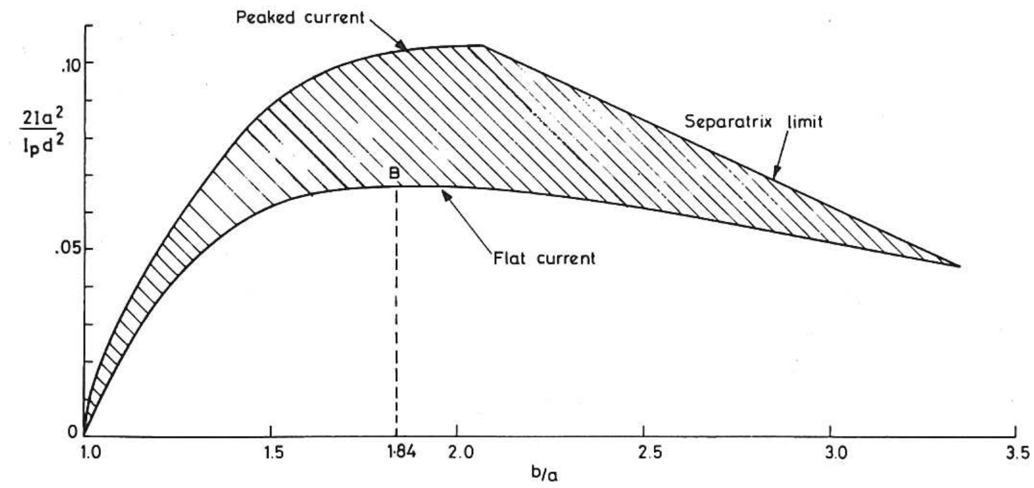

Fig.10 Plots of $\frac{2Ia^2}{I_p d^2}$ against b/a obtained from Figs.7 and 8. All results for mixed current profiles lie in the hatched region.

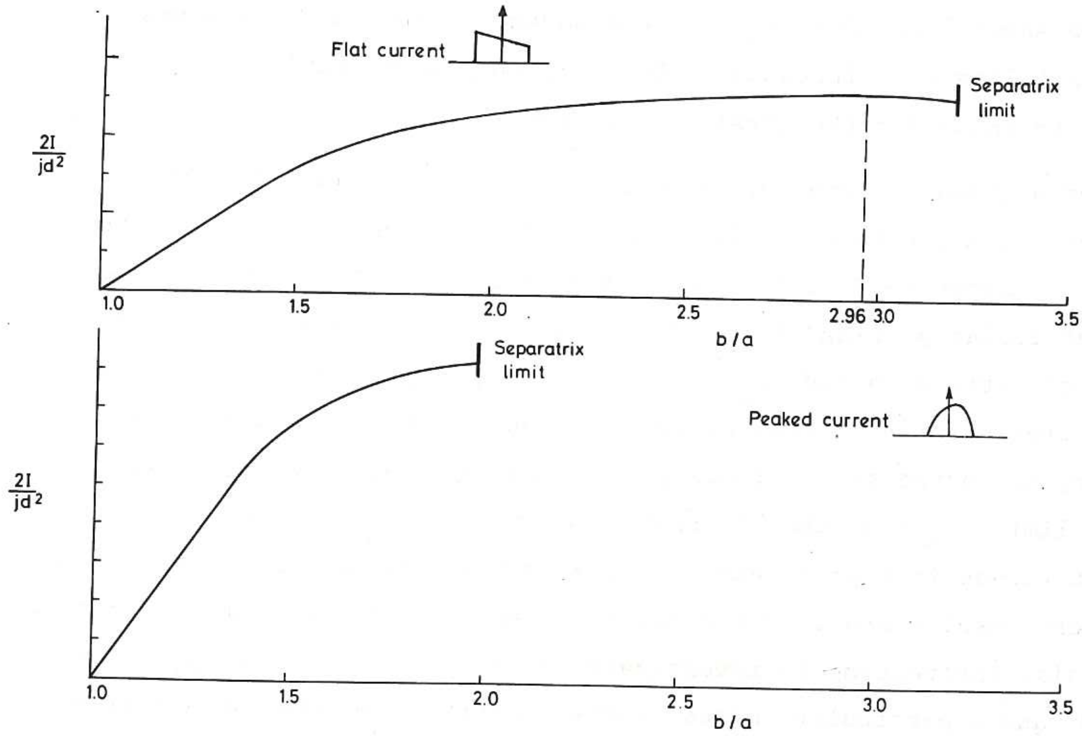

Fig.11 Plots of $\frac{2I}{jd^2}$ against b/a for toroidal flat and peaked current profiles. In the computations I_p is held fixed and b varied.

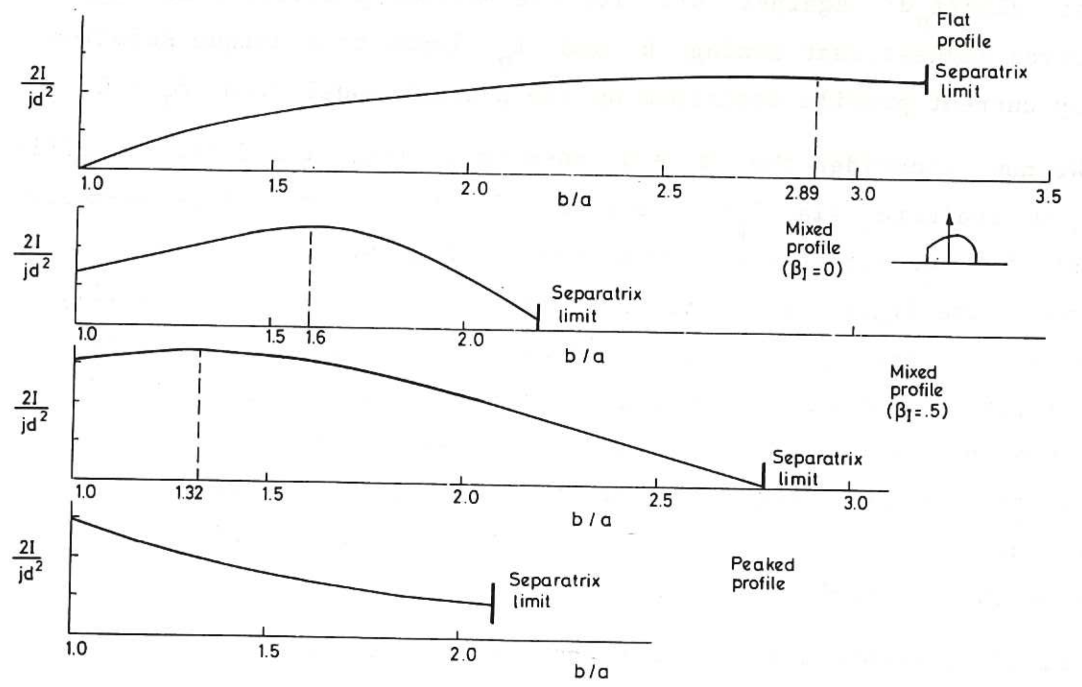

Fig.12 Plots of $\frac{2I}{jd^2}$ against b/a for toroidal flat, mixed and peaked current profiles. In the computations b is held fixed and I_p and I_w varied simultaneously.

falls to about 2.1. For $j_B = 0$, the advancing separatrix removes the possibility of bifurcation. Overall, the above results are very similar to those for the straight quadrupole (see Fig.7).

For a given I and d, and fixing b, we now vary I_p and I_w (together) from run to run. Plotting $2I/jd^2$ against b/a we obtain the curves shown in Fig.12. Each point on a given curve corresponds to a particular pairing of I_p and I_w. Again, the results are for an aspect ratio of 5 and $\beta_I = 0$. As for Fig.11, the b/a for bifurcation decreases with decreasing current density at the boundary. Now, however, as in the straight case, b/a can approach unity, and finally, in the limit $j_B = 0$ the bifurcation point disappears. The flat current curves in Figs 11 and 12 are almost identical. It is clear that all these results are very similar to those for the straight quadrupole. It is also interesting to investigate the effect of raising β_I. For $\beta_I = 0$ and a particular 'mixed' current profile, we find bifurcation at b/a = 1.6. Taking essentially the same current profile and setting $\beta_I = 0.5$, however, we find the b/a for bifurcation to be reduced to 1.32. Our results are summarised in the appendix.

By analogy with Fig.3, we use the results of Figs.11 and 12 ($\beta_I = 0$) to plot $2Ib^2/I_pd^2$ against b/a for the various profiles (see Fig.13). The curves suggest that fixing b and I_p leads to a unique solution for any current profile described by the present model with $\beta_I = 0$.

We now reconsider the $\beta_I = 0$ results of Figs.11 and 12. Specifically, we evaluate $2Ia^2/I_pd^2$ from both figures and plot this quantity against b/a; we obtain the curves shown in Fig.14. As described in Section 2, the figure is interpretable in two ways. Using a rail-limiter, the points B are <u>not</u> to be regarded as bifurcation points. Using a point-limiter, however, the points B denote a second class of bifurcation, the b/a values for which, are profile-dependent. Given I_p, we observe that the point B for each profile implies a corresponding upper-bound on the value of "a". For a point limiter, this sets a bound on the limiter's position.

Finally, maintaining the same current profiles as for $\beta_I = 0$, we have repeated the runs shown in Fig.14 but for increased poloidal-beta. Our results, which are presented in Fig.15, indicate the value of β_I to be relatively unimportant for the point limiter. Once again we find that fixing b and I_p leads to a unique solution.

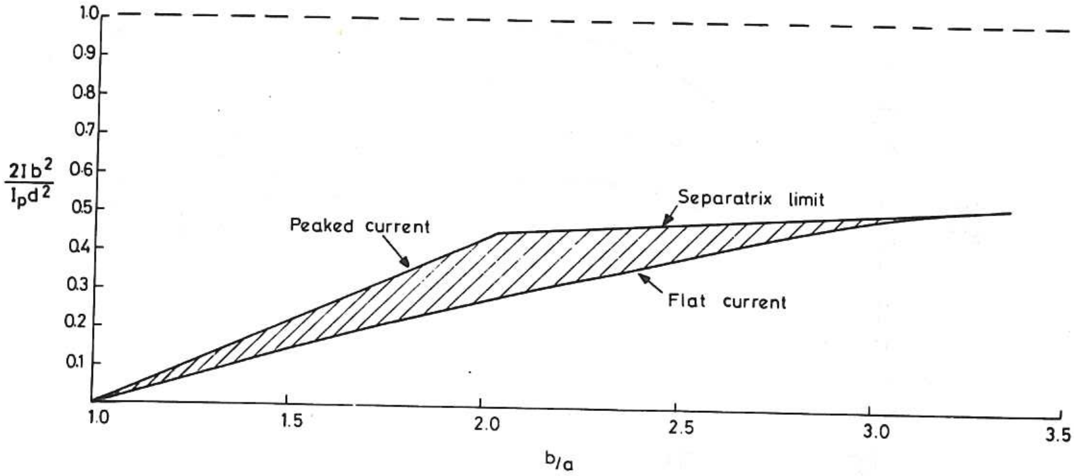

Fig.13 Plots of $\frac{2Ib^2}{I_pd^2}$ against b/a obtained from Figs.11 and 12. All results for mixed current profiles lie in the hatched region.

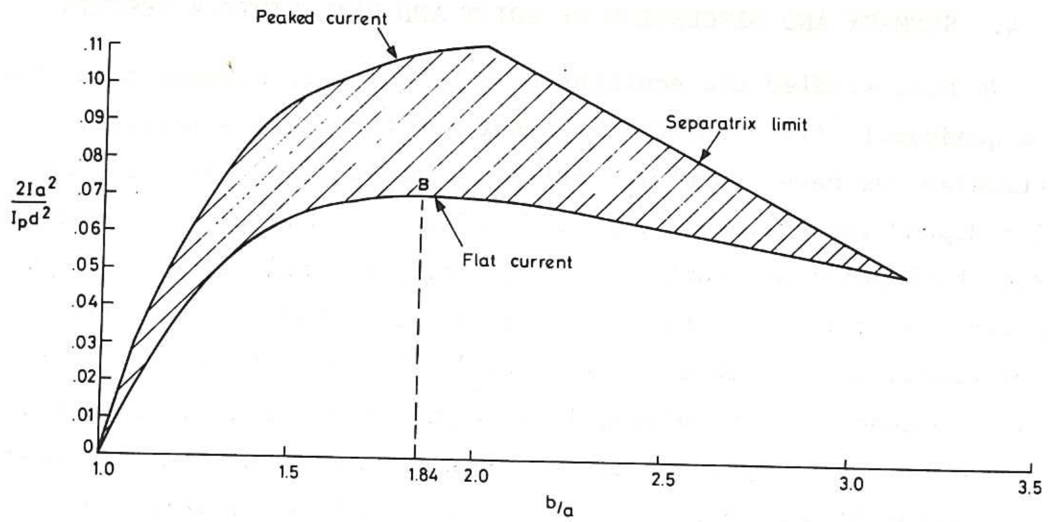

Fig.14 Plots of $\frac{2Ia^2}{I_pd^2}$ against b/a obtained from Figs.11 and 12. All results for mixed current profiles lie in the hatched region.

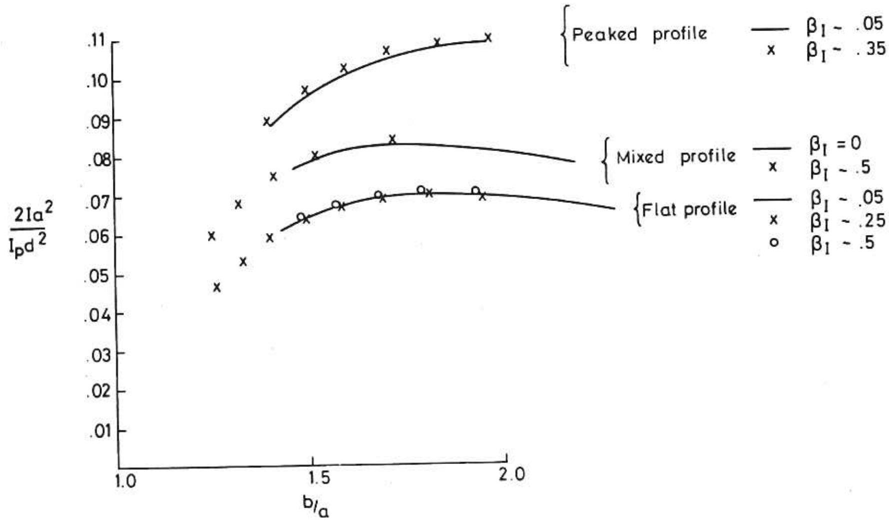

Fig.15 Plots of $\frac{2Ia^2}{I_pd^2}$ against b/a for various current profiles and different β_I, as obtained from Figs.11 and 12.

4. SUMMARY AND DISCUSSION OF POINT AND RAIL LIMITER RESULTS

We have studied the equilibrium properties of plasmas maintained by a quadrupole field in both straight and toroidal geometries. In particular, we have restricted ourselves to longitudinal currents which depend linearly on the poloidal-flux ψ. Although we have discussed both point and rail-limiters, the principal numerical work has been for rail-limiters only. For the toroidal calculations, it is necessary to introduce current (I_W) carrying conductors over and above the quadrupole windings, if an equilibrium is to be established; such conductors are present in experiments like TOSCA. A comparison of the straight and force-free ($\beta_I = 0$) toroidal calculations show the results to be virtually indistinguishable.

For a quadrupole of given dimension, d, current I, and plasma aspect ratio 5, we have investigated two alternative schemes of solution: (a) we fix the total plasma current, I_p, and I_w, and stride through a succession of prescribed limiter values, or (b) we fix the limiter and stride through pairs of values for I_p and I_w. Plotting $2I/jd^2$ (j is defined in Eq.(7)) against b/a we have found that the two procedures lead to different results. For a chosen j, both schemes reveal, that in general, there are two values of b/a compatible with the equilibrium equations and boundary conditions. The bifurcation points which occur depend on both the form of current profile and β_I. With scheme (a) the bifurcations (Type I) run from b/a = 2.9 ('flat' current) to approximately 2.2 for a mixed profile; for the profile which vanishes at the boundary, the bifurcation point disappears through the intervention of the separatrix. For scheme (b), however, the bifurcation points (Type II) can approach unity; the limit of b/a = 1.0 is reached for profiles which vanish at the boundary. We note that the process whereby the bifurcation points vanish, is very different for the two schemes. We further note, that the flat-current is degenerate, in the sense that both schemes yield identical results for this profile.

It must be emphasised that the above types of bifurcation are observed when the results are analysed in terms of j. When the quantity $2Ib^2/I_pd^2$ is plotted against b/a, however, the resulting curves increase monotonically for all current profiles investigated. Thus we conclude that for all profiles described by our model, fixing a rail-limiter and prescribing the total plasma current gives a unique solution for the equilibrium.

Interpreting our results in terms of a point-limiter, that is, plotting $2Ia^2/I_pd^2$ against b/a, we again find that there are two values for b/a which satisfy the equilibrium. The bifurcation points (type III) run from approximately 2.1 to 1.75; for the flat current b/a = 1.8. These results are found to be very insensitive to changes in poloidal-beta.

We further note, that for our peaked profile (j_B = 0), which is probably the most realistic and for which we found no bifurcations, there is a maximum value of b/a. The latter is approximately 2.0, irrespective of β_I, for a plasma of aspect ratio 5.

We now relate the present work to that of earlier authors. Feneberg and Lackner [3] have published results for a 'flat' current profile in the belt pinch. Although apparently very different, asymptotic analysis for the straight analogue, shows their system of conductors to behave essentially as a quadrupole. This is supported by the fact that they found a bifurcation at $b/a \sim 2.9$. Using a fixed toroidal plasma current and varying the position of a rail-limiter, we have repeated their work, and find a bifurcation at $b/a \sim 1.9$; this clearly corresponds to our $b/a = 1.84$. Cenacchi et al [2] have discussed a toroidal, flat-current, circular cross-section plasma with a point-limiter; they show that for convergence of an iterative scheme such as our own, the limiter must be placed on the inside of the torus.

Computations using the point limiter have been performed for the uniform current. It has been found that only equilibria with $b/a < 1.75$ are computable, and that the iterations diverge when we seek equilibria with $b/a > 1.75$. With the point-limiter, it is clear from Eq.(4) that $\frac{\partial}{\partial I}(b/a) \to \infty$ as $b/a \to 1.84$, and hence convergence difficulties must be expected. It is for this reason that extensive computations using a point-limiter have not been carried out. A similar convergence phenomenon was found in the preliminary calculations reported by Papaloizou et al [5], where using a rail-limiter, equilibria with $b/a > 2.9$ were not computable. These two observations suggest that if particular constraints produce a formulation in which the solution is not unique, then using standard numerical methods, the range of computable solutions will always be limited.

Papaloizou et al [5] have considered a plasma with a δ-function current distribution. They found no bifurcation and showed the maximum value of b/a to be 1.95; this corresponds very closely with $b/a = 2.1$ for the peaked current used in the present paper. It has been suggested [5] that there is a critical profile above which bifurcation cannot occur. Since bifurcation is very dependent on the form of profile, further investigation is required to elucidate this feature. We note in passing, that Field and Papaloizou [15] have considered a straight cylindrical vacuum-plasma configuration with $j_z \propto \psi$, and maintained by a perfectly conducting wall. They find the solution to be unique for $j_z = 0$ at the boundary.

5. RECTANGULAR-LIMITER

In producing vertical elliptic plasmas, TOSCA, which has a point limiter, shows two distinct types of configuration; the plasma either contacts the limiter or it touches the enclosing steel vacuum vessel at top and bottom. This suggests that we reconsider our analysis of Section 2, taking account of the above observations. Thus we investigate the equilibrium of a straight plasma with uniform current in the presence of a rectangular limiter. In fact, Toyama et al. [9] have carried out experiments using such a limiter. Defining $E = B/A$, where A and B are the half-width and height of the limiter, respectively, then for $b/a \leqslant E$ the plasma touches the sides, that is $a = A$; for $b/a \geqslant E$, however, the plasma touches the top and bottom, that is $b = B$. Now we have shown that all equilibria must satisfy $2Ia^2/I_p d^2 \leqslant 0.067$, and therefore for the plasma to touch the sides

$$I_p \geqslant \frac{2IA^2}{0.067\, d^2} \tag{9}$$

From Fig. 3 and Eq. (2), for a plasma to touch the top and bottom of the limiter, we must have

$$I_p \leqslant \frac{2IB^2}{d^2E^2}\, \frac{(1 + E)(1 + E^2)}{(E - 1)} \; . \tag{10}$$

Combining Eqs. (9) and (10), we see that the complete range of b/a is permissible provided that

$$0.067 \geqslant \frac{E - 1}{(1 + E)(1 + E^2)} \; . \tag{11}$$

As can be inferred from Fig. 5 and Eq. (2), the above condition is always satisfied.

We now consider the implications of the above discussion; there are two separate cases, namely, $E \leqslant 1.84$ and $E \geqslant 1.84$. Given I, d and I_p, and a limiter with $E \leqslant 1.84$, we examine the uniqueness properties of the solution. From the arguments of Section 2, there is one solution with $b = B$ and one solution with $a = A$. We must ascertain whether both these solutions can occur (non-uniqueness), or whether they represent one solution in different regimes of equilibrium. Thus by Eqs.(2) and (4)

$$\frac{1}{B^2}\,\frac{\left(\frac{B}{a}\right)^2\left(\frac{B}{a}-1\right)}{\left(1+\frac{B}{a}\right)\left(1+\frac{B^2}{a^2}\right)} = \frac{1}{A^2}\,\frac{\left(\frac{b}{A}-1\right)}{\left(1+\frac{b}{A}\right)\left(1+\frac{b^2}{A^2}\right)} . \tag{12}$$

Setting $b = B$

$$\frac{\left(\frac{B}{a}\right)^2\left(\frac{B}{a}-1\right)}{\left(1+\frac{B}{a}\right)\left(1+\frac{B^2}{a^2}\right)} = \frac{E^2(E-1)}{(1+E)(1+E^2)} . \tag{13}$$

Since the left-hand side of Eq. (13) is a monotonically increasing function of $1/a$, there is only one solution, namely, $a = A$. Similarly, setting $a = A$, Eq. (12) gives

$$\frac{E-1}{(1+E)(1+E^2)} = \frac{\left(\frac{b}{A}-1\right)}{\left(1+\frac{b}{A}\right)\left(1+\frac{b^2}{A^2}\right)} . \tag{14}$$

This also has only one solution (see Eq. (4) and Fig. 5), which is $b = B$. Since consideration of Eq. (12) has led only to the degenerate solution $a = A$, $b = B$, we conclude that for a prescribed plasma current (I and d fixed) with a rectangular limiter such that $E \leqslant 1.84$, then the solution is unique. Whether or not the plasma touches the top or sides of the limiter depends on the magnitude of I_p.

Finally, we consider the case $E \geqslant 1.84$; the appropriate parts of Figs. 3 and 5 are shown in Figs. 16 and 17. For a given I_p such that

$$0.067 \geqslant \frac{2IA^2}{I_p d^2} \geqslant \frac{F(E)}{E^2} , \tag{15}$$

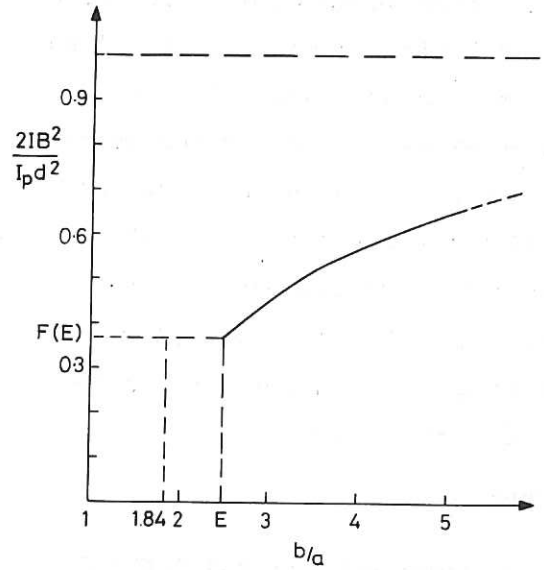

Fig.16 Plot of $\frac{2IB^2}{I_p d^2}$ against b/a showing the range of values of b/a for which the plasma boundary touches top and bottom of a rectangular limiter.

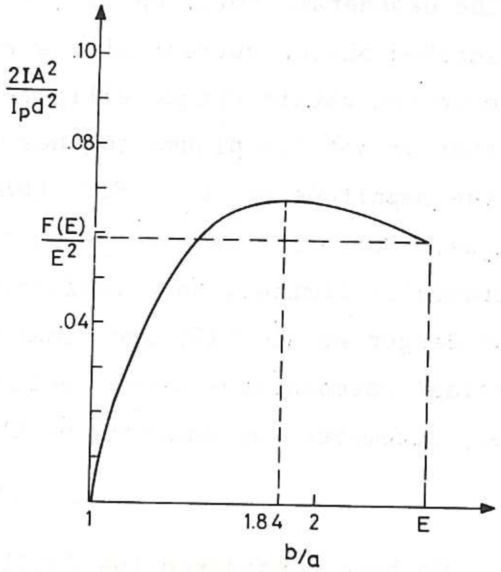

Fig.17 Plot of $\frac{2IA^2}{I_p d^2}$ against b/a showing the range of values of b/a for which the plasma boundary touches the sides of a rectangular limiter.

then there are two solutions which touch the limiter sides. Thus for $(b/A)_1 \leqslant b/A \leqslant E$ the solutions are not unique. For $B/a \geqslant E$ there is one solution which touches the top and bottom of the limiter, and for $b/A \leqslant (b/A)_1$ there is one solution touching the sides. Again we have to ascertain whether or not both these solutions can occur or whether they represent one solution in different regimes. Since we have

$$\frac{2IB^2}{I_p d^2} = \frac{\left(\frac{B}{a}\right)^2\left(\frac{B}{a} - 1\right)}{\left(1 + \frac{B}{a}\right)\left(1 + \frac{B^2}{a^2}\right)} \geqslant \frac{E^2(E-1)}{(1+E)(1+E^2)} \equiv F(E)$$

and

$$\frac{2IA^2}{I_p d^2} = \frac{\left(\frac{b}{A} - 1\right)}{\left(1 + \frac{b}{A}\right)\left(1 + \frac{b^2}{A^2}\right)} \leqslant \frac{E-1}{(1+E)(1+E^2)} \equiv \frac{F(E)}{E^2} \qquad (16)$$

then

$$\frac{2IB^2}{I_p d^2} \geqslant F(E) \geqslant \frac{2IB^2}{I_p d^2} . \qquad (17)$$

Since (17) can only be satisfied by the equality signs, we are again led to the degenerate solution $a = A$, $b = B$. Thus we conclude that for a prescribed plasma current with a rectangular limiter such that $E \geq 1.84$, then we can obtain unique ellipses with $b/a \leq (b/A)_1$ and $b/a \geq E$; whether or not the plasma touches the top or sides of the limiter depends on the magnitude of I_p. For $(b/A)_1 \leq b/a \leq E$ the solutions are not unique. Computations for the straight, uniform current plasma with a rectangular limiter, have confirmed our analytic conclusions. We note that Berger et al. [10] and Cenacchi et al. [2] have given examples of toroidal calculations using rectangular limiters. They have not, however, attempted any analysis of the type presented here.

6. CONCLUSIONS

We have considered the equilibria of plasmas with simple current profiles and maintained by a quadrupole field. For the straight uniform-current, model, we have used asymptotic analysis to investigate the roles of point and rail-limiters. We have found that prescribing the plasma current, I_p, with a rail-limiter always ensures uniqueness, whereas prescribing I_p with a point-limiter does not. Numerical work confirms that the same conclusions can be drawn for non-uniform currents. Furthermore, computations reveal the existence of three types of bifurcation. The corresponding values of b/a depend on the current profile and poloidal-beta. For non-vanishing current at the boundary the bifurcation b/a can approach unity. For a vanishing current however, bifurcation disappears through the intervention of the separatrix; there is then a maximum value of b/a attainable, which for an aspect ratio of 5 (TOSCA), is approximately 2.0 irrespective of poloidal-beta. Finally, we have investigated the straight uniform-current plasma in the presence of a rectangular-limiter. Asymptotic analysis shows that there are two cases to be considered: (a) $E \leq 1.84$ and (b) $E \geq 1.84$, where E is the ratio of height to width of the limiter. For $E \leq 1.84$, prescription of E and I_p ensures uniqueness. Whether or not the plasma touches the sides or top and bottom of the limiter, depends on the magnitude of I_p. For $E \geq 1.84$, however, there is a range of b/a for the plasma for which the solutions are not unique.

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APPENDIX

CONDUCTOR CONFIGURATION AND NUMERICAL WORK

The filamentary conductors used in the TOSCA simulations are shown in Fig. 18. The plasma current, I_p, is chosen to be negative, and the four vertical field winding currents are of the same value, namely, I_w. The latter currents correspond to the vertical field, as calculated from Shafranov's formula [11], required to maintain a centred circular plasma in equilibrium. The quadrupole currents, I, I_1, I_2, are selected such that the quadrupole field vanishes at the centre of the plasma. Experimentally, this ensures that the plasma is not moved sideways as the quadrupole windings are energised. Since the decay index of the vacuum field is negative for $b/a > 1.05$, we would expect TOSCA (in the absence of feedback) to be unstable to axisymmetric modes [12], and this has been confirmed experimentally.

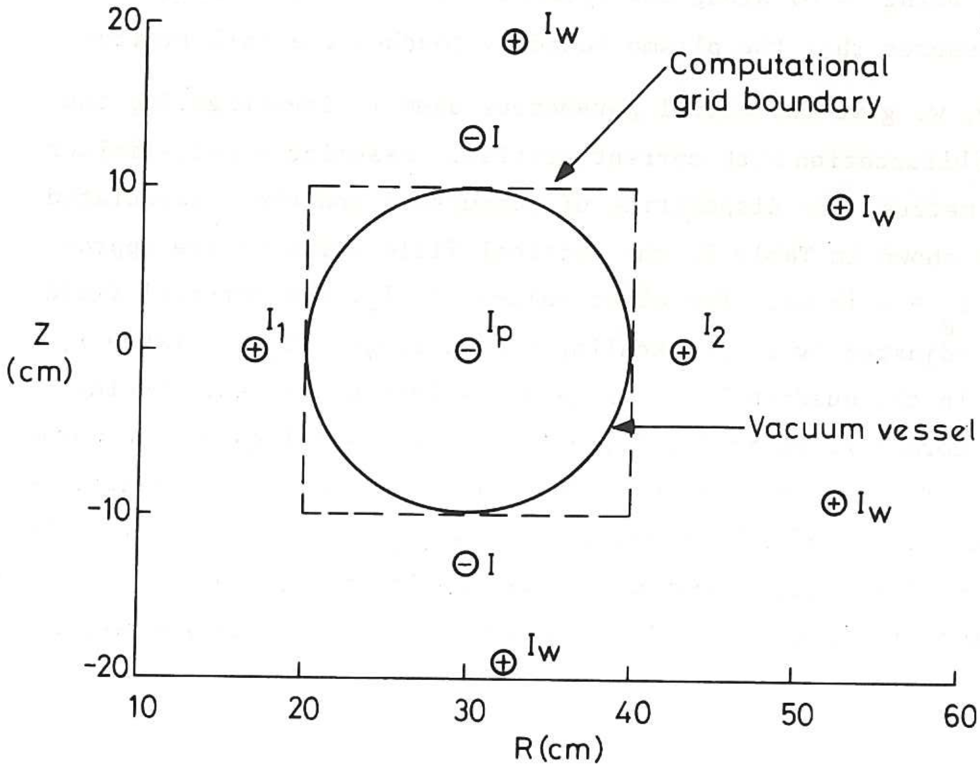

Fig.18 Disposition of conductors as used in computations to simulate TOSCA.

The computations are performed using the discrete conductors MHD equilibrium codes as implemented at Culham, see [4] and [13]. These codes work in a manner similar to that described by Lackner [14]. The Bunemann algorithm is used to solve Poisson's equation and the boundary conditions are computed by Lackner's Green's function technique. As pointed out by Cenacchi et al [2], symmetry about $z = 0$ has to be imposed in order to obtain convergence for significant values of b/a. The initial shape of the plasma is always taken as a circle although this is not essential for convergence. Two constraints are always used - a limiter (point or rail) is introduced and the plasma current has a specified value. As confirmed by Cenacchi et al [2] the point limiter, if used, has to be on the inside of the plasma (on $z = 0$). At each iteration, the flux at the plasma boundary is defined as the flux at the point limiter position and the plasma boundary is the contour ψ = constant which passes through this point. When a rail limiter is used, the flux at the plasma boundary is defined as the minimum value of the flux (I_p being < 0) along the specified z = constant plane. This definition ensures that the plasma boundary touches the rail limiter.

Finally, we give the actual parameters used in investigating the variation of bifurcation with current profile. Assuming a rail-limiter at $z = 0.05$ metres, the disposition of conductors and their associated currents are shown in Table I; the vertical field currents are appropriate for $I_p = - 18$ kA. For other values of I_p, the vertical field is suitably adjusted by simply scaling the currents, I_w in Table I, the current in the quadrupole windings being left unchanged. Taking the current form defined in Eq.(7), a few typical settings of the parameters M, N, P and Q are listed in Table II, together with the corresponding values of b/a for bifurcation ($(b/a)_{crit}$) and $I_{p\ crit}$. With one exception, the results quoted are for the force-free case ($P = Q = 0$). For $P = 7.0 \times 10^2$ and $Q = - 10^7$ the poloidal-beta is approximately 0.5.

Table I Details of Conductor Configuration

R(m)	Z(m)	I(kA)	
.525	.09	4.05	(I_w)
.325	.19		
.325	- .19		
.525	- .09		
.3	.13	- 2.5	(I)
.3	- .13		
.17	0	3.06875	(I_1)
.43	0	1.93125	(I_2)

Table II Variation of Bifurcation Point ($(b/a)_{crit}$) as a function of current profile ($I_{p\ crit}$ is within 0.25 kA of the true value, results having been obtained by fitting a quadratic between 3 values close to the bifurcation point).

M	N	P	Q	$I_{p\ crit}$	$(b/a)_{crit}$
1	0	0	0	- 1.5 kA	2.89
$3 \cdot 10^{-4}$	- 1	0	0	- 1.75 kA	2.65
$2 \cdot 10^{-4}$	- 1	0	0	- 1.75 kA	2.54
10^{-4}	- 1	0	0	- 2.25 kA	2.07
$9 \cdot 10^{-5}$	- 1	0	0	- 2.5 kA	1.97
$8 \cdot 10^{-5}$	- 1	0	0	- 2.75 kA	1.82
$7 \cdot 10^{-5}$	- 1	0	0	- 3.5 kA	1.60
$7 \cdot 10^{-5}$	- 1	$7 \cdot 10^{2}$	$- 10^{7}$	- 6.25 kA	1.32
$6.5 \cdot 10^{-5}$	- 1	0	0	- 4.5 kA	1.45
0	- 1	0	0	no bifurcation	

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