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DYNAMICS OF HIGH-β TOKAMAKS WITH ANISOTROPIC PRESSURE

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ABSTRACT

On the basis of a high-β, long wavelength ordering, the C.G.L. equations are used to discuss the dynamics and linear stability of a general anisotropic, plasma-vacuum Tokamak. The multiple time-scale method is used to derive a reduced set of non-linear M.H.D. equations. To lowest-order, the perpendicular component of pressure, $p_\perp$, is not necessarily constant on the flux-surfaces, ψ. We give a simple example of such an equilibrium, and a heuristic treatment of the Fokker-Planck equation shows that equilibria of this type can only be established by near-perpendicular injection. Comparison with the Kruskal-Oberman energy principle shows that the C.G.L. principle never overestimates stability. For $\bar{p} = \frac{1}{2}(p_\parallel + p_\perp)$ constant on flux-surfaces, the M.H.D. linear stability of an anisotropic tokamak to long wavelength modes, is identical (within the ordering) to that for the equivalent scalar pressure tokamak. A similar result has recently been obtained in the limit of short wavelengths for fixed-boundary modes (ballooning) [20].

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I. INTRODUCTION

This paper is concerned with the gross dynamics of a high-β, anisotropic pressure tokamak. In particular, we derive a set of reduced non-linear magnetohydrodynamic equations of motion in the long wavelength approximation. A thorough investigation of these equations will require extensive numerical computations. Here, we discuss only those features and results which can be obtained analytically.

The present work has been stimulated by the current interest in neutral injection. Apart from its use as an additional heat source in conventional experiments, neutral injection has recently been proposed as a method for "pumping-up" β in the flux-conserving tokamak concept [1]; it is also fundamental to the counterstreaming-ion tokamak ($p_{\parallel} \gg p_{\perp}$) [2]. These applications lead to general questions concerning the gross dynamics and stability of anisotropic high-pressure tokamaks.

Our analysis is based on the single-fluid M.H.D. equations as derived from the Vlasov equation by Chew, Goldberger and Low [3] in the small Larmor radius limit. Use of the C.G.L. equations implies neglect of energy transport parallel to the magnetic field, and would appear to be justified only for rapid motions growing on timescales short compared to a typical trapped-particle bounce-period [4]. However, we show that within the ordering to be described, linear stability on the basis of C.G.L. generally implies stability in the more refined Kruskal-Oberman theory [5]. Thus, as far as nearly-marginal displacements are concerned, the C.G.L. theory does not over-estimate stability; this suggests that the C.G.L. equations be used to describe long-wavelength motions in general.

Defining a and R_o to be respectively the typical minor and major radii of the plasma, the inverse aspect-ratio $\epsilon = a/R_o$ is assumed to be small. Then the time t_o in which a magnetosonic wave traverses the minor cross-section is short compared to the time t_1, ($\sim t_o/\epsilon$) for a shear Alfvén wave to complete a circuit of the torus. This separation in timescales may be used to simplify the equations of fluid motion in a large aspect-ratio tokamak. In the case of low plasma pressure, with $\beta \sim \epsilon^2$, the scalar pressure M.H.D. equations reduce to a set involving only two field variables, namely, the magnetic field and fluid flow, the pressure having been eliminated from leading order [6]. By adopting the high-β ($\sim \epsilon$) tokamak ordering, Strauss [7] has incorporated the effects of scalar pressure into the reduced equations. Following the general ordering procedure of ref. [7], we set $p_\perp \sim p_\parallel \sim \epsilon B^2$, and for completeness, include a vacuum region. In view of the wide separation between the "fast" and "slow" characteristic times t_o and t_1, we use the multiple time-scale method to solve the anisotropic fluid equations.

On the fast scale the equations of motion yield a quasi-static straight "theta-pinch" configuration where the dia-magnetic part of the toroidal field maintains pressure balance. Perturbation of equilibrium on this time-scale shows it to be marginally stable to incompressible modes. The dynamical development of these modes must be investigated on the slow time-scale. At this order the poloidal magnetic field and toroidicity influence the motion, and we derive a set of reduced equations for both the plasma and vacuum region, together with boundary conditions appropriate to the plasma-vacuum interface and the enclosing perfectly conducting wall. The plasma equations are the same as in the isotropic case, but with scalar pressure replaced by an effective pressure $\bar{p} = \frac{1}{2}(p_\perp + p_\parallel)$. There is, in fact, an interesting difference from the scalar-pressure case. In the latter, if the pressure, p, is initially constant on a flux-surface, then it remains so to the appropriate order. For the anisotropic case we find an analogous result for $p_\parallel$, and this ensures that there is no toroidal acceleration of the plasma; for $p_\perp$ it is not possible to deduce such a result. We shall show later, however, that for $p_\perp$ to vary round flux-surfaces, the particle distribution function must be strongly anisotropic. In particular, if we suppose that such anisotropy in velocity space results from neutral injection, then leading-order variation of $p_\perp$ round flux-surfaces can only arise for large angles of injection relative to the magnetic field. Thus, except for this special case, $\bar{p}$ will indeed be constant on flux surfaces at all times, so that the plasma behaves as if the pressure were scalar and equal to $\bar{p}$. In these circumstances, we can immediately infer, that within the long wavelength, high-β tokamak ordering, the equilibrium and stability properties of anisotropic systems are identical with those for scalar pressure.

The reduced equations possess an energy integral and linearisation in the flow variable leads to an energy principle (δW_{CGL}) for small displacements. The same result can be obtained by applying our long-wavelength high-β ordering directly to the general C.G.L. energy principle; for wall-on-plasma the result is the same as that derived by Strauss [7], but with scalar pressure replaced by $\bar{p}$. It is of considerable interest to directly apply our ordering procedure to the more rigorously based energy principle of Kruskal and Oberman (δW_{KO}), and then to compare the result with δW_{CGL} . In the event of perpendicular injection leading to a strongly anisotropic distribution in the trapped particle band, then we have been unable to draw any definite conclusion. However, if the trapped band is, at most, weakly anisotropic, then to leading-order we demonstrate that

$$\delta W_{CGL} \leqslant \delta W_{KO} \ ,$$

where the equality sign refers to equilibria for which $\bar{p}$ is constant on flux-surfaces. Thus for the present ordering, together with the proviso on trapped particles, the C.G.L. principle gives the correct marginal stability for $\bar{p}$ constant on flux-surfaces; for $\bar{p}$ varying round flux-surfaces, the C.G.L. principle gives an under-estimate of stability.[1]

In section II of our paper, we describe the basic notation, equations and ordering. Section III contains the derivation of the reduced equations and the boundary conditions. In section IV we obtain the equilibrium equations and describe a particular example with $p_\perp$ varying round flux-surfaces; the circumstances under which neutral injection can produce such a situation are also discussed. In

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[1]It might appear that the above inequality is inconsistent with the inequalities given by Rosenbluth and Rostoker [4] for scalar pressure equilibria, namely,

$$\delta W_{MHD} \leqslant \delta W_{KO} \leqslant \delta W_{CGL} \ .$$

In the long wavelength high-β approximation, however, their equality signs now become appropriate, as does the equality in our result for the case of scalar pressure.

section V we discuss linear stability and compare the C.G.L. and Kruskal-Oberman energy principles. Section VI states our conclusions.

II. NOTATION, BASIC EQUATIONS AND ORDERING

The plasma-vacuum configuration is shown in Fig.1. We use rectangular coordinates x,y centred on the point O , which is distance R_o from the major axis of the torus. Position in major azimuth is determined by the angular coordinate φ . The inverse aspect-ratio $\epsilon(\sim x/R_o)$, is assumed small, and is our basic expansion parameter. We suppose that the plasma has a sharp boundary Γ , which in general, is separated from the perfectly-conducting, axisymmetric wall, W , by a vacuum region. The trajectory of a point tied to the fluid at the plasma-vacuum interface, with initial position vector $\underset{\sim}{\rho}_b$, will be denoted by $\underset{\sim}{r}_b(t|\underset{\sim}{\rho}_b)$. Then from the kinematics of the interface

$$\frac{d\underset{\sim}{r}_b}{dt} = \underset{\sim}{v}(\underset{\sim}{r}_b, t) \tag{1}$$

where $\underset{\sim}{v}$ is the fluid velocity. The family of such trajectories describes the motion of the plasma surface.

The anisotropic M.H.D. equations are

$$\rho\left(\frac{\partial \underset{\sim}{v}}{\partial t} + (\underset{\sim}{v}\,.\,\nabla)\underset{\sim}{v}\right) = -\nabla\,.\,\underset{=}{p} + j \times \underset{\sim}{B} \tag{2}$$

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{v} \times \underset{\sim}{B}) \tag{3}$$

$$\nabla \times \underset{\sim}{B} = \underset{\sim}{j} \tag{4}$$

$$\nabla\,.\,\underset{\sim}{B} = 0 \tag{5}$$

$$\frac{\partial \rho}{\partial t} + \nabla\,.\,(\rho\underset{\sim}{v}) = 0 \tag{6}$$

where the pressure tensor $\underset{=}{p}$ is of the form

$$\underset{=}{p} = \underset{=}{I}p_\perp + \underset{\sim}{b}\underset{\sim}{b}(p_\parallel - p_\perp) ,$$

and

$$\underset{\sim}{b} = \underset{\sim}{B}/|B| \quad .$$

The components of $\underline{\underline{p}}$ are determined from the Chew, Goldberger and Low [3] equations of state, namely

$$\frac{\partial p_\perp}{\partial t} + \underset{\sim}{v} \cdot \nabla p_\perp = - p_\perp \{2\nabla . \underset{\sim}{v} - [\underset{\sim}{b} \cdot \nabla) \underset{\sim}{v}] \cdot \underset{\sim}{b}\} \tag{7}$$

and

$$\frac{\partial p_\parallel}{\partial t} + \underset{\sim}{v} \cdot \nabla p_\parallel = - p_\parallel \{\nabla . \underset{\sim}{v} + 2[(\underset{\sim}{b} \cdot \nabla) \underset{\sim}{v}] \cdot \underset{\sim}{b}\} \ . \tag{8}$$

These equations are only valid provided that transport of parallel and perpendicular energy along the magnetic field can be neglected. As we shall see later, however, we only require Eqs. (7) and (8) to hold up to $O(\epsilon^2)$.

In order to model conditions in a tokamak, the poloidal magnetic field $\underset{\sim}{B}_\perp = \hat{\varphi} \times (\underset{\sim}{B} \times \hat{\varphi})$ is taken to be smaller than the toroidal component by a factor of order ϵ . We adopt the high-β ordering $p_\perp/B^2 \sim p_\parallel/B^2 \sim \epsilon$, and as a consequence, the toroidal field contains a diamagnetic term in first order; the density ρ is taken to be of zeroth order. Normalising the plasma flow velocity $\underset{\sim}{v}$ relative to the characteristic Alfvén velocity, we take $|\underset{\sim}{v}| \sim \epsilon$, an ordering which we shall show to be self-consistent. Our analysis requires that the flow velocity be expanded as

$$\underset{\sim}{v} = \underset{\sim}{v}^{(1)} + \underset{\sim}{v}^{(2)} + \ldots\ldots$$

where the superscripts denote the order in ϵ . The components of pressure and magnetic field are similarly expanded. It proves convenient to write

$$B_\varphi = (I_o + I)/R \ ,$$

where the field I_o/R is generated by the external winding currents, I_o being constant, and I is due to diamagnetic currents. With the ordering chosen for $p_\perp$, I is of first order relative to I_o . We shall use a suffix V to denote values of a function in the vacuum region. In parallel with the expansion of field quantities, we must also expand the differential operators. Restricting the present analysis to long-wavelength motions, we assume

$$\frac{1}{R_o}\frac{\partial}{\partial\varphi} \sim 0\left(\epsilon\frac{\partial}{\partial x}\, ,\ \epsilon\frac{\partial}{\partial y}\right) ,$$

and, for example, the gradient operator can be written

$$\nabla = \nabla^{(0)} + \nabla^{(1)} + \nabla^{(2)} \ldots .$$

where

$$\nabla^{(0)} \equiv \nabla_\perp = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\, ,\ \nabla^{(1)} = \hat{\varphi}\,\frac{1}{R_o}\frac{\partial}{\partial\varphi}\, ,\ \text{etc.}$$

The divergence and curl operators are similarly expanded.

Magnetosonic fluctuations occurring on the fast time-scale are compressible and stable. On the long time-scale, however, incompressible motions develop and can lead (in the presence of dissipation) to the evolution of a toroidal equilibrium, and possible subsequent long wavelength instability. The difference in time-scales suggests use of the multiple time-scale formalism. Thus, we replace direct functional dependence on a single time variable t (normalised with respect to t_o) by dependence on a sequence of variables $\{\tau_o , \tau_1 , \ldots .\}$, each of which depends parametrically on t according to

$$\frac{d\tau_o}{dt} = 1\, , \qquad \frac{d\tau_1}{dt} = \epsilon\, ,\ \ldots\, , \qquad \frac{d\tau_n}{dt} = \epsilon^n\, .$$

Then for any function f ,

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial\tau_o} + \epsilon\frac{\partial f}{\partial\tau_1} + \epsilon^2\frac{\partial f}{\partial\tau_2} \ldots$$

Clearly, changes on the t_o and t_1 time-scales are reflected in the τ_o and τ_1 dependences.

III. REDUCED EQUATIONS AND BOUNDARY CONDITIONS

(a) Reduced Equations

The pressure tensor $\underline{\underline{p}}$ gives rise to the fluid volume force

$$\nabla\cdot\underline{\underline{p}} = \nabla p_\perp + \underline{b}[\underline{b}\cdot\nabla(p_\parallel - p_\perp)] + (p_\parallel - p_\perp)\ \nabla\cdot(\underline{b}\underline{b})$$

where $\underline{b}$ is the unit vector along the magnetic field. Using our ordering we expand Eqs. (2)-(6), and in lowest order obtain the fast-timescale momentum equation

$$\rho \frac{\partial \underset{\sim}{v}}{\partial \tau_o} = - \nabla_\perp (p_\perp + I_o I/R_o^2) \tag{9}$$

and induction equation

$$\left(\frac{\partial I}{\partial \tau_o}\right) = - I_o \nabla_\perp \cdot \underline{v} \, . \tag{10}$$

In order not to complicate the notation unduly, we have omitted the first-order superscripts from $\underline{v}$, $\underline{\underline{p}}$ and I . Similarly, ρ denotes the zeroth-order density, which by Eq. (6) is constant on the τ_o-timescale. From Eqs. (7) and (8) we deduce that

$$\frac{\partial p_\perp}{\partial \tau_o} = \frac{\partial p_\parallel}{\partial \tau_o} = 0 \tag{11}$$

For the vacuum region, Eq. (4) leads to

$$\nabla_\perp I_v = 0 \tag{12}$$

in first-order, where as before we omit the superscript from I_v . In the absence of skin currents, I is continuous with I_v at the lowest-order interface; this implies that $p_\perp$ must vanish at the interface. For equilibrium on the fast-timescale $\partial/\partial \tau_o = 0$, so that from Eqs. (9) and (10)

$$p_\perp + \frac{I_o I}{R_o^2} = P(\tau_1, \ldots |\varphi) \tag{13}$$

and

$$\nabla_\perp . \underline{v} = 0 \tag{14}$$

Applying Eq. (13) to the interface, and making use of the continuity in I , we obtain

$$I_v = \frac{R_o^2}{I_o} P(\tau_1, \ldots |\varphi) \, , \tag{15}$$

which is compatible with the fact, that according to Eq. (12), I_v does not vary in the poloidal plane.

We now consider the response of the above equilibrium to small perturbation on the fast time-scale. We find the subsequent motion to be compressible and to correspond to stable magnetosonic oscillations of frequency $\omega^2 \sim B^2/\rho a^2$, where "a" is the characteristic minor radius. To proceed further we make the physically-reasonable assumption that such modes, if excited, have amplitudes very much smaller than those of the slow-timescale flows. Thus, at all stages in the slow evolution, the conditions of fast-flow equilibrium, namely Eqs. (13) and (14), will apply.

The dynamical equations for the slow-timescale motion are obtained in next order. We introduce a flux function ψ for the first-order poloidal magnetic field so that

$$B_\perp = \nabla_\perp \psi \times \hat{\varphi} ,$$

and making use of Eq. (14), we define a stream-function U for the poloidal flow in first order, such that

$$\underset{\sim}{v} = \nabla U \times \hat{\varphi} + v_\varphi^{(1)} \hat{\varphi} \quad .$$

By removing terms secular in τ_o from the second-order momentum equation, and annihilating the transverse gradient operator with $\hat{\varphi}.\nabla x$, we obtain

$$\rho \frac{d}{d\tau_1} \nabla_\perp^2 U + \nabla\rho.\nabla \frac{\partial U}{\partial \tau_1} - \hat{\varphi}.\nabla\rho \times \left\{ \frac{1}{2} \nabla_\perp (\nabla_\perp U)^2 - \nabla_\perp^2 U \; \nabla_\perp U \right\}$$
$$= (\underline{B}.\nabla)^{(1)} \nabla_\perp^2 \psi + \frac{2}{R_o} \frac{\partial \overline{p}}{\partial y} \tag{16}$$

where

$$\frac{d}{d\tau_1} = \frac{\partial}{\partial \tau_1} + \underline{v}^{(1)}.\nabla_\perp , \quad \text{and} \quad (\underline{B}.\nabla)^{(1)} = \underline{B}_\perp^{(1)}.\nabla_\perp + \frac{B_o}{R_o} \frac{\partial}{\partial \varphi} \quad .$$

The equation of continuity in first order, is

$$\frac{d\rho}{d\tau_1} = 0 \tag{17}$$

From the transverse components of the induction equation, Eq. (3), removal of secularity in τ_o at the second order, leads to

$$\frac{\partial \psi}{\partial \tau_1} = (\underline{B}.\nabla)^{(1)} U \tag{18}$$

where we have chosen an appropriate time-dependent gauge for ψ .

Similarly, the toroidal component of Eq. (3) leads to

$$\frac{dI}{d\tau_1} = - I_o \left\{ (\nabla.\underline{v})^{(2)} - \frac{2v_x^{(1)}}{R_o} - \frac{1}{B_o} (\underline{B}.\nabla)^{(1)} v_\varphi^{(1)} \right\} \qquad (19)$$

As observed by Strauss [7], this is in fact an equation determining $\nabla.\underline{v}$ to lowest non-vanishing order, for by axial magnetic flux conservation, it follows that

$$\frac{dP}{d\tau_1} = \frac{d}{d\tau_1} \left(p_\perp + \frac{I_o I}{R_o^2} \right) = 0 \qquad (20)$$

Equations (7) and (8) lead to

$$\frac{dp_\perp}{d\tau_1} = 0 \qquad \text{and} \qquad \frac{dp_\parallel}{d\tau_1} = 0 \qquad (21)$$

in second order. Using the first of these it follows that

$$\frac{dI}{d\tau_1} = 0 \ .$$

Thus Eq.(19) determines $\nabla . \underline{v}$ up to second-order. A consequence of Eq.(20), is that for initially axisymmetric configurations (of interest here), P is constant. Thus, redefining I_o if necessary, we can set $P \equiv 0$ without any loss of generality.

Making use of the dynamical equilibrium condition, Eq.(13), the toroidal component of the equation of motion yields

$$\rho \frac{dv_\phi^{(1)}}{d\tau_1} = - (\underline{B} . \nabla)^{(1)} p_\parallel \qquad (22)$$

Following the discussion for scalar pressure in ref.[7], we can make use of Eqs.(18), and (21) to show that

$$\frac{d}{d\tau_1} [(\underline{B} . \nabla)^{(1)} p_\parallel] = 0 \qquad (23)$$

Equations (22) and (23) reveal an important property of the first order equilibrium necessary for consistency in the ordering of the toroidal flow: if initially, $(\underline{B} . \nabla)^{(1)} p_\parallel \neq 0$, a fluid element will experience a constant toroidal acceleration, so that on the τ_1 - timescale v_ϕ will grow without limit. Thus, it is a necessary condition that

$$(\underline{B} \cdot \nabla)^{(1)} p_{\parallel} \equiv 0 \tag{24}$$

We note, however, that although

$$\frac{d}{d\tau_1} [(\underline{B} \cdot \nabla)^{(1)} p_{\perp}] = 0 \ , \tag{25}$$

there is no '$p_{\perp}$' counterpart to Eq.(22), and thus $(\underline{B} \cdot \nabla^{(1)}) p_{\perp}$ is not necessarily zero; we shall return to the consequences of this feature later in the paper. If, as we shall assume, there is no initial toroidal motion, then $v_{\phi}^{(1)} \equiv 0$ on the slow-timescale, and U completely determines the first-order flow. Finally, since the vacuum magnetic field is curl-free, we have

$$\nabla_{\perp}^2 \psi_v = 0 \ . \tag{26}$$

In summary, Eqs.(16), (17), (18) and (21), together with the supplementary condition Eq.(24), give a closed description of the plasma dynamics on the slow timescale.

(b) Boundary and Interface Conditions

Having prescribed the initial conditions, the subsequent motion is fully determined once the matching conditions at the interface, and the boundary condition at the conducting wall, are specified. We note that the motion of the zeroth-order interface, $\Gamma^{(o)}$, is given by Eq.(1), that is

$$\frac{d}{d\tau_1} \underline{r}_b^{(o)} = \underline{v}^{(1)}(\underline{r}_b^{(o)}, \tau_1) \tag{27}$$

As observed previously, in the absence of skin-currents, $p_{\perp}^{(1)}$ must vanish on $\Gamma^{(o)}$. In addition, we shall make the physically reasonable assumption that $p_{\parallel}^{(1)}$ vanishes on $\Gamma^{(o)}$. Thus, dropping superscripts

$$p_{\perp}(\underline{r}_b, \tau_1) = p_{\parallel}(\underline{r}_b, \tau_1) = 0 \tag{28}$$

By continuity of the poloidal magnetic field

$$\psi(\underline{v}_b, \tau_1) = \psi_v(\underline{r}_b, \tau_1) \ . \tag{29}$$

At the interface it only remains to prescribe the free-boundary condition for U. From the equation of motion, Eq.(2), and the kinematics of the interface, we deduce

$$\rho \frac{d}{d\tau_1} (\hat{n} \cdot \nabla_\perp U) = \tfrac{1}{2}\rho \hat{s} \cdot \nabla_\perp |\nabla_\perp U|^2 + (\hat{n} \cdot \underline{B}_\perp^{(1)}) \nabla_\perp^2 \psi \qquad (30)$$

for points on $\Gamma^{(o)}$, where $\hat{n}$ is the unit vector both normal to $\Gamma^{(o)}$ and lying in the poloidal plane; $\hat{s} = \hat{n} \times \hat{\phi}$ is tangential to the interface cross-section. The normal component of magnetic field vanishes at the conducting wall, and by an appropriate choice of gauge (consistent with that which led to Eq.(18)) we may write this condition as

$$\psi_v \Big|_{wall} = 0 \qquad (31)$$

Given the initial shape of the interface $\Gamma^{(o)}$, and initial values for the profiles of U, ψ, $p_\perp$ and $p_\parallel$, the latter satisfying Eq.(24), the subsequent motion is determined by Eqns (16), (17), (18), (21) and (26), subject to the matching conditions (27)-(30) and boundary condition (31). Note that U is obtained from a Poisson equation with a Neumann boundary condition determined by Eq.(30). An arbitrary ϕ-dependent part of U which is left undetermined, may be specified in any convenient manner; for example, by setting $U = 0$ along the geometric axis.

IV. EQUILIBRIUM ON SLOW-TIMESCALE

(a) MHD Equilibrium

Setting $\partial/\partial\tau_1 = 0$ and $U = 0$, we derive the condition for toroidal equilibrium. Thus from Eq.(16),

$$(\underline{B} \cdot \nabla)^{(1)} \nabla_\perp^2 \psi + \frac{2}{R_o} \frac{\partial \bar{p}}{\partial y} = 0 \ ,$$

where $p_\parallel$ must satisfy $(\underline{B} \cdot \nabla)^{(1)} p_\parallel = 0$. For the interesting case of axisymmetry, the equilibrium condition may be integrated once, and employing the usual (ψ, ϕ, χ) coordinates we obtain

$$\nabla_\perp^2 \psi + \frac{2x}{R_o} \frac{\partial \bar{p}}{\partial \psi} + \frac{\partial g}{\partial \psi} = 0 \qquad (32)$$

where

$$p_\parallel \equiv p_\parallel(\psi) \tag{33}$$

and the function $g(\psi,\chi)$ is undetermined, but must satisfy

$$\frac{\partial g}{\partial \chi} = \frac{-x}{R_o}\frac{\partial p_\perp}{\partial \chi} \tag{34}$$

For $p_\perp$ constant along $\underline{B}$, that is $p_\perp \equiv p_\perp(\psi)$, then $g \equiv g(\psi)$, and in this case Eq.(32) is just the $\beta \sim \varepsilon$ approximation of the Grad-Shafranov equation, with the scalar pressure $p(\psi)$ replaced by $\bar{p}(\psi)$. Eqs. (32)-(34) can of course, be directly obtained by an ε-expansion of the general axisymmetric equilibrium equations for anisotropic plasma [8]; the dynamical equilibrium condition, Eq.(13), can be similarly obtained. As for the scalar pressure case [7], we expect that the addition of some velocity damping to the equations of Section III, would give a simple and direct means of computing high-β toroidal equilibria.

We now turn to the question of equilibria with $p_\perp$ not constant along $\underline{B}$, that is, $p_\perp = p_\perp(\psi,\chi)$. In particular, we outline a simple analytic model of such an equilibrium. We choose the linear forms $p_\parallel = \varepsilon B_o^2\hat{p}_\parallel(\psi/\psi_o)$, and $p_\perp = \varepsilon B_o^2\hat{p}_\perp(1 + \alpha\,\frac{x}{a})(\psi/\psi_o)$ where ψ_o is a convenient normalisation for the flux function ψ; $\hat{p}_\parallel$, $\hat{p}_\perp$ and α are constants; and we require that ψ vanishes on the plasma boundary, which is of circular cross-section and radius 'a'. Then consistent with the choice of $p_\perp$ and Eq.(34), we may set

$$g = \varepsilon^2 B_o^2\,\frac{\psi}{\psi_o}\left(\hat{G} - \frac{1}{2}\,\alpha\hat{p}_\perp\left(\frac{x}{a}\right)^2\right),$$

where $\hat{G}$ is a further constant. Substituting these choices into Eq.(32) leads to

$$\nabla_\perp^2\psi + 2\hat{P}x(1 + \tfrac{1}{2}\delta x) + \hat{G} = 0 \tag{35}$$

where we have expressed lengths in units of a, ψ in units of $\psi_o = \varepsilon B_o a$, and we have defined $\hat{P} = \frac{1}{2}(\hat{p}_\perp + \hat{p}_\parallel)$, and $\delta = \dfrac{\alpha\hat{p}_\perp}{2\hat{p}}$. Since the toroidal current density $j_\phi = -\frac{\varepsilon B_o}{a}\nabla_\perp^2\psi$, the total current is given by

$$I = -\left(\frac{\varepsilon B_o}{a}\right)\pi a^2 \hat{G}(1 + \tfrac{1}{4}k\delta)$$

where we define $k = \hat{P}/\hat{G}$. The solution of (35) subject to the given boundary condition is easily obtained, and employing polar coordinates with $x = r\cos\theta$, $y = r\sin\theta$, we find

$$\psi = \frac{1}{2\bar{q}} \frac{(1 - r^2)}{(1 + \frac{1}{4}k\delta)} \left\{1 + kr\cos\theta + \frac{1}{6} k\delta\left[\frac{3}{4}(1 + r^2) + r^2\cos 2\theta\right]\right\} \qquad (36)$$

where the 'mean' safety factor

$$\bar{q} = \frac{2\pi a\varepsilon B_o}{I} .$$

The effective pressure

$$\bar{p} = \varepsilon B_o^2 \hat{P}(1 + \delta x)\psi \qquad (37)$$

so that if we define a poloidal beta by $\beta_p = \frac{8\pi a^2}{I^2}\int \bar{p}\,r\,d\theta\,dr$ then $\beta_p = \varepsilon^{-1}k(1 + k\delta/3)/(1 + k\delta/4)^2$. Thus k is a measure of β_p. The equilibrium has a β-limit due to the stagnation point in the poloidal-flux entering the plasma. This occurs if

$$k > \frac{1}{(1 - 5\delta/12)} .$$

Surfaces of constant pressure and flux surfaces are shown in Figure 2 for the case $k = 0.5$, $\delta = -0.2$. The current profiles for equilibria of this type are of a rather idealised form, for

$$j_\phi = \frac{I}{\pi a^2(1 + \frac{1}{4}k\delta)} \left\{1 + 2kr\cos\theta(1 + \tfrac{1}{2}\delta r\cos\theta)\right\} .$$

We see that the current density varies only in the direction of the major radius, and in the low-β limit becomes constant. When $\delta = 0$, so that $\bar{p} \equiv \bar{p}(\psi)$, the equilibrium (36) corresponds precisely to that of the isotropic plasma with scalar pressure equal to $\bar{p}$ [9]. Variation on magnetic surfaces of $p_\perp$, occurs for $\delta \neq 0$. It will be shown in the next Section that δ is related to the intensity and angle of neutral injection, and is generally small. From the condition that $p_\perp \geqslant 0$ we see that

$$|\delta| < 1/(1 + \hat{P}_{\parallel}/\hat{P}_{\perp}) \; .$$

Negative δ produces constant $\bar{p}$ surfaces displaced inwards relative to the flux surfaces; positive δ corresponds to an outward relative shift.

(b) Equilibrium Velocity-Space Distributions and Injection

We now discuss the implications of equilibria with $p_{\perp} = p_{\perp}(\psi,\chi)$, as regards the properties of the component distribution functions, $f_j(\underline{r},\underline{v})$. The parallel and perpendicular pressures at a given point are

$$p_{\parallel} = \sum_j m_j \sum_{\sigma=-1}^{1} \int \frac{B}{|v_{\parallel}|} v_{\parallel}^2 f_j \; d\mu d\omega$$

and

$$p_{\perp} = \sum_j m_j \sum_{\sigma=-1}^{1} \int \frac{B}{|v_{\parallel}|} \mu B \; f_j \; d\mu d\omega$$

where m_j is the j^{th} species particle mass, $\omega = \frac{1}{2}v^2$, $\mu = \frac{1}{2}v_{\perp}/B^2$ and $v_{\parallel} = \sigma(2[\omega - \mu B])^{\frac{1}{2}}$ is the velocity parallel to $\underline{B}$. Recalling that in equilibrium, f_j depends on $\underset{\sim}{v}$ only through ω,μ and σ, and that $\underset{\sim}{B} \cdot \nabla f_j = 0$, it follows [10] that

$$\underset{\sim}{B} \cdot \nabla p_{\parallel} + \frac{(p_{\perp} - p_{\parallel})}{B^2} \underset{\sim}{B} \cdot \nabla(\tfrac{1}{2}B^2) = 0 \qquad (38)$$

and

$$\underset{\sim}{B} \cdot \nabla p_{\perp} - \frac{(2p_{\perp} + C)}{B^2} \underset{\sim}{B} \cdot \nabla(\tfrac{1}{2}B^2) = 0 \qquad (39)$$

where we have introduced the pressure - like moment C, defined to be

$$C = \sum_j m_j \int \frac{B}{|v_{\parallel}|} (\mu B)^2 \left.\frac{\partial f_j}{\partial \omega}\right|_{\mu} d\mu d\omega \; .$$

Applying our previous ordering, the lowest order part of Eq.(38) immediately leads to Eq.(24). Similarly, making use of Eq.(13), Eq.(39) yields

$$\underset{\sim}{B} \cdot \nabla p_{\perp} = \frac{-\,\varepsilon C}{(1 + C/B_o^2)} \underset{\sim}{B}^{(1)} \cdot \nabla \frac{\chi}{a} + O(\varepsilon^3) \qquad (40)$$

Thus for leading-order variation of $p_{\perp}$ along $\underset{\sim}{B}$, the pressure-like moment C must be a zeroth-order quantity. If f_j is quasi-Maxwellian, C cannot be zeroth-order; for even if $p_{\perp} \neq p_{\parallel}$, then $C \sim p_{\perp}, p_{\parallel}$ in all situations. C can only be large for distributions which are strongly non-isotropic in some part of velocity space. We now discuss how neutral injection might lead to such a distribution.

Consider the anisotropy produced by injection of energetic ions with sufficient intensity that the beam-ion energy density is comparable to the background plasma pressure. The density of energetic particles will always be much smaller than that of the bulk plasma, so we assume that f_b, the hot-ion distribution, is small compared to the background, Maxwellian distribution. If it is also assumed that $v_i \ll v \ll v_e$ for the hot ions, where v_i, v_e are thermal velocities of the background ions and electrons, then the Fokker-Planck collision operator for f_b simplifies to [11]

$$\left.\frac{\partial f_b}{\partial t}\right|_{coll} = \frac{1}{\tau_s}\left[\frac{1}{v^2}\frac{\partial}{\partial v}\left\{(v^3 + v_c^3)f_b\right\} + \frac{m_i}{2m_h}\left(\frac{v_c}{v}\right)^3\frac{\partial}{\partial\xi}\left((1-\xi^2)\frac{\partial f_b}{\partial\xi}\right)\right] \quad (41)$$

where $\xi = v_\parallel/v$ is the cosine of the angle between the magnetic field and the velocity vector, τ_s denotes the hot-ion slowing-down time, and m_i, m_h are the background ion and hot ion masses, respectively. The first term in Eq.(41) represents the frictional drag of the background plasma on the hot ions, which for velocities greater than $v_c = v_e\left\{\frac{3\sqrt{\pi}}{4}\frac{m_e}{m_i}\right\}^{1/3}$ is mainly due to the electrons. The effect of pitch-angle scattering is represented by the second term. By neglecting the guiding-centre motion perpendicular to $\underset{\sim}{B}$, the drift kinetic equation [12] for hot ions is simply

$$v_\parallel \underline{b} \cdot \nabla f_b(\omega,\mu,\underset{\sim}{r};\sigma) = \left.\frac{\partial f_b}{\partial t}\right|_{coll} + \tilde{S}(\psi, v - v_o)\delta(\xi - \xi_o)\,\delta(\chi - \chi_o) \quad (42)$$

where the second term on the right represents a source of hot ions, which on a flux surface ψ = constant, are created at position $\chi = \chi_o$ in minor azimuth, with some distribution of speeds $\tilde{S}$ about v_o. The ions are all produced with the same pitch angle $\xi_o(\psi)$. With the exception of a very small group of particles whose drift trajectories are at, or almost at, the transition between trapping and circulation round a flux surface, the hot ions will travel many times around their drift orbits before slowing down appreciably. Of course, there will also be some pitch-angle diffusion during a slowing down time, but particles scattered into the transitional band will diffuse rapidly out of this region of velocity space. Thus, we may solve Eq.(42) by expanding in $1/(\omega_{tr}\tau_s)$, where ω_{tr} denotes the transit frequency. Then to lowest order,

$$\underline{b} \cdot \nabla f_b^{(o)} = 0, \qquad \text{so} \qquad f_b^{(o)} \equiv f_b^{(o)}(\psi,\omega,\mu;\sigma)$$

Since the orbit topology of circulating particles is not sensitively-dependent on pitch angle, we can follow Cordey and Houghton [11] and neglect the pitch-angle scattering term in Eq.(41) for velocities $v > v_c$, when the particle source lies outside the trapped band. In this case, the first-order part of Eq.(42) leads to

$$f_b^{(o)}(\omega,\lambda,\psi;\sigma) = \frac{2J(\psi,\chi_o)}{I(\lambda_o)} \tau_s \delta_{\sigma\sigma_o} \left(\frac{\int_v^\infty v^2 \tilde{S}\, dv}{(v^3 + v_c^3)} \right) \delta(\lambda - \lambda_o) \tag{43}$$

where $J = \dfrac{R}{|\nabla\psi||\nabla\chi|}$ is the jacobian, σ_o denotes the sign of $v_\parallel$ for the source ions, $\lambda = \mu/\omega$, $\lambda_o = \dfrac{1}{B(\psi,\chi_o)}(1 - \xi_o^2)$ and

$$I(\lambda) = \oint \frac{JBd\chi}{\sqrt{1 - \lambda B}} .$$

On the other hand, if the injection source lies inside the trapped-particle zone of velocity-space, the λ-variation of f_b cannot be approximated by a delta-function; for the latter would give rise to a divergence in the density (and other moments of $f_b^{(o)}$) at the turning points of the injection orbit, $\lambda = \lambda_o$. We shall not examine this case further, for it will be shown later that the presence of strong anisotropy in the trapped-particle band can lead to important differences between C.G.L. and Kruskal-Oberman stability. For circulating particles, we can use Eq.(43) to evaluate $p_\perp$ and C directly. Thus we find that

$$C = \frac{1}{2} \left\{ \frac{(\lambda_o B)^2}{(1 - \lambda_o B)} \right\} p_{\perp b}^{(1)} + O(\varepsilon) \tag{44}$$

where $p_{\perp b}^{(1)}$ is the beam-ion perpendicular pressure, which is of first order in ε. We exclude the narrow transitional layer in velocity space where $\omega_{tr}\tau_s < 1$. Clearly then, for C to be $O(1)$ we require $1 - \lambda_o B \sim O(\varepsilon)$. Injected particles satisfying this condition have $v_\parallel/v \sim O(\varepsilon^{\frac{1}{2}})$, showing that $p_\perp$ can only vary significantly on flux surfaces if the angle of injection is high. Physically, the pressure variation arises from the fact that circulating particles move around their drift orbits with variable speed (since $v_{\parallel} = |v|\sqrt{1 - \lambda B}$), so that on average, particles 'spend' more time in the region of high magnetic field than in the low-field region. For this modulation in pressure to be comparable in magnitude to the surface-average, fractional variations in $|v_{\parallel}|$ must be substantial. This is the case only if $|v_{\parallel}| \sim \varepsilon^{\frac{1}{2}}v$.

Variation of $p_{\perp}$ round flux-surfaces cannot be produced with parallel or low-angle, oblique ($\xi_o \sim O(1)$) injection; under these circumstances, the plasma behaves as if the pressure were scalar and given by $\bar{p}(\psi)$. We note, from Eqs.(39) and (44), that if large-angle injection is applied such that $C \sim O(1)$, then $p_{\perp}$ increases inwards on flux-surfaces, provided that the diamagnetic well is not of sufficient depth to reverse the direction in which B increases.

For the model equilibrium described in Section IV(a), it is clear that only negative values of δ are compatible with Eqs.(40) and (44) and that $|\delta|$ decreases as the angle of injection becomes smaller.

Furthermore, in the limit $1 \ll T^2 \ll \varepsilon^{-1}$ where $\xi_o = T(\psi)\varepsilon^{\frac{1}{2}}$, the modulation of $p_{\perp}$ becomes small (but still of leading order in ε), and noting that $1 - \lambda_o B = \varepsilon T^2(1 + O(1/T^2))$, we find that $p_{\perp b} = \varepsilon B_o^2\, P_{\perp b}(\psi)(1 + O(1/T^2))$, and $C = P_{\perp b}(\psi)/2T^2 + O(1/T^4)$. Using this value for C in Eq.(40), it is seen that $p_{\perp}$ takes the form

$$p_{\perp} = \varepsilon\, B_o^2\left(p_o(\psi) + P_{\perp b}(\psi)\left(1 - \frac{1}{2T^2}\,\frac{x}{a}\right)\right) + O\left(\frac{1}{T^4}\right),$$

where $\varepsilon B_o^2 p_o$ denotes the pressure of the isotropic background plasma. The parallel pressure remains unaffected at leading order because of the high angle of injection, and is also given by p_o. When p_o and $p_{\perp b}$ are linear functions of ψ, and T is constant, we obtain the model forms leading to the equilibrium of Eq.(36), where we now identify $\xi = \hat{p}_{\perp}/\hat{p}_{\parallel}$, and δ according to

$$\xi \doteq 1 + p_{\perp b}/p_o, \quad \text{and} \quad \delta = \frac{1}{2T^2}\,\frac{(\xi - 1)}{(\xi + 1)}\;.$$

We see that ξ is a measure of the injection power; and δ, which determines the amount of variation in $\bar{p}$ round flux surfaces, depends on the angle of injection as well as on the injection power.

V. LINEAR STABILITY

The linear stability of anisotropic equilibria to hydromagnetic disturbances may be tested at two different levels of approximation. In the simpler of these, the C.G.L. equations are used to construct the potential energy (δW_{CGL}) associated with small displacements of the fluid [13]; the necessary and sufficient condition for stability is $\delta W_{CGL} > 0$. A more sophisticated theory, due to Kruskal and Oberman [5], takes into account the transport of energy along field lines. In obtaining their necessary and sufficient condition, $\delta W_{KO} > 0$, Kruskal and Oberman assumed that $\frac{\partial f}{\partial \omega} < 0$. Given the latter restrictions they also deduced the inequality

$$\delta W_{CGL} \geqslant \delta W_{KO} \tag{45}$$

which indicates that use of the G.C.L. theory leads to an overestimate of stability. Using neutral injection, however, the ion distribution will not be a monotonic function of ω, and thus Eq.(45) is inapplicable. Although Grad [15] has shown that a necessary condition for stability of a guiding centre plasma is $\frac{\partial f}{\partial \omega}(\omega,\mu,\underline{r}) < 0$ [2], the pertubations which grow as a result of non-monotonic f do not preserve the single-particle, second adiabatic invariant, $J = \oint v_{\parallel}\, d\ell$. For the long wave-length modes of interest here, and for growth rates small compared to a typical particle bounce frequency, we can assume the adiabatic invariance of J, and in this case Andreoletti [14] has derived δW_{KO} without restriction on the form of f. We now demonstrate that, within our ordering, and provided the trapped band is only weakly anisotropic, then $\delta W_{CGL} \leqslant \delta W_{KO}$. The reduced equations, Eqs.(16)-(26) of Section III, possess an energy integral, W_{CGL}, which is constant on the toroidal Alfven timescale. This is $O(\varepsilon^2)$ and given by

$$W_{CGL} = \int_{v_p} d\tau \left\{ \frac{1}{2}|\nabla_\perp \psi|^2 + \frac{1}{2}\rho\, |\nabla_\perp U|^2 - \frac{2x}{R_o}\bar{p} \right\} + \int_{v_v} d\tau\, \frac{1}{2}\, |\nabla_\perp \psi|^2$$

where v_p and v_v denote the zeroth-order plasma and vacuum volumes, respectively. Linearising about a given equilibrium, we obtain the energy principle (to $O(\varepsilon^2)$) for small-amplitude disturbances with exponential time-dependence $e^{\gamma\tau_1}$:

---

[2] We note that a similar condition, namely, $\frac{\partial f}{\partial \omega}(\omega, \mu, J) < 0$, has been shown by Taylor [16] to be sufficient for stability to interchanges, the theory applying in the limit of low-β.

$$
\begin{aligned}
- \gamma^2 \int_{V_{po}} \rho\, |\nabla_\perp U|^2\, d\tau = 2\delta\, W^{(2)}_{CGL} = \int_{V_{po}} d\tau\, \Big\{ & |\nabla\{(\underset{\sim}{B}_o \cdot \nabla)^{(1)} U\}|^2 \\
& + \{(\underset{\sim}{B}_o \cdot \nabla)^{(1)} U\}(\nabla_\perp J_{\phi o} \times \nabla_\perp U) \cdot \hat{\phi} \\
& + \frac{2}{R_o} \frac{\partial U}{\partial y} (\nabla_\perp \bar{p}_o \times \nabla_\perp U) \cdot \hat{\phi} \Big\} + \int_{V_{vo}} d\tau |\nabla_\perp \psi_v|^2 \\
& - \int_{\Gamma_o^{(o)}} dS \left\{ \frac{2x}{R_o} \left(\frac{\partial U}{\partial \ell} \right)^2 \hat{n} \cdot \nabla \bar{p}_o + J_{\phi o} \frac{\partial U}{\partial \ell} \{\underset{\sim}{B}_o \cdot \nabla)^{(1)} U\} \right\} .
\end{aligned}
\tag{46}
$$

Here, U is related to the displacement ξ by $\xi = \nabla U \times \hat{\phi}$, equilibrium quantities are denoted by the subscript 0, and $\Gamma_o^{(o)}$ denotes the zeroth-order interface, to which $\hat{n}$ is the outward unit normal. Derivatives along the minor cross section of $\Gamma_o^{(o)}$ are denoted by $\partial/\partial\ell$. If the plasma touches the perfectly conducting wall, then the vacuum term vanishes, as does the surface integral, since $\xi \cdot \hat{n} = 0$ implies $\frac{\partial U}{\partial \ell} = 0$. In this case Eq.(46) reduces to the result given by Strauss, with scalar pressure replaced by $\bar{p}$. We note that the last term in Strauss' result is missing a factor 2.

The first term in Eq.(46) is stabilising and represents the work done by the perturbation in bending the field-lines. Kink modes may be driven by the current-density gradient and this effect is contained in the second term. If we rewrite the third term in the form $- 2(\underset{\sim}{\xi} \cdot \underset{\sim}{\kappa}^{(1)})(\underset{\sim}{\xi} \cdot \nabla\bar{p})$, where $\underset{\sim}{\kappa}^{(1)} = - \hat{x}/R_o$ is the curvature of the magnetic field to leading order, then it is clear that this contribution represents the interaction of field curvature and pressure gradient; this can give rise to the interchange [17] and ballooning [18] instabilities.

Let us now compare the CGL result, Eq.(46), with the more complete Kruskal-Oberman theory. In zeroth order, δW_{KO} is positive-definite, and vanishes only if $\nabla \cdot \underset{\sim}{\xi} = O(\varepsilon)$. This enables us to introduce a stream function U as before. Now the so-called trapped-particle, or kinetic term in δW_{KO} is given by [10]

$$- \int d\tau \sum_j m_j \iint \frac{B d\mu d\omega}{|v_\parallel|} \frac{\partial f_j}{\partial \omega} \left\langle v_\parallel^2 \; \underset{\sim}{\xi} \cdot \underset{\sim}{\kappa} + \mu B (\nabla \cdot \underset{\sim}{\xi} + \underset{\sim}{\xi} \cdot \underset{\sim}{\kappa}) \right\rangle^2$$

where the velocity space integration is performed over the trapped-particles only,

$$\langle g \rangle = \int \frac{d\ell}{|v_\parallel|} g \Bigg/ \int \frac{d\ell}{|v_\parallel|}$$

is the field-line average of g, and κ is the curvature vector. We assume that neutral injection, when present, does not create strong anisotropy in the velocity distribution of trapped particles. In this situation, the kinetic term is $O(\varepsilon^{7/2})$ for minimising displacements whereas the remaining contributions to δW_{KO} are $O(\varepsilon^2)$. Thus, we can neglect the kinetic term, and at $O(\varepsilon^2)$, minimising over the value of $(\nabla \cdot \xi)^{(1)}$, we obtain

$$\delta W^{(2)}_{KO} = \delta W^{(2)}_{CGL} + \int_{V_{po}} \frac{C^{(o)} B_o^2}{B_o^2 + C^{(o)}} \left(\frac{1}{R_o} \frac{\partial U}{\partial y} \right)^2 d\tau \tag{47}$$

where $\delta W^{(2)}_{CGL}$ is given by Eq.(46). In deriving this result we assume that the 'mirror' stability criterion, $1 + (2p_\perp + C)/B^2 > 0$, is satisfied [10]. At leading order, this is equivalent to

$$B_o^2 + C^{(o)} > 0 \tag{48}$$

From Eq.(44) we see that this is always satisfied since $C^{(o)} > 0$. Thus, $\delta W^{(2)}_{CGL}$ is <u>never</u> greater than $\delta W^{(2)}_{KO}$; the two expressions coincide if $\bar{p} \equiv \bar{p}(\psi)$. There is no conflict between this conclusion and Eq.(45): if $\frac{\partial f}{\partial \omega} \nless 0$ then Eq.(45) is inapplicable, whilst if $\frac{\partial f}{\partial \omega} < 0$, then $\bar{p} \equiv \bar{p}(\psi)$ and our result agrees with the equality of Eq.(45). Thus given the restriction of weak anisotropy in the trapped band, then $\delta W^{(2)}_{CGL} \leqslant \delta W^{(2)}_{KO}$, and use of the CGL formalism never over-estimates stability. We note that the above discussion is applicable to the general plasma-vacuum configuration.

Although $p_\perp$-variation round flux-surfaces is not expected to be important for gross modes, it is known that such equilibria can lead to interesting results as regards stability to ballooning, that is, modes of high toroidal mode number [19],[20]. Thus, beneficial stabilising effects can be obtained by an inward weighting $p_\perp$. For such a configuration to be possible, the total B must increase towards the major axis, despite the diamagnetic effect. Now the dynamic equilibrium condition (13) leads to

$$\underline{B} \cdot \nabla B_\phi = - \frac{\varepsilon B_o}{(1 + C^{(o)}/B_o^2)} \underline{B} \cdot \nabla\left(\frac{x}{a} \right) + O(\varepsilon^3)$$

which together with Eq.(48), ensures that B_ϕ, and hence B, increases inwards as required.

In discussing the relationship between δW_{CGL} and δW_{KO} we excluded the case where injection deposits hot ions in the trapped particle band. This enabled us to neglect the kinetic terms. Under the same conditions which lead to $C \sim O(1)$, anisotropy within the trapping zone gives rise to kinetic terms which are of the same order as the fluid terms, that is $O(\varepsilon^2)$. For the model distribution function [10]

$$f_b = g(\psi,\omega)(\lambda B_{min})^{\ell} ,$$

which can be taken to approximate the case of exactly perpendicular injection on the outside of the mid-plane of the torus, where $B = B_{min}(\psi)$, it is easily shown that

$$C = - (\ell + 2)p_{\perp b} + O(\varepsilon)$$

where $p_{\perp b}$ is the beam perpendicular pressure. Then $C \sim O(1)$ for $\ell \sim \varepsilon^{-1}$, and is always negative, so that apart from uncertainty over the effect of the kinetic terms at leading order, the sign of the second term in Eq.(47) is negative. Thus it is possible that for injection into the trapped zone, use of CGL equations could give an over-estimate of stability. Also, for the model considered above we see from Eq.(40) that maxima of $p_\perp$ can be situated on the outside of the torus, a situation which expected to be unfavourable for stability against pressure-driven modes. For the above reasons the CGL equations are unsatisfactory for the description of plasma dynamics with perpendicular injection into the trapped band.

VI. CONCLUSIONS

On the basis of the $\beta \sim \varepsilon$, long wavelength ordering, we have used the CGL equations to discuss the dynamics and linear stability of a general axisymmetric, anisotropic, plasma-vacuum system. In particular, we have derived reduced equations on the toroidal Alfven time-scale. They can be used for the numerical investigation of non-linear motions in tokamaks; addition of velocity damping would provide a simple means of evolving axisymmetric equilibria.

To avoid toroidal acceleration of the plasma, it is necessary that $p_\parallel$ be constant round a flux-surface, that is $p = p_\parallel(\psi)$. If this condition holds initially, then it is satisfied for all time on the toroidal Alfvèn time-scale. For $p_\perp$, however, there is no such restriction, and $p_\perp$ can take the form $p_\perp = p_\perp(\psi,\chi)$. We have found a simple analytic example of a tokamak equilibrium with a χ-dependent $p_\perp$. A heuristic discussion of the Fokker-Planck equation shows that equilibria of this type can only arise from large-angle injection; it is required that the trapped band be only weakly anisotropic to avoid trapped particle effects in the energy principle.

We have derived an energy integral from the reduced equations, and linearisation leads to an energy principle for small displacements; this is identical with the $\beta \sim \varepsilon$, long wavelength version of the CGL energy principle. We have also applied our ordering directly to the Kruskal-Oberman energy principle. If particles in the trapped band are weakly anisotropic, then

$$\delta W_{CGL} \leq \delta W_{KO} \ ,$$

where the equality sign refers to equilibria for which $\bar{p}$ is constant on flux-surfaces. Thus, given our ordering and the above proviso on trapped particles, the CGL principle never overestimates stability.

For $\bar{p}$ constant on flux-surfaces, the MHD linear stability of an axisymmetric $\beta \sim \varepsilon$, anisotropic plasma-vacuum configuration to long wavelength $(n \sim 1)$ modes, is identical to that of the equivalent scalar pressure system, as has been shown also for fixed-boundary modes in the limit of large toroidal mode number $(n \gg 1)$ [19, 20]. Note that both these results are independent of the degree of anisotropy.

REFERENCES

[1] Clarke, J.F., High Beta Flux-Conserving Tokamaks, ORNL/Tm 5429 (1976).

[2] Cordey, J.G., Core, W.G.F., Nucl. Fusion 15 (1975), 710.

[3] Chew, G.F., Goldberger, M.L., Low, F.E., Proc, Roy. Soc. 236, (1956), 112.

[4] Rosenbluth, M.N., Rostoker, N., Phys. Fluids, 2, (1959),23.

[5] Kruskal, M.D., Oberman, C.R., Phys. Fluids, 1, (1958), 275.

[6] Rosenbluth, M.N., Monticello, D.A., Strauss, H.R., White, R.B., Phys. Fluids, 19, (1976), 1987.

[7] Strauss, H.R., Phys. Fluids, 20, (1977), 1354.

[8] Mercier, C., Cotsaftis, M., Nucl. Fusion, 1 (1961), 121.

[9] Haas, F.A., Phys. Fluids, 15, (1972), 141

[10] Connor, J.W., Hastie, R.J., Phys. Fluids 19, (1976), 1732.

[11] Cordey, J.G., Houghton, M.J., Nucl. Fusion, 13 (1973), 215.

[12] Frieman, E.A., Phys. Fluids 13 (1970) 490.

[13] Bernstein, I.B., Frieman, E.A., Kruskal, M.D., Kulsrud, R.M., Proc. Roy. Soc. A244, (1958) 17.

[14] Andreoletti, J., Comptes Rendus Acad. Sc., 256 (1963), 1251.

[15] Grad, H, Phys. Fluids 9 (1966) 225.

[16] Taylor, J.B., Phys. Fluids 7 (1964) 767.

[17] Mercier, C., Nucl. Fusion, 1, (1960), 47.

[18] Dobrott, D., Nelson, D.B. Greene, J.M., Glasser, A.H., Chance, M.S., Frieman, E.A., Phys. Rev. Letts., 39 (1977) 943.

[19] Fielding, P.J., Haas, F.A., Phys. Rev. Letts., 41 (1978), 801.

[20] Sykes, A.A., Turner, M., Fielding, P.J., Haas, F.A., 7th Int. Conf. on Plasma Physics and Controlled Nuclear Fusion Research, Innsbruck, IAEA/CN/37 K-5, (1978).

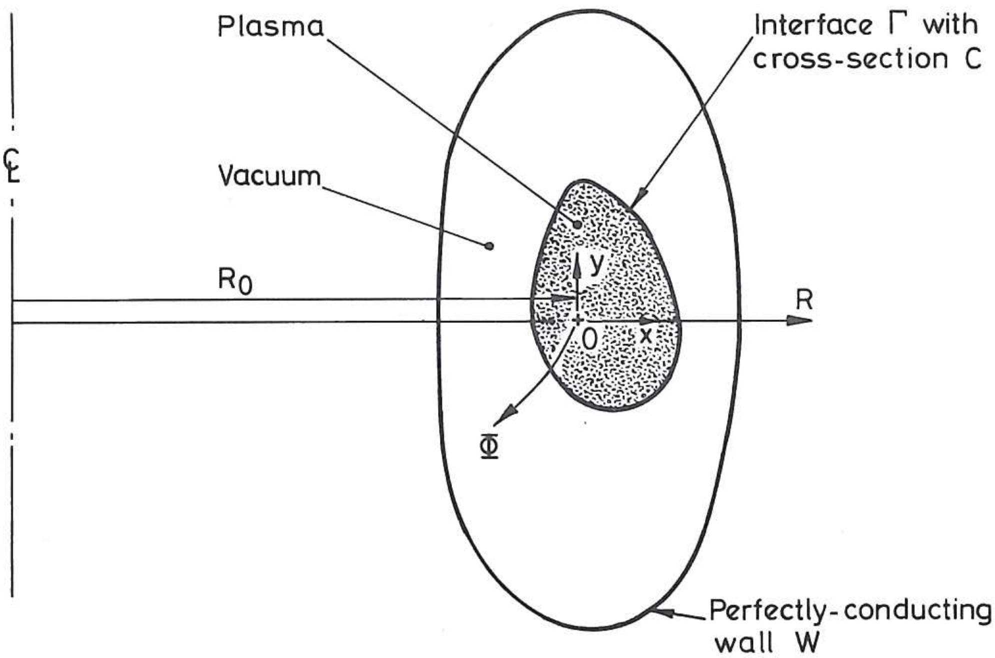

Fig.1 Coordinate systems.

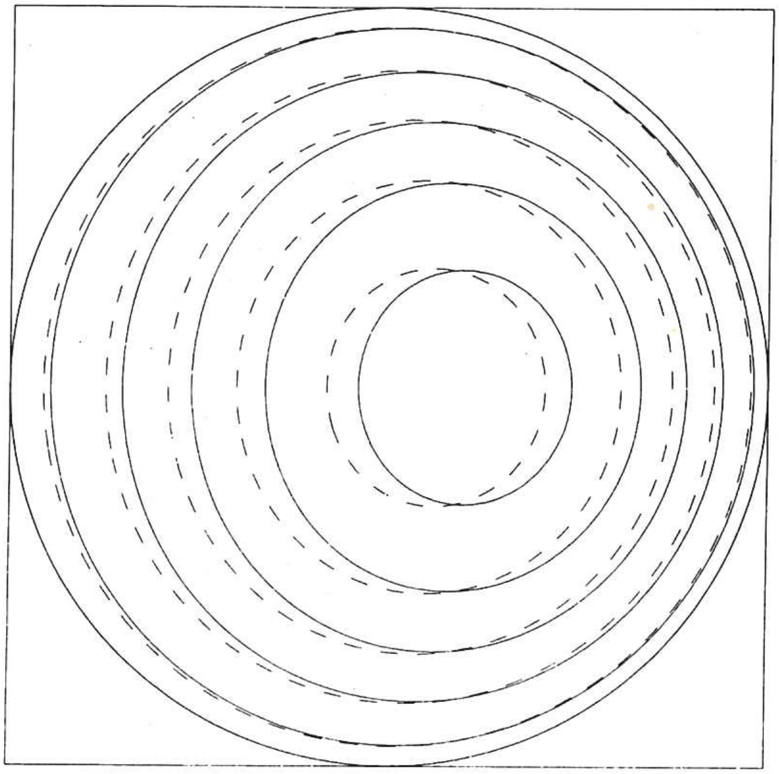

Fig.2 Flux (solid lines) and constant $\overline{p}$ surfaces (dashed lines) for equilibrium with $k = 0{\cdot}5$ and $\delta = -0{\cdot}2$. The major axis lies to the left.