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MAXIMUM PLASMA PRESSURE FOR STABILITY IN MAGNETIC FIELDS WITH FINITE MINIMA

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ABSTRACT

This paper discusses the maximum plasma pressure for stable containment in magnetic fields which possess a non-zero minimum in field strength, (minimum-B fields or magnetic wells). The basic limitations are ones on the pressure gradient and are calculated exactly for a special class of equilibria and more generally by an expansion procedure based on a shallow-well approximation. Transcribed into estimates of the critical pressure itself, these results indicate a maximum pressure equal to the depth of the magnetic well. If B_1 is the field strength at the largest closed $|\underset{\sim}{B}|$ contour and B_2 the field strength at the lowest point of the well then $p_{\perp}\max \simeq \frac{1}{2}(B_1^2 - B_2^2)$.

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I. INTRODUCTION

It has been shown[1,2] that hydromagnetically stable equilibria exist in magnetic fields which possess a non-zero minimum in field strength; such fields are now referred to as "minimum-B fields" or "magnetic wells". The stability of these equilibria was demonstrated only in the limit of vanishing β (where β is the ratio of plasma pressure to magnetic pressure, i.e. $\beta = {}^{2p_\perp}/_{B^2}$) and the present paper is concerned with the problem of determining the limiting β for plasma stability in magnetic wells. It is shown that in general this is related to the depth of the magnetic well, being given by

$$\beta_c \simeq \frac{(B_1^2 - B_2^2)}{B_1^2}$$

where B_1 is the value of $|\underset{\sim}{B}|$ at the plasma surface and B_2 is the minimum value of $|\underset{\sim}{B}|$ (so that $(B_1^2 - B_2^2)$ is a measure of the depth of the magnetic well.)

The main calculation is based on an expansion procedure - somewhat similar to that developed by the Princeton Group[3] for the calculation of critical β in the Stellarator. That is, we expand both the equilibrium configuration and the energy integral δW, (the minimum of which determines stability by its sign), as asymptotic series in powers of small quantities. There are, however, important differences between the present work and that of the Princeton Group. For example the requirement of periodicity of the perturbations played an important role in the Stellarator work whereas it plays none here. On the other hand the anisotropy of the pressure tensor is vital to our calculation while it did not enter that on the Stellarator. In addition to the main calculation related topics are discussed in appendices; Appendix A gives a fuller analysis of the equilibrium than that in the main text and discusses the relation of the vacuum field to that in the presence of plasma, Appendix B gives a rearrangement of the Kruskal-Oberman energy principle which is more convenient than the original form, and Appendix C discusses some exact results obtained for special equilibria and which are not restricted to shallow wells.

The calculations of stability in Reference 1 are already essentially based on an expansion procedure, the expansion being in powers of the small quantity β; that is one writes

$$\delta W = \delta W_o + \delta W_\beta + \delta W_{\beta\beta} + \delta W_{\beta\beta\beta} + \dots \qquad \dots (1.1)$$

and then shows that the minimum $\delta W_o = 0$ and the minimum $\delta W_\beta > 0$. At sufficiently small β, $\delta W_{\beta\beta}$ is negligible and the system is therefore stable. Instabilities may arise when β is large enough for successive terms $\delta W_{\beta\beta}$ etc. to become comparable with δW_β and one method of determining the critical β might be to evaluate $\delta W_{\beta\beta}$. However, there are practical and logical objections to this. For example, if $\delta W_{\beta\beta}$ is comparable with δW_{β} then one must expect that $\delta W_{\beta\beta\beta}$ is also comparable with δW_{β} and the β-expansion is apparently invalid.

This objection can be overcome, and the problem much simplified if the expansion of δW is made not just in the single small quantity β but in several small quantities, which are then grouped appropriately. One can then obtain an expansion of δW in which stability is again determined by the sign of the first non-zero term in the expansion rather than by comparison of several terms.

To perform this sort of expansion we regard the magnetic field as being made up of a number of constituents. The basic zero-order magnetic field is a uniform field $\underset{\sim}{B}_o$ in the z-direction and magnetic well is created by the addition of a "mirror" component $\underset{\sim}{B}_m$ and a "stabilising" component $\underset{\sim}{B}_s$. This terminology is chosen because the superposition of a mirror field and a multipole (stabilising) field is a well known way of producing the desired form of minimum-B field[1,4]. However the significant properties of the components which we call "mirror" and "stabilising" are that the "mirror" component is principally parallel to $\underset{\sim}{B}_o$ while the "stabilising" component is purely perpendicular to $\underset{\sim}{B}_o$. A further contribution to the field is produced by the plasma itself and is denoted by $\underset{\sim}{B}_\beta$ so that

$$\underset{\sim}{B} = \underset{\sim}{B}_o + \underset{\sim}{B}_m + \underset{\sim}{B}_s + \underset{\sim}{B}_\beta \qquad \ldots (1.2)$$

where $\underset{\sim}{B}_m$, $\underset{\sim}{B}_s$, $\underset{\sim}{B}_\beta$ are all small compared to $\underset{\sim}{B}_o$. This approximation corresponds to considering the stability of plasma in a "shallow" magnetic well. The pressure tensor

$$\underset{\approx}{P} = p_\perp \underset{\approx}{I} + (p_\parallel - p_\perp)\, \underset{\sim}{n}\, \underset{\sim}{n} \qquad \ldots (1.3)$$

is likewise treated as a small quantity compared to the magnetic pressure $B_o^2/2$, and we therefore have to consider a number of small quantities

$$\frac{B_m}{B_o} = \mu,\ \frac{B_s}{B_o} = \sigma,\ \frac{B_\beta}{B_o} = \beta,\ \frac{2p_\perp}{B_o^2} = \beta_\perp,\ \frac{2p_\parallel}{B_o^2} = \beta_\parallel \qquad \ldots (1.4)$$

In principle we should now expand δW in powers of these small quantities and then group terms together to obtain a stability criterion. However it is more convenient to do the grouping first by attributing relative orders of magnitude to the small quantities σ, μ, β, $\beta_\perp$, $\beta_\parallel$ in terms of a single expansion parameter λ and then to expand in powers of this single quantity.

Our first task is therefore to assign the relative orders of the small quantities. The appropriate ordering is that in which all the relevant quantities appear in the eventual stability criterion and one can arrive at this ordering as follows. The earlier work shows that $\nabla|\underset{\sim}{B}|$ plays an important role in determining stability at low β, so that $\underset{\sim}{B}_s$, $\underset{\sim}{B}_m$, $\underset{\sim}{B}_\beta$ must all contribute to $|\underset{\sim}{B}|$ in the same order in λ. Because $\underset{\sim}{B}_s$ is perpendicular to $\underset{\sim}{B}_o$ it contributes to $|\underset{\sim}{B}|$ only as $\underset{\sim}{B}_s^2$ and must be chosen of lower order in λ than $\underset{\sim}{B}_m$ or $\underset{\sim}{B}_\beta$. Hence the appropriate ordering of the field components is

$$B_s \sim \lambda B_o,\ B_m \sim \lambda^2 B_o,\ B_\beta \sim \lambda^2 B_o \ . \qquad \dots (1.5)$$

The remainder of the ordering is determined by consideration of the equilibrium equations which are examined briefly in the next section. A more detailed discussion of equilibrium is given in the appendix.

II. THE EQUILIBRIUM SITUATION

If we expand the fields in the equilibrium equation

$$\underset{\sim}{j} \times \underset{\sim}{B} = \nabla\cdot\underset{\approx}{P} \qquad \dots (2.1)$$

then in lowest order

$$\underset{\sim}{j}^\beta \times \underset{\sim}{B}_o = \nabla p_\perp + \underset{\sim}{n}_o\left[\underset{\sim}{n}_o\cdot\nabla(p_\parallel - p_\perp)\right] \qquad \dots (2.2)$$

where $\underset{\sim}{n}_o$ is a unit vector along $\underset{\sim}{B}_o$. The component of $\underset{\sim}{j}^\beta$ perpendicular to $\underset{\sim}{n}_o$ is given by

$$j_\perp^\beta = \frac{\underset{\sim}{B}_o \times \nabla p_\perp}{B_o^2} \qquad \dots (2.3)$$

and as $\nabla\cdot j_\perp^\beta$ vanishes identically the parallel component of current $j_\parallel^\beta$ is zero in this order so that

$$\nabla \times \underset{\sim}{B}_\beta = \underset{\sim}{j}^\beta = \frac{\underset{\sim}{B}_o \times \nabla p_\perp}{B_o^2} \qquad \dots (2.4)$$

If we are to take $\underset{\sim}{B}_\beta$ of order λ^2 then clearly $p_\perp$ and $\underset{\sim}{j}^\beta$ are of order λ^2. The appropriate order for $p_\parallel$ could be obtained by examining the equilibrium in a higher approximation but it is simpler to return to the exact equation (2.1) and take the component parallel to $\underset{\sim}{B}$, that is

$$\frac{\partial p_\parallel}{\partial s} = \frac{-(p_\perp - p_\parallel)}{B}\frac{\partial B}{\partial s} \qquad \dots (2.5)$$

then, since $p_\perp/B^2$ and $\frac{1}{B}\frac{\partial B}{\partial s}$ are both of order λ^2, we must take $p_\parallel/B^2 \sim \lambda^4$.

To the order in which it is required for the stability analysis the equilibrium configuration is completely specified by giving $p_\perp$ using (2.5) to determine $p_\parallel$ and (2.4) to determine $\underset{\sim}{j}^\beta$ and $\underset{\sim}{B}^\beta$. The description of the equilibrium, therefore, requires only the solution of (2.4) for $\underset{\sim}{B}^\beta$. This is a classical problem whose solution is known for a variety of distributions $p_\perp$ and is discussed in Appendix A.

III. EXPANSION OF ENERGY INTEGRAL

In the following sections we examine the stability of the equilibria outlined in sections 1 and 2. This is done by investigating the sign of the change of potential energy δW of the system when subjected to an infinitesimal arbitrary displacement $\underset{\sim}{\xi}$. If this δW is positive for all possible displacements the system is stable, while if δW can be made negative for some choice of $\underset{\sim}{\xi}$, it is unstable.

A number of different energy principles, corresponding to different models of the plasma, exist. The simplest of these, the Magnetohydrodynamic Energy Principle[5], is not applicable to our problem since there can be no equilibrium with scalar pressure in minimum-B systems. The Double-Adiabatic[5] (or Chew-Goldberger-Low[6]) Energy Principle is based on a fluid model of the plasma in which the pressure is a diagonal tensor with two independent components

$$\underset{\approx}{P} = p_\perp \underset{\approx}{I} + (p_\parallel - p_\perp)\, \underset{\sim}{n}\underset{\sim}{n}$$

and heat flow along the magnetic field is neglected.

The Kruskal-Oberman (δW_{KO}) Energy Principle[7] is derived from an individual particle approach, in the limit of small larmor radius. This theory neglects the effect of a perturbed electric field parallel to $\underset{\sim}{B}$ (which plays a somewhat similar role to $\xi_\parallel$ in the fluid model), and it ignores the requirement of quasi-neutrality. These effects have been incorporated by Newcomb[8] in possibly the most correct form (δW_N) of Energy Principle. Various authors have obtained the following inequalities among these Energy Principles

$$\delta W_{KO} \leq \delta N_N \leq \delta W_{CGL}$$

and since δW_N is the most nearly correct form available, one may regard δW_{CGL} as an upper bound for δW_N giving necessary criteria for stability, and δW_{KO} as a lower bound for δW_N giving sufficient criteria for stability. The Kruskal-Oberman theory is thus the most conservative and, according to the other theories, yields an underestimate of the critical β, but it requires a more detailed description of the plasma than δW_{CGL}, including some terms not immediately expressible in terms of macroscopic quantities such as $p_\perp$ or $p_\parallel$. Before tackling the Kruskal-Oberman theory, therefore, the method was applied to the simpler "double adiabatic" theory. A brief account of this simpler calculation has already been published[9] and here we will concentrate on the calculation based on the Kruskal-Oberman energy principle.

Kruskal-Oberman Theory

The Kruskal-Oberman energy principle treats correctly the motion of particles in the small larmor radius limit, however, the constraint of charge neutrality is neglected and in this respect it leads to sufficient rather than necessary conditions. The appropriate function is

$$\Delta W = \tfrac{1}{2} \int \delta W \, d\tau$$

with

$$\begin{aligned} \delta W = \; & \underset{\sim}{Q}^2 - \underset{\sim}{j}\cdot\underset{\sim}{Q} \times \underset{\sim}{\xi} + (2p_\perp + c)(\nabla\cdot\underset{\sim}{\xi} - q)^2 + (\underset{\sim}{\xi}\cdot\nabla p_\perp)(\nabla\cdot\underset{\sim}{\xi} - q) \\ & + (\underset{\sim}{\xi}\cdot\nabla p_\parallel)q - (p_\parallel - p_\perp)[\underset{\sim}{n}\cdot(\underset{\sim}{a}\cdot\nabla)\underset{\sim}{\xi} + \underset{\sim}{a}\cdot(\underset{\sim}{n}\cdot\nabla)\underset{\sim}{\xi} - q^2 - q\,\nabla\cdot\underset{\sim}{\xi}] \\ & - \sum_i m_i \iint \frac{B}{v_\parallel} \frac{\partial f}{\partial \varepsilon} \langle v_\parallel^2 q + \mu B(\nabla\cdot\xi - q)\rangle^2 \, d\mu d\varepsilon \end{aligned} \qquad \ldots (3.1)$$

where

$$\underset{\sim}{Q} = \underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times \underset{\sim}{B}) \qquad \underset{\sim}{a} = (\underset{\sim}{n}\cdot\nabla)\underset{\sim}{\xi} - (\underset{\sim}{\xi}\cdot\nabla)\,\underset{\sim}{n} \qquad \ldots (3.2)$$

$$q = \underset{\sim}{n}\cdot\underset{\sim}{a} \;.$$

The displacement parallel to $\underset{\sim}{B}$, $\xi_\parallel$, can be taken to be zero, since δW can be shown to be independent of this component of $\underset{\sim}{\xi}$. <u>Henceforth therefore $\underset{\sim}{\xi}$ always denotes a vector which is perpendicular to $\underset{\sim}{B}$.</u> The average appearing in (3.1) is defined by

$$\langle g \rangle = \int \frac{d\ell}{v_\parallel} g \Bigg/ \int \frac{d\ell}{v_\parallel} \qquad \ldots (3.3)$$

and

$$C = \sum_i m_i \iint \frac{B}{v_\parallel} \frac{\partial f}{\partial \varepsilon} (\mu B)^2 \, d\mu d\varepsilon \qquad \ldots (3.4)$$

where $f(\mu,\varepsilon,\underset{\sim}{x}_\perp)$ is the particle distribution expressed in terms of the magnetic moment μ, the energy ε, and the parallel velocity $v_\parallel$ is given by $\tfrac{1}{2}v_\parallel^2 = \varepsilon - \mu B$. We shall restrict ourselves to distributions for which $\frac{\partial f}{\partial \varepsilon} < 0$; this is in any case necessary for the derivation of the energy principle given by Kruskal and Oberman[7].

The appearance of terms which depend on the particle distribution function rather than on purely macroscopic quantities, causes some difficulty and the ordering procedure must first be extended to include such terms.

At first sight it may appear that because terms such as 'C' depend on the presence of plasma they are of at least as high an order in λ as the pressure components, i.e. $\sim \lambda^2$. However this is not the case, as we may see from the following argument. In terms of the distribution function $f(\mu,\varepsilon,\underset{\sim}{x}_\perp)$ the perpendicular pressure is

$$p_\perp = \sum_i m_i \iint \frac{B}{v_\parallel} \mu B f d\mu d\varepsilon = -\sum_i m_i \iint B v_\parallel \mu B \frac{\partial f}{\partial \varepsilon} d\mu d\varepsilon \qquad \ldots (3.5)$$

and the parallel pressure is

$$p_\parallel = -\sum_i m_i \iint \frac{B v_\parallel{}^3}{3} \frac{\partial f}{\partial \varepsilon} d\mu d\varepsilon \qquad \ldots (3.6)$$

so that according to Schwartz' inequality

$$-C \geqslant \frac{p_\perp{}^2}{3p_\parallel} \qquad \ldots (3.7)$$

and $p_\perp{}^2/p_\parallel$ is a <u>zero-order</u> quantity. Again if we take the derivative of $p_\perp$ along the magnetic field

$$\frac{\partial p_\perp}{\partial s} = (C + 2p_\perp) \frac{1}{B} \frac{\partial B}{\partial s} \qquad \ldots (3.8)$$

which, because $\frac{\partial p_\perp}{\partial s}$ and $\frac{\partial B}{\partial s}$ are both of order λ^2, again indicates that C must be treated as a zero-order quantity. The order of other terms involving $\partial f/\partial \varepsilon$ can be gaged from comparison with the expressions (3.5) or (3.6).

We have now to expand the energy integral (3.1) in powers of λ by writing

$$\begin{aligned} \underset{\sim}{B} &= \underset{\sim}{B}_o + \lambda \underset{\sim}{B}_s + \lambda^2 \underset{\sim}{B}_m + \lambda^2 \underset{\sim}{B}_\beta + O(\lambda^3) \\ p_\perp &= \lambda^2 p_\perp + O(\lambda^3) \\ p_\parallel &= \lambda^4 p_\parallel + O(\lambda^5) \\ \underset{\sim}{j} &= \lambda^2 \underset{\sim}{j}_\perp + O(\lambda^3) \; . \end{aligned} \qquad \ldots (3.9)$$

The displacement $\underset{\sim}{\xi}$ which minimises δW will depend on the configuration, i.e. on λ, so that it must also be expanded in λ and so

$$\underset{\sim}{\xi} = \underset{\sim}{\xi}_0 + \lambda\underset{\sim}{\xi}_1 + \lambda^2\underset{\sim}{\xi}_2 + O(\lambda^3) \quad . \qquad \ldots (3.10)$$

We find that it is necessary to carry the expansion to fourth order and that certain contributions of lower order vanish because of cancellation between many terms of 3.1. In order to avoid carrying these unnecessary terms it is convenient to rewrite δW in a modified form. We introduce the notation

$$\frac{\underset{\sim}{\xi}\cdot\underset{\sim}{\nabla}B}{B} \equiv s \qquad \ldots (3.11)$$

then, as shown in appendix B, δW can be written in the more convenient form

$$\begin{aligned} \delta W = \ & \underset{\sim}{Q}_\perp^2 \left\{1 + \frac{p_\perp - p_\parallel}{B^2}\right\} + Q_\parallel^2 \left\{1 + \frac{2p_\perp + C}{B^2}\right\} \\ & + \left(\frac{2Q_\parallel}{B} + s\right)\left\{(2p_\perp + C)s - \underset{\sim}{\xi}\cdot\nabla p_\perp\right\} \\ & + q\,(\underset{\sim}{\xi}\cdot\nabla p_\parallel + (p_\perp - p_\parallel)s) - (\underset{\sim}{n}\cdot\underset{\sim}{Q} \times \underset{\sim}{\xi})j_\parallel \left\{1 + \frac{(p_\perp - p_\parallel)}{B^2}\right\} \\ & - \sum m_i \iint \frac{B}{v_\parallel}\frac{\partial f}{\partial \varepsilon}\left\langle v_\parallel^2 q - \mu B\left(\frac{Q_\parallel}{B} + s\right)\right\rangle^2 d\mu d\varepsilon \end{aligned} \qquad \ldots (3.12)$$

The advantage of using this form of δW is that q, s have no contributions of lower than second order and $j_\parallel$ none of lower than fourth order. (Appendices A, B).

The contributions to δW in various orders of the λ expansion are now easily obtained. In zero order

$$\delta W_0 = [\underset{\sim}{Q}_\perp^0]^2 + \left(1 + \frac{C}{B_0^2}\right)[Q_\parallel^0]^2 - \sum m_i \iint \frac{B_0}{v_\parallel}\frac{\partial f}{\partial \varepsilon}\,\mu^2\langle Q_\parallel^0 \rangle^2 \quad . \qquad \ldots (3.13)$$

Because $\frac{\partial f}{\partial \varepsilon} < 0$, δW_0 is certainly non-negative if

$$\left(1 + \frac{C}{B_0^2}\right) > 0 \qquad \ldots (3.14)$$

However, even if this condition is satisfied δW_0 can still be minimised to zero by displacements such that,

$$\underset{\sim}{Q}_0 = \nabla \times (\underset{\sim}{\xi}_0 \times \underset{\sim}{B}_0) = 0 \quad . \qquad \ldots (3.15)$$

that is,

$$Q_{\perp}^{0} = B_0 \frac{\partial \xi_0}{\partial z} = 0$$

$$Q_{\parallel}^{0} = - B_0 (\nabla \cdot \xi_0) = 0 \; . \qquad \ldots (3.16)$$

If condition (3.14) is satisfied, then, we must proceed to higher order in the expansion scheme in order to ascertain the stability of the system. However if (3.14) is <u>not</u> satisfied then the system is certainly unstable since δW_0 can then be made negative by some ξ_0. A suitable ξ_0 can be found as follows[8]: Consider a localised displacement

$$\xi_0 = m_{\perp} \operatorname{Cos} k_{\parallel} z \operatorname{Cos} k_{\perp} [m_{\perp} \cdot (x - x_0)] \exp \left\{ \frac{-(x - x_0)^2}{2\delta^2} \right\} \qquad \ldots (3.17)$$

where $m_{\perp}$ is a unit vector perpendicular to n_0. Then by choosing $k_{\perp} \gg k_{\parallel} \gg \delta^{-1}$, $Q_{\parallel}^{2}$ and, because of the oscillatory behaviour of ξ_0, it will also be greater than $\langle Q_{\parallel} \rangle^2$. The energy integral δW_0 is then dominated by the term

$$(1 + \frac{C}{B_0^2}) \; [Q_{\parallel}^{0}]^2 \qquad \ldots (3.18)$$

and so will be negative if $(1 + C/B_0^2) < 0$.

At the zero-order stage, therefore, we obtain a <u>necessary</u> condition for stability, according to Kruskal-Oberman theory, namely that (3.14) be satisfied.

In the next order in λ, $\delta W_1 \equiv 0$ and

$$\delta W_2 = [Q_{\perp}^{1}] + (1 + \frac{C}{B_0^2}) \; [Q_{\parallel}^{1}]^2 - \sum m_i \iint \frac{B_0}{v_{\parallel}} \; \mu^2 \frac{\partial f}{\partial \varepsilon} \langle Q_{\parallel}^{1} \rangle^2 \; . \qquad \ldots (3.19)$$

If (3.18) is satisfied this is again non-negative but is zero for ξ_1 satisfying

$$Q^1 = \nabla \times (\xi_1 \times B_0) + \nabla \times (\xi_0 \times B_s) = 0 \; . \qquad \ldots (3.20)$$

A ξ_1 which satisfies (3.20) and ensures that ξ remains perpendicular to B in this order is

$$\xi_1 = \frac{B_0 \times \nabla \psi}{B_0^2} - \frac{(\xi_0 \cdot B_s) B_0}{B_0^2} \; , \qquad \ldots (3.21)$$

with ψ defined by

$$B_0 \cdot \nabla \psi = (\xi_0 \times B_s) \cdot n_0 \; . \qquad \ldots (3.22)$$

Proceeding further one finds $\delta W_3 \equiv 0$ and

$$\delta W_4 = Q^2 - (2\frac{Q_\parallel}{B} + s)(\underset{\sim}{\xi}_o \cdot \nabla p_\perp) + C(\frac{Q_\parallel}{B} + s)^2 \qquad \ldots (3.23)$$

$$- \sum m_i \iint \frac{B_o^3}{v_\parallel} \frac{\partial f}{\partial \varepsilon} \mu^2 \left\langle \frac{Q_\parallel}{B} + s \right\rangle^2 d\mu d\varepsilon$$

where the subscripts on $\underset{\sim}{Q}_2 \equiv \underset{\sim}{Q}$ and $s_2 \equiv s = \dfrac{\underset{\sim}{\xi} \cdot \nabla B}{B_o}$ have been suppressed.

This expression (3.23) leads to sufficient criteria for stability which can be obtained by writing it in the form

$$\delta W_4 = \underset{\sim}{Q}_\perp^2 + (1 + C/B^2) \left[Q_\parallel + \frac{Cs - \underset{\sim}{\xi}_o \cdot \nabla p_\perp}{B(1 + C/B^2)} \right]^2$$

$$- \sum_i m_i \iint \frac{B}{v_\parallel} \frac{\partial f}{\partial \varepsilon} (\mu B)^2 \left\langle \frac{Q_\parallel}{B} + s \right\rangle^2 d\mu d\varepsilon \qquad \ldots (3.24)$$

$$+ \frac{(Cs - \underset{\sim}{\xi}_o \cdot \nabla p_\perp)\,[\underset{\sim}{\xi}_o \cdot \nabla (p_\perp + \tfrac{1}{2} B^2)]}{B^2 + C} \quad .$$

For example, in the special equilibria[1,10] with the pressure of the form $p_\perp = p_\perp(B)$, $p_\parallel = p_\parallel(B)$, it can be shown that

$$\underset{\sim}{\xi}_o \cdot \nabla p_\perp = Cs_2 \equiv Cs$$

and

$$C = B \frac{dp_\perp}{dB}(B) \quad ,$$

hence in this case

$$\delta W_4 = \underset{\sim}{Q}_\perp^2 + (1 + \frac{C}{B_o^2})\, Q_\parallel^2 - \sum_i m_i \iint \frac{B_o^3}{v_\parallel} \frac{\partial f}{\partial \varepsilon} \mu^2 \left\langle \frac{Q_\parallel}{B_o} + s \right\rangle^2 . \qquad \ldots (3.25)$$

As in the lower orders, this is non-negative when $(1 + C/B_o^2) > 0$, however, in this order δW is <u>actually positive definite</u>. This is because the presence of the extra term "s" in the last expression means that, unlike δW_o and δW_2, the fourth order expression δW_4 cannot be minimised to zero by $\underset{\sim}{Q} \equiv 0$ (because $s = \dfrac{\underset{\sim}{\xi}_o \cdot \nabla B}{B_o}$ and $\dfrac{\partial \underset{\sim}{\xi}_o}{\partial z} = 0$ are sufficient to ensure that the last integral does not vanish even when $Q_\parallel = 0$). That is, δW_4 consists of three non-negative terms which cannot simultaneously be zero. Hence δW_4 is positive definite and the system is stable whenever $(1 + \frac{C}{B_o^2}) > 0$.

Combining this last result with the necessary criterion obtained in zero order we see that a necessary and sufficient criterion for stability of the special equilibria $p_\perp \equiv p_\perp(B)$, $p_\parallel \equiv p_\parallel(B)$ is simply

$$(1 + C/B^2) = (1 + \frac{1}{B}\frac{dp_\perp}{dB}) > 0 \qquad \ldots (3.26)$$

In fact, as shown in appendix C, (3.26) is an exact result[11] for these special equilibria, and is not restricted by the limitations of our expansion scheme.

IV. APPLICATION

Returning to the general result (3.24) we see that a sufficient condition for δW_4 to be positive is that (3.14) be satisfied together with

$$\int \frac{[\underset{\sim}{\xi}_o\cdot\nabla(p_\perp + \tfrac{1}{2}B^2)]\ [\frac{C}{B}\underset{\sim}{\xi}_o\cdot\nabla B - \underset{\sim}{\xi}_o\cdot\nabla p_\perp]}{C + B^2}\, d\tau > 0 \qquad \ldots (4.1)$$

and by (3.16) $\underset{\sim}{\xi}_o$ is independent of z so that (4.1) can be written as

$$\int \frac{1}{(C + B^2)}\left[\frac{\partial}{\partial x_\perp}(p_\perp + \tfrac{1}{2}B^2)\right]\left[\frac{C}{B}\frac{\partial B}{\partial x_\perp} - \frac{\partial p_\perp}{\partial x_\perp}\right] ds > 0 \qquad \ldots (4.2)$$

where the integration is along a field line. This condition, which is valid for small β and small field curvature, has some resemblance to the well known[2,12,13] zero-β criterion

$$\int \frac{1}{B^2}\frac{\partial B}{\partial\psi}\frac{\partial p}{\partial\psi}\, ds < 0 \quad ,$$

which holds for axial symmetry.

A more restrictive requirement which ensures stability is that the integrand of (4.1) itself be positive,

$$[\underset{\sim}{\xi}_o\cdot\nabla(p_\perp + \tfrac{1}{2}B^2)]\ [Cs - \underset{\sim}{\xi}_o\cdot\nabla p_\perp] > 0 \quad . \qquad \ldots (4.3)$$

Now to the required order in λ

$$\nabla_\perp\, p_\perp = \frac{C}{B}\,\nabla_\perp B + \overline{\nabla_\perp f} \qquad \ldots (4.4)$$

where

$$\overline{\nabla_\perp f} \equiv \sum m_i \iint \frac{\mu B^2}{v_\parallel}\ \nabla f\ d\mu d\varepsilon \quad ,$$

and (4.1) can be written

$$[\underset{\sim}{\xi}_0\cdot\nabla_\perp(p_\perp + \tfrac{1}{2}B^2)]\ [\underset{\sim}{\xi}_0\cdot\overline{\nabla_\perp f}] < 0 \ . \qquad \ldots (4.5)$$

If the vectors $\nabla_\perp(p_\perp + \frac{1}{2}B^2)$ and $\overline{\nabla_\perp f}$ are not parallel it is always possible to choose a $\underset{\sim}{\xi}_0$ which violates (4.5), and so no simple sufficient condition can be obtained; each case must be discussed individually using the less restrictive form (4.1).

If, however, we consider equilibria such that the current along magnetic field lines is zero $(j_\parallel \equiv 0)$, we have from ref. (1) and appendix A that, to λ^4 order,

$$(\nabla_\perp p_\perp \times \nabla_\perp B)\cdot\underset{\sim}{B}_0 = 0$$

so that the vectors $\nabla_\perp p_\perp$ and $\nabla_\perp B$ are parallel to each other and hence to $\overline{\nabla_\perp f}$. Then we can write

$$\overline{\nabla_\perp f} = \alpha\nabla_\perp B$$

for some scalar function α, and a simple sufficient condition because

$$\alpha\nabla_\perp B\cdot\nabla_\perp(p_\perp + \tfrac{1}{2}B^2) \leqslant 0 \ . \qquad \ldots (4.6)$$

Together with (3.14), which can be written with the aid of (3.8) in the form

$$\nabla_\parallel B\cdot\nabla_\parallel(p_\perp + \tfrac{1}{2}B^2) > 0 \ , \qquad \ldots (4.7)$$

equation (4.6) gives a complete sufficient condition for stability.

Equations (4.6) and (4.7) limit the maximum pressure gradients in the system and hence limit the maximum plasma pressure for which stability can be maintained. Equation (4.7) is necessary and the pair together are sufficient. (These same results can be obtained from the energy integral due to Newcomb[8], hence (4.7) really is a necessary condition even in that more accurate theory.)

Although the basic limitation is on pressure gradient, it is convenient to have simple direct estimates of the maximum pressure (or maximum β) which is allowed in a magnetic well. Such estimates are obtained by integrating the gradient condition. For example for magnetic wells in which B increases monotonically (and we consider no other) the condition (4.7) shows that the maximum plasma pressure is restricted to

$$p_\perp^{max} = \tfrac{1}{2}\Delta_\parallel\ B^2 \qquad \ldots (4.8)$$

where $\Delta_\parallel B^2$ measures the depth of the magnetic well taken along the line of force through the minimum point to the edge of the plasma. To obtain a similar result from (4.6) we must first consider the coefficient α which connects the variation of $f(\mu,\varepsilon,x_\perp)$ with the variation of B. If f is independent of $x_\perp$ then we return[1] to the special equilibria $p_\perp \equiv p_\perp(B)$, $p_\parallel \equiv p_\parallel(B)$ and for this case $\alpha \equiv 0$ so that (4.6) is automatically satisfied. Equation (4.8) then provides an adequate (and, in fact, exact[11]) estimate of $p_\perp^{max}$.

In general α is non-zero; if we try to obtain greater plasma densities by increasing f near the centre of the well then f will increase as B decreases and α will normally be negative. Hence in most cases of interest (4.6) can be replaced by

$$\nabla_\perp B \cdot \nabla_\perp(p_\perp + \tfrac{1}{2}B^2) \geqslant 0 \qquad \dots (4.9)$$

and so leads to a maximum pressure

$$p_\perp^{max} = \tfrac{1}{2}\Delta_\perp B^2 \qquad \dots (4.10)$$

where $\Delta_\perp B^2$ is a measure of the depth of the magnetic well transverse to the field lines. In many situations the transverse depth of the well is less than the longitudinal depth so that (4.10) is a more restrictive condition than (4.8).

It should be borne in mind that the depths of well referred to above are the actual depths in the presence of plasma. Because the plasma is diamagnetic it tends to increase the depth of the magnetic well, by an amount which is proportional to $p_\perp^{max}$ but depends on the shape of the plasma distribution. Hence it is possible to write (4.10) as

$$p_\perp^{max} = \tfrac{1}{2}k\ \Delta_\perp B^2_{vac} \qquad \dots (4.11)$$

where $\Delta_\perp B^2_{vac}$ is the depth of the magnetic well in the absence of plasma and k is a shape dependent coefficient which is greater than unity. It is related to the classical demagnetising factor and can be calculated exactly for certain simple shapes (appendix A). This last form (4.11) relates the limiting pressure to the depth of the depression in the basic vacuum field and so is particularly useful for a quick survey of proposed containment systems.

V. DISCUSSION

In the preceding sections the stability of a plasma in a minimum-B field has been analysed by an expansion procedure. This procedure corresponds to examining plasma stability in a shallow magnetic well. The maximum pressure is restricted by conditions (4.6) and (4.7) on the pressure gradient. These can be translated into direct estimates of the critical pressure such as (4.8) and (4.10) which show that $p_\perp^{max}$ is equal to the "transverse depth" of the magnetic well or to its "longitudinal depth" whichever is less. The maximum β which can be contained in a magnetic well is

$$\beta_c \simeq \frac{(B_1^2 - B_2^2)}{B_1^2}$$

where B_1 is the value of $|\underset{\sim}{B}|$ on the largest closed $|\underset{\sim}{B}|$ contour (see eg Ref.1) and B_2 is the actual minimum of $|B|$ in the containment region. This is an encouraging result since β_c can easily attain values as high as one-third or more in practical systems. This is significantly greater than other confinement systems such as the Stellarator[3]. (Strictly the well depths are those measured in the presence of the plasma and are even deeper than those of the original vacuum well, to which they can be related as in appendix A.) Restrictions on plasma pressure seem to arise in two different ways, one by a restriction on the parallel gradient (4.7) and the other on the perpendicular gradient (4.6). These two criteria can be related to two different forms of instability, though the distinction is not clear cut.

The restriction on the parallel gradient stems from the condition $(1 + C/B^2) > 0$, and the corresponding instability (3.17) is closely related to the "mirror" instability in a uniform plasma. This is normally regarded as imposing a limitation on $p_\perp^2/p_\parallel$ and such a limitation does indeed follow from (3.14) and (3.7); namely $p_\perp^2/3p_\parallel < B^2$.

The restriction imposed on the perpendicular gradient is given by (4.6) which, using (4.4) becomes

$$\left(\frac{\alpha}{C + \alpha B}\right) \nabla_\perp p_\perp \cdot \nabla_\perp \left(p_\perp + \tfrac{1}{2}B^2\right) < 0 \quad . \qquad \ldots (5.1)$$

The interesting situation is $\alpha < 0$, $C < 0$ so we can write this as[14]

$$\nabla_\perp p_\perp \cdot \nabla_\perp \left(p_\perp + \tfrac{1}{2}B^2\right) < 0 \quad . \qquad \ldots (5.2)$$

The last factor in this expression is related to the radius of curvature and hence violation of this criterion occurs when the plasma diamagnetic currents have so modified the vacuum field that the relative signs of $\nabla_\perp p_\perp$ and the curvature have been reversed compared to the zero-β situation. In the low-β limit the lines of force in a minimum-B system are everywhere convex toward the plasma - that is have "stable" curvature. If one now adds some plasma to the centre of the system then, being diamagnetic, it causes the lines of force to "bulge out", that is it tends to create an unstable curvature.

The corresponding instability is akin to the "kink" instability, but the anisotropy of the pressure, and the close connection between pressure gradient and pressure anisotropy in mirror systems, makes it difficult to relate the instabilities in a magnetic well to the simpler form of instability in infinite media or infinite cylinders.

In order to obtain these results we have introduced a specific ordering of the various expansion parameters and it may be asked why our choice is the "correct" one. Briefly the reason is that with any other choice of ordering one particular parameter would be dominant and no critical condition would have been obtained. Perhaps the most arbitrary element in the ordering is the relation of $p_\perp^\beta$ to λ. We have taken $p_\perp^\beta \sim \lambda^2$ and have obtained the criterion given above. If $p_\perp$ were larger, say of order λ, then the equilibrium condition (2.5) would require $p_\parallel \sim \lambda^2$. Then in zero order we would again obtain the expression (3.13) so that a necessary condition would again be $(1 + C/B^2) > 0$ which for example with $p_\perp = p_\perp(B)$ leads, as before to a critical pressure

$$p_\perp^{max} \lesssim B_0 \Delta B_1 \quad . \qquad \ldots (5.3)$$

However in this case ΔB_1 would be simply ΔB_β, the depression due to the plasma itself, and because $\Delta B_\beta < p_\perp/B_0^2$ equation (5.3) could never be satisfied. Hence if the orders of magnitude are such that $p_\perp^\beta \sim \lambda$ the system is always unstable.

If on the other hand $p_\perp$ were much smaller, say of order λ^3, then for equilibrium $p_\parallel \sim \lambda^5$ and $C \sim \lambda$. It would be found that δW_4 can be minimised to zero in the same way that δW_0 and δW_2 were before. Then stability is determined by δW_5 which would be given by

$$\delta W_5 = - s_2 (\underset{\sim}{\xi}_0 \cdot \nabla p_\perp) + Cs_2^2 - \sum m_i \iint \frac{B_0^3}{v_\parallel} \frac{\partial f}{\partial \varepsilon} \mu^2 \langle s_2 \rangle^2 \qquad \ldots (5.4)$$

and so is positive if

$$\underset{\sim}{\xi}_0 \cdot \nabla B (Cs - \xi_0 \cdot \nabla p_\perp) > 0 \qquad \ldots (5.5)$$

which is clearly just what is obtained from (4.3) when $\nabla p \ll \nabla B^2$. In fact with this ordering δW_5 is simply proportional to $p_\perp$ and so cannot lead to a critical pressure; it is the same as the $\beta \to 0$ case discussed in Ref. 1.

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APPENDIX A

THE EQUILIBRIUM CONFIGURATION

In this section some features of the equilibrium are examined in more detail. In particular we consider the solution of the equilibrium equation and the calculation of $\underset{\sim}{B}_\beta$, the magnetic field due to the plasma in λ^2 order. We also consider some higher order equilibrium quantities and the way in which these lead to constraints on the equilibrium pressure distribution.

The Plasma Field

To second order in λ the equilibrium equation gives

$$\underset{\sim}{j}_\perp = \frac{\underset{\sim}{B}_0 \times \nabla p_\perp}{B_0^2} \qquad \ldots \text{(A.1)}$$

where $p_\perp$ is the (arbitrary) second order perpendicular pressure. It follows that $\nabla \cdot j_\perp \equiv 0$ so that $j_\parallel$ is constant along lines of force and for plasma surrounded by vacuum $j_\parallel \equiv 0$. The field due to plasma currents is therefore given by

$$\nabla \times \underset{\sim}{B}_\beta = \underset{\sim}{j}_\perp = - \nabla \times \left(\frac{p_\perp \underset{\sim}{B}_0}{B_0^2}\right) \qquad \ldots \text{(A.2)}$$

$$\nabla \cdot \underset{\sim}{B}_\beta = 0 \qquad \ldots \text{(A.3)}$$

and the problem is simply that of finding the magnetic field due to a distribution of magnetisation

$$\underset{\sim}{M} = - \frac{p_\perp \underset{\sim}{B}_0}{B_0^2} \quad . \qquad \ldots \text{(A.4)}$$

The general solution of (A.2), (A.3) is

$$\underset{\sim}{B}_\beta = - p_\perp \frac{\underset{\sim}{B}_0}{B_0^2} + \nabla \varphi \qquad \ldots \text{(A.5)}$$

where

$$\nabla^2 \varphi = \frac{1}{B_0} \frac{\partial p_\perp}{\partial z} = (- \nabla \cdot \underset{\sim}{M}) \quad . \qquad \ldots \text{(A.6)}$$

A simple illustration is the case of a spherical plasma pressure distribution $p_\perp \equiv p(r)$ when

$$\varphi = \frac{z}{B_0 r^3} \int_0^r r^2 p(r)\, dr \quad . \qquad \ldots \text{(A.7)}$$

Only the z component of $\underset{\sim}{B}_\beta$ is really required since to second order that is what determines the change in the depth of the magnetic well. This is given by

$$\underset{\sim}{B}_o \cdot \underset{\sim}{B}_\beta = -\left(1 - \frac{z^2}{r^2}\right) p(r) + \left(1 - \frac{3z^2}{r^2}\right) \frac{1}{r^3} \int_o^r r^2 p(r) \qquad \ldots \text{(A.8)}$$

from which we can calculate the effect of plasma on the well depth. At the centre of the well the field is decreased by

$$\tfrac{1}{2}\, \Delta^\beta B^2 = \frac{2}{3}\, p(o) \quad . \qquad \ldots \text{(A.9)}$$

Because the field is also affected at the plasma boundary the transverse well depth is actually increased by more than (A.9) while the longitudinal well depth is increased by less than (A.9).

Similar results can be obtained for an ellipsoidal pressure distribution by using the known solutions of (A.6) corresponding to the problem of a uniformly magnetised ellipsoid (or spheroid).

If a spheroid of semi axes a, b, is uniformly magnetised in a direction parallel to the Oa axis then the field at its centre[15] is

$$\underset{\sim}{B}(o) = \underset{\sim}{M}\, [1 - A(e)] \qquad \ldots \text{(A.10)}$$

where $\underset{\sim}{M}$ is the magnetisation and A is the "demagnetising factor" which depends only on the eccentricity of the spheroid. If the long axis of the spheroid is in the direction of magnetisation $(a > b)$ then

$$A = \left\{\frac{1}{2e} \log \frac{1 + e}{1 - e} - 1\right\} \frac{(1 - e^2)}{e^2} \qquad \ldots \text{(A.11)}$$

where $e^2 = 1 - b^2/a^2$. If the short axis is in the direction of magnetisation $(a < b)$ then

$$A = \left\{\frac{1}{e^2} - \frac{(1 - e^2)^{\frac{1}{2}}}{e^3} \sin^{-1} e\right\} \qquad \ldots \text{(A.12)}$$

where $e^2 = 1 - a^2/b^2$. When the pressure isobars are all similar spheroids the depression of the field at the centre is just

$$\underset{\sim}{B}_o \cdot \underset{\sim}{B}_\beta = p_\perp(o)\, [1 - A(e)] \quad . \qquad \ldots \text{(A.13)}$$

When the isobars are spheroidal but not similar an appropriate average of $A(e)$ is required. If the spheroids are very elongated in the field direction, $A(e) \to 0$, and the full diamagnetic effect of the plasma is experienced.

Higher Order Equilibrium Currents

In the preceding paragraphs the second order equilibrium current and the field resulting from it were calculated. In this order of approximation the pressure $p_\perp^{(2)}$ is entirely arbitrary. However the earlier work[1] showed that the pressure tensor must satisfy a constraint if $\nabla\cdot\underset{\sim}{j}$ is to be zero. As we have seen, this is the case in second order for any $p_\perp$ and it is of interest, therefore, to see how a constraint reappears in higher order approximations. We will find that although the constraint appears only in fourth order it nevertheless imposes a restriction on the second order pressure tensor.

We have already seen that in second order the current perpendicular to $\underset{\sim}{B}_o$ is

$$j_\perp^{(2)} = \frac{\underset{\sim}{B}_o \times \nabla p_\perp^{(2)}}{B_o^2} \qquad \text{... (A.14)}$$

and

$$j_\parallel^{(2)} = 0 \ .$$

In the third order the equilibrium equation gives

$$\underset{\sim}{j}_3 \times \underset{\sim}{B}_o + \underset{\sim}{j}_2 \times \underset{\sim}{B}_1 = \operatorname{div} \underset{\approx}{p}^{(3)} \ . \qquad \text{... (A.15)}$$

so that we can again find the current perpendicular[16] to $\underset{\sim}{B}_o$ in this order. In fact

$$j_\perp^{(3)} = -\frac{(\underset{\sim}{B}_o \times \underset{\sim}{B}_1)}{B_o^2} \cdot \nabla\left\{\frac{1}{B_o}\frac{\partial p_\perp^{(2)}}{\partial z}\right\} \qquad \text{... (A.16)}$$

and the requirement that $\nabla\cdot\underset{\sim}{j}^{(3)} = 0$,

$$\nabla_\perp\cdot j_\perp + \frac{\partial j_\parallel}{\partial z} = 0 \qquad \text{... (A.17)}$$

leads to

$$j_\parallel^{(3)} = \frac{\underset{\sim}{B}_o \times \underset{\sim}{B}_1}{B_o^2} \cdot \nabla p_\perp^{(2)} \ . \qquad \text{... (A.18)}$$

(Here again it should be recalled that $j_\parallel^{(3)}$ is the current parallel to $\underset{\sim}{B}_o$; the current parallel to $\underset{\sim}{B}$ in this order is $(j_\parallel^{(3)} + \underset{\sim}{B}_1\cdot\underset{\sim}{j}^{(2)}/B_o)$ and this vanishes identically.)

Continuing to fourth order we can similarly find $j_\perp^{(4)}$ in terms of the lower order fields and pressures, but in this order no satisfactory solution to (A.17) can be found. This is because we must have $j_\parallel$ vanishing at the plasma boundary and this means that we must have

$$\int dz\ (\nabla\cdot j_\perp^{(4)}) = 0 \qquad \text{... (A.19)}$$

where the integral is along any line parallel to z-axis from one plasma boundary to the other. This condition, which was automatically satisfied in lower order, appears as a constraint in fourth order. After some lengthy albegra (A.19) can be reduced to the constraint

$$\int dz\ \nabla p_\perp^{(2)}\cdot(\underset{\sim}{B}_o \times \nabla B) = 0 \ . \qquad \text{... (A.20)}$$

This constraint is clearly satisfied by taking $p_\perp^{(2)} \equiv p_\perp(B)$: in fact it can be shown that if $p_\perp$ and $p_\parallel$ are functions only of the field strength, then equilibrium is satisfied to all orders[10].

APPENDIX B

THE ENERGY FUNCTION

In this appendix the Kruskal-Oberman energy integral is rearranged into a form in which the dependence of δW on $\underset{\sim}{\xi}$ is contained in terms involving $\underset{\sim}{Q}$, s and q only, where

$$\underset{\sim}{Q} \equiv \underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times \underset{\sim}{B}); \quad s = \frac{\underset{\sim}{\xi}\cdot\nabla B}{B}; \quad q \equiv \underset{\sim}{n}\cdot(\underset{\sim}{n}\cdot\underset{\sim}{\nabla})\,\underset{\sim}{\xi} \qquad \ldots \text{(B.1)}$$

and it is shown, that although Q has contributions to all orders in the λ expansion, s and q have no contribution of lower order than λ^2. This, together with the known ordering of $p_\parallel$, $p_\perp$, C and $j_\parallel$ (see appendix A), makes it a straightforward matter to write down δW in the various orders of the λ expansion. The original form for δW is,

$$\begin{aligned}\delta W \;=\; & Q^2 - \underset{\sim}{j}\cdot\underset{\sim}{Q} \times \underset{\sim}{\xi} + (2p_\perp + C)(\nabla\cdot\underset{\sim}{\xi} - q)^2 + \underset{\sim}{\xi}\cdot\nabla p_\perp\,(\nabla\cdot\underset{\sim}{\xi} - q) \\ & + (\underset{\sim}{\xi}\cdot\nabla p_\parallel)q - (p_\parallel - p_\perp)\,\left[\underset{\sim}{n}\cdot(\underset{\sim}{a}\cdot\nabla)\,\underset{\sim}{\xi} + \underset{\sim}{a}\cdot(\underset{\sim}{n}\cdot\nabla)\,\underset{\sim}{\xi} - q^2 - q\nabla\cdot\underset{\sim}{\xi}\right] \qquad \ldots \text{(B.2)} \\ & - \sum_i m_i \iint \frac{B}{v_\parallel}\frac{\partial f}{\partial \varepsilon}\,\left\langle v_\parallel^2\, q + \mu B(\nabla\cdot\underset{\sim}{\xi} - q)\right\rangle^2 d\mu d\varepsilon\end{aligned}$$

where, as in the main text,

$$\underset{\sim}{a} = (\underset{\sim}{n}\cdot\nabla)\,\underset{\sim}{\xi} - (\underset{\sim}{\xi}\cdot\nabla)\underset{\sim}{n} \quad ; \quad q = \underset{\sim}{n}\cdot\underset{\sim}{a} \equiv \underset{\sim}{n}\cdot(\underset{\sim}{n}\cdot\nabla)\,\underset{\sim}{\xi} \qquad \ldots \text{(B.3)}$$

and

$$\langle g \rangle = \int \frac{d\ell}{v_\parallel}\, g \Bigg/ \int \frac{d\ell}{v_\parallel} \;.$$

Then, with $Q_\parallel \equiv \underset{\sim}{n}\cdot\underset{\sim}{Q}$, we have the following identities,

$$\frac{\underset{\sim}{Q}}{B} = \underset{\sim}{a} - \underset{\sim}{n}\,(\nabla\cdot\underset{\sim}{\xi} + s) \quad ; \quad \frac{Q_\parallel}{B} = q - \nabla\cdot\underset{\sim}{\xi} - s \qquad \ldots \text{(B.4)}$$

On substituting (B.4) into (B.2) we obtain

$$\begin{aligned}\delta W = \; & \underset{\sim}{Q}_\perp^2 + Q_\parallel^2 - \underset{\sim}{j}\cdot\underset{\sim}{Q} \times \underset{\sim}{\xi} + (2p_\perp + C)\,\left(\frac{Q_\parallel}{B} + s\right)^2 - \underset{\sim}{\xi}\cdot\nabla p_\perp\left(\frac{Q_\parallel}{B} + s\right) \\ & + (\underset{\sim}{\xi}\cdot\nabla p_\parallel)q - (p_\parallel - p_\perp)\left[\frac{\underset{\sim}{n}\cdot(\underset{\sim}{Q}_\perp\cdot\nabla)\,\underset{\sim}{\xi} + \underset{\sim}{Q}_\perp\cdot(\underset{\sim}{n}\cdot\nabla)\underset{\sim}{\xi}}{B} + q\,\left(\frac{Q_\parallel}{B} + s\right)\right] \qquad \ldots \text{(B.5)} \\ & - \sum_i m_i \iint \frac{B}{v_\parallel}\frac{\partial f}{\partial \varepsilon}\left\langle v_\parallel^2\, q - \mu B\left(\frac{Q_\parallel}{B} + s\right)\right\rangle^2 d\mu d\varepsilon \;.\end{aligned}$$

The term in $\underset{\sim}{j}$ can be conveniently transformed using the equilibrium equation to replace

$$\underset{\sim}{j}_\perp = \frac{\underset{\sim}{B} \times \underset{\sim}{\nabla}\cdot\underset{\approx}{P}}{B^2} .$$

Thus

$$\underset{\sim}{j}\cdot\underset{\sim}{Q} \times \underset{\sim}{\xi} = j_\parallel\, \underset{\sim}{n}\cdot\underset{\sim}{Q} \times \underset{\sim}{\xi} + \frac{Q_\parallel}{B}\, \underset{\sim}{\xi}\cdot\nabla p_\perp + (p_\perp - p_\parallel)\, \frac{Q_\parallel}{B}\, q \qquad \text{... (B.6)}$$

where we have made use of the fact that $\underset{\sim}{\xi} \equiv \underset{\sim}{\xi}_\perp$ so that $\underset{\sim}{\xi}\cdot\underset{\sim}{n} = 0$ and, for example

$$\underset{\sim}{\xi}\cdot(\underset{\sim}{n}\cdot\underset{\sim}{\nabla})\underset{\sim}{n} = -\,\underset{\sim}{n}\cdot(\underset{\sim}{n}\cdot\underset{\sim}{\nabla})\underset{\sim}{\xi} \equiv -\,q \ . \qquad \text{... (B.7)}$$

Also from (B.3)

$$\underset{\sim}{Q}_\perp\cdot(\underset{\sim}{n}\cdot\nabla)\underset{\sim}{\xi} = \frac{Q_\perp^2}{B} - \underset{\sim}{n}\cdot(\underset{\sim}{\xi}\cdot\nabla)\underset{\sim}{Q}_\perp$$

so that in the $(p_\parallel - p_\perp)$ term of (B.5)

$$\frac{\underset{\sim}{n}\cdot(\underset{\sim}{Q}_\perp\cdot\nabla)\underset{\sim}{\xi} + \underset{\sim}{Q}_\perp\cdot(\underset{\sim}{n}\cdot\nabla)\underset{\sim}{\xi}}{B} = \frac{1}{B^2}\left[Q_\perp^2 - j_\parallel\, \underset{\sim}{n}\cdot\underset{\sim}{Q} \times \underset{\sim}{\xi}\right] \qquad \text{... (B.8)}$$

Finally, on substituting (B.6) and (B.8) into (B.5) the following expression is obtained,

$$\begin{aligned}\delta W = {} & \underset{\sim}{Q}_\perp^2 \left(1 + \frac{p_\perp - p_\parallel}{B^2}\right) + Q_\parallel^2 \left(1 + \frac{2p_\perp + C}{B^2}\right) - j_\parallel\, \underset{\sim}{n}\cdot\underset{\sim}{Q}_\perp \times \underset{\sim}{\xi}\left(1 + \frac{p_\perp - p_\parallel}{B^2}\right) \\ & + q\left[\underset{\sim}{\xi}\cdot\nabla p_\parallel + (p_\perp - p_\parallel)s\right] - \left[\underset{\sim}{\xi}\cdot\nabla p_\perp - (2p_\perp + C)\, s\right]\left(\frac{2Q_\parallel}{B} + s\right) \qquad \text{... (B.9)} \\ & - \sum_i m_i \iint \frac{B}{v_\parallel}\frac{\partial f}{\partial \varepsilon} \left\langle v_\parallel^2 q - \mu B\left(\frac{Q_\parallel}{B} + s\right)\right\rangle^2 d\mu d\varepsilon \ .\end{aligned}$$

The expression (B.9) is exact, and equivalent to the original form (B.2) but more convenient to use. The terms of various order in the λ expansion of δW can easily be obtained from (B.9) by noting e.g. that s has contributions only of second and higher order (since $\nabla B \sim \lambda^2$). Although less obvious this is also true of q, as can be shown from the perpendicular component of the equilibrium equation.

$$\underset{\sim}{B} \times \frac{[(\underset{\sim}{\nabla} \times \underset{\sim}{B}) \times \underset{\sim}{B}]}{B^2} = \frac{\underset{\sim}{B} \times \nabla p_\perp}{B^2} + \frac{(p_\parallel - p_\perp)}{B^2}\, \underset{\sim}{B} \times (\underset{\sim}{n}\cdot\underset{\sim}{\nabla})\underset{\sim}{n} \ ,$$

that is

$$\underset{\sim}{B} \times (\underset{\sim}{n}\cdot\underset{\sim}{\nabla})\underset{\sim}{n}\ [B^2 + p_\perp - p_\parallel] = \underset{\sim}{B} \times \nabla(p_\perp + \tfrac{1}{2}B^2) \quad . \qquad \ldots \text{(B.10)}$$

Taking the cross product of equation (B.10) with $\underset{\sim}{\xi}$ gives

$$\underset{\sim}{\xi}\cdot(\underset{\sim}{n}\cdot\underset{\sim}{\nabla})\underset{\sim}{n}\ [B^2 + p_\perp - p_\parallel] = \underset{\sim}{\xi}\cdot\nabla(p_\perp + \tfrac{1}{2}B^2)$$

and therefore

$$q = - \frac{\underset{\sim}{\xi}\cdot\nabla(p_\perp + \tfrac{1}{2}B^2)}{B^2 + p_\perp - p_\parallel} \qquad \ldots \text{(B.11)}$$

which, like s has no contribution of lower order than λ^2.

APPENDIX C

AN EXACT RESULT

It is shown in this appendix that exact necessary and sufficient stability criteria can be obtained for the special equilibria of the form[1,10] $p_\perp \equiv p_\perp(B)$, $p_\parallel \equiv p_\parallel(B)$ without approximation or restriction as to depth of well, and that the limiting β is indeed given by

$$\beta_{max} = \frac{B_1^2 - B_2^2}{B_2^2} \qquad \dots (C.1)$$

where B_1 is the value of $|B|$ on the plasma boundary, (which is a contour of constant $|B|$ in this case), and B_2 is the value of $|B|$ at the centre of the well.

The derivatives of $p_\parallel$ and $p_\perp$ in any direction are given by

$$\frac{\partial p_\parallel}{\partial s} = p_\parallel' \frac{\partial B}{\partial s} \quad ; \quad \frac{\partial p_\perp}{\partial s} = p_\perp' \frac{\partial B}{\partial s}$$

where the prime denotes differentiation with respect to $|B|$, and on substituting these results into (3.8) and (2.5) respectively we obtain

$$\begin{aligned} Bp_\perp' &= 2p_\perp + C \\ Bp_\parallel' &= p_\parallel - p_\perp \end{aligned} \qquad \dots (C.2)$$

Then

$$\underset{\sim}{\xi} \cdot \nabla p_\parallel = p_\parallel' Bs = (p_\parallel - p_\perp)s$$

$$\underset{\sim}{\xi} \cdot \nabla p_\perp = p_\perp' Bs = (2p_\perp + C)s$$

and since $j_\parallel \equiv 0$ for these equilibria[10], the expression (B.9) for δW reduces to

$$\begin{aligned} \delta W = {} & \underset{\sim}{Q}_\perp^2 \left(1 + \frac{p_\perp - p_\parallel}{B^2}\right) + Q_\parallel^2 \left(1 + \frac{2p_\perp + C}{B^2}\right) \\ & - \sum m_i \iint \frac{B}{v_\parallel} \frac{\partial f}{\partial \varepsilon} \left\langle v_\parallel^2 q - \mu B \left(\frac{Q_\parallel}{B} + s\right)\right\rangle^2 d\mu d\varepsilon \, . \end{aligned} \qquad \dots (C.4)$$

Hence sufficient conditions for stability are

$$\begin{aligned} B^2 - p_\parallel + p_\perp &> 0 \\ B^2 + 2p_\perp + C &> 0 \end{aligned} \qquad \dots (C.5)$$

However these are also necessary conditions, because just as in Section 3 it is always possible to find displacements for which δW is dominated by the terms in $Q_\perp^2$ or in $Q_\parallel^2$.

Using (C.2), we may rewrite (C.5) as conditions on the pressure gradient, namely

$$B - \frac{dp_\parallel}{dB} > 0 \qquad \qquad \ldots \text{(C.6)}$$

$$B + \frac{dp_\perp}{dB} > 0$$

and by integrating the second of these inequalities from the centre of the well to the plasma boundary the exact result (C.1) is obtained for β_{max}.