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Abstract

The equations of MHD equilibrium are solved by including plasma pressure and current in a large aspect-ratio ordering scheme first used by Dobrott and Frieman (1971) for the calculation of toroidal, $\ell = 3$ stellarator vacuum fields. The extended ordering unifies the low-beta equilibrium theory for tokamaks and $\ell = 3$ stellarators, and allows solutions to be obtained simply for arbitrarily prescribed pressure and current density profiles. Expressions are given for the equilibrium magnetic field and the equation for the flux surfaces is calculated, including the effects of $\ell = 3$ shaping and toroidal displacement. These results are used to calculate equilibria for the parameters of CLEO stellarator, and we examine the role of an externally applied vertical field in reducing pressure-induced flux surface distortion and destruction.

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1. Introduction

Many equilibrium calculations have been performed for Stellarators (e.g. Solovev and Shafranov, 1970). In these, two main approaches are employed, given the difficulty of exact solution in toroidal geometry. Either we can expand quantities in Taylor-series about a known magnetic axis (Mercier and Luc, 1974; Lortz and Nührenberg, 1976), or we can adopt an ordering scheme in which the helical fields, pressure, etc., will be small in some parameter. The former approach is of greatest value in dealing with systems in which the magnetic axis is a three-dimensional curve, as in Spitzer's original figure-8 stellarator. With an ordering scheme, however, we often have greater flexibility, validity no longer being restricted to the region close to the axis.

In the 'conventional' stellarator ordering, the helical field amplitude is the basic expansion parameter, the axial current, plasma pressure and toroidicity all being treated as second order. The equilibrium configuration in a toroidal $\ell = 3$ stellarator was calculated by Greene and Johnson (1961) using this method; however, the triangularity of the flux surfaces was suppressed at the chosen level of approximation by the averaging technique used for their determination, and two-dimensional numerical solution of the equilibrium equation was necessary because of the equal relative orderings of pressure and toroidicity.

In many low pressure stellarator experiments, the helical flux surface distortions are substantial, and typically these devices operate in a tokamak-like mode with regard to the size of the total rotational transform and the plasma pressure. Thus, it is of interest to derive an approximate solution to the equilibrium equations by means of an ordering scheme which better reflects the conditions in these experiments. In this way we can also unify the theory of equilibria at low pressure in tokamaks and in stellarators.

Considering only $\ell = 3$ stellarators, we investigate the toroidal, large aspect-ratio case using the ordering introduced by Dobrott and Frieman (1971) to define the vacuum field. Since the number of helical field periods round the torus is large and the pressure is low, the rotational transform is composed of independent additive contributions from the vacuum field and from the plasma current separately (Ohasa et al., 1977). Like the vacuum transform, that due to the current is of order unity in general, and we incorporate the effects of pressure at second order in the inverse aspect ratio, as in the low pressure tokamak theory. With these orderings, a satisfactory level of approximation can be attained in solution of the general low-beta problem for arbitrary pressure and current density profiles.

In section 2, we introduce the coordinates and basic characteristics of the model and describe the ordering scheme. Section 3 is devoted to the calculation of the equilibrium magnetic fields and in section 4 we derive the equation for the flux surfaces. The boundary value relation between the external vertical field and the plasma surface displacement is calculated in section 5 , and in section 6 several interesting special cases are examined. Finally the use of these results is illustrated by application to the case of Cleo stellarator (Atkinson et al., 1976).

2. Coordinates and Ordering Scheme

It proves convenient to use quasi-cylindrical polar coordinates (r, θ, ξ) as shown in figure 1, where r is measured from the minor axis of the torus. The inverse aspect ratio ε is assumed small and in order to recover the basic tokamak-like features of the system, it is supposed that the plasma current generates a poloidal field of order ε relative to the main field $\underset{\sim}{B}_o$ which is directed in the ξ direction, and that the plasma pressure $P \sim \varepsilon^2$. We assume that the number of $\ell = 3$ helical field periods round the torus, p is large and that the helical wavelength $\frac{2\pi}{h}$ is large compared with a typical minor radius a , so that $ha \ll 1$: for example, the plasma radius, or the stellarator winding radius can be used for the normalising length a . Note that $p = hR_o$ where R_o is the major radius of the torus, so $p = \varepsilon^{-1} ha$. Thus, if $ha \sim \varepsilon^{\alpha}$, then $0 < \alpha < 1$. By requiring that the toroidal and helical modulations of B_ξ be of the same order and that the rotational transform be of order unity, we determine $\alpha = \frac{1}{3}$, so that $p = \bar{p}\varepsilon^{-2/3}$, where $\bar{p} \sim O(1)$. This is the ordering proposed originally by Dobrott and Frieman (1971).

We introduce the scaled variables $\rho = r/a$ and $\bar{s} = hR_o\xi = \bar{p}\varepsilon^{-2/3}\xi$. The magnetic field and plasma pressure are normalised with respect to B_o , so we have $\underset{\sim}{b} = \underset{\sim}{B}/B_o$ and $\hat{P} = P/B_o^2$. Further defining $\hat{\underset{\sim}{J}} = \underset{\sim}{J}ga/B_o$, where $g = 1 - \varepsilon\rho\cos\theta$, and $\underset{\sim}{J}$ is the plasma current density, and $\hat{\nabla} = ag\underset{\sim}{\nabla}$, we write the MHD equilibrium equations

$$\hat{\nabla}\hat{P} = \hat{\underset{\sim}{J}} \times \underset{\sim}{b} \quad (i); \qquad \hat{\underset{\sim}{J}} = \hat{\nabla} \times \underset{\sim}{b} \quad (ii); \qquad \hat{\nabla} \cdot \underset{\sim}{b} = 0 \quad (iii) \qquad (1)$$

The field, current and pressure are now expanded with the orderings indicated, in powers of $\lambda = \epsilon^{\frac{1}{3}}$, so that

$$\hat{P} = \lambda^6 \hat{p}^{(6)} + \ldots ; \qquad \hat{\underset{\sim}{J}} = \lambda^3 \hat{\underset{\sim}{J}}^{(3)} + \ldots ; \qquad \underset{\sim}{b} = \underset{\sim}{b}^{(o)} + \lambda^2 \underset{\sim}{b}^{(2)} \ldots$$

In addition, we must expand the differential operators, so that for example

$$\hat{\nabla} f = \hat{\nabla}^{(o)} f + \lambda \hat{\nabla}^{(1)} f + \lambda^3 \hat{\nabla}^{(3)} f ,$$

where

$$\hat{\nabla}^{(o)} = \hat{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho}\hat{\theta}\frac{\partial}{\partial\theta} \quad ; \quad \hat{\nabla}^{(1)} = \hat{\xi}\,\bar{p}\frac{\partial}{\partial\bar{s}} ,$$

and

$$\hat{\nabla}^{(3)} = -\rho\cos\theta\,\hat{\nabla}^{(o)} .$$

The plasma current can be represented in the form

$$\hat{\underset{\sim}{J}} = \frac{ga\,\underset{\sim}{J}}{B_o} = g\,h\,\underset{\sim}{b} + \frac{\underset{\sim}{b} \times \hat{\hat{\nabla}}P}{\underset{\sim}{b}^2} \tag{2}$$

where the force-free current h is determined by

$$g\,\underset{\sim}{b} \,.\, \hat{\nabla}h = - \left(\underset{\sim}{b} \times \hat{\nabla}\hat{P}\right) . \hat{\nabla}\left(\frac{1}{\underset{\sim}{b}^2}\right). \tag{3}$$

As the right-hand side of (3) is small, it will be convenient to write

$$h = \lambda^3 h_c^{(3)} + \lambda^4 h_c^{(4)} + \lambda^5 h_c^{(5)} + \lambda^6 h_c^{(6)} + \lambda^6 h_p^{(6)} + \ldots.$$

where

$$g\,\underset{\sim}{b} \,.\, \hat{\nabla}h_c = 0 \tag{4}$$

and

$$g\,\underset{\sim}{b} \,.\, \hat{\nabla}h_p = - \left(\underset{\sim}{b} \times \hat{\nabla}\hat{P}\right) . \hat{\nabla}\left(\frac{1}{\underset{\sim}{b}^2}\right) \tag{5}$$

We can now proceed to the solution, order by order.

3. Equilibrium Calculation

Our objective is the determination of the magnetic fields to sufficiently high order (in fact, to 6^{th} order) that effects associated with the toroidal displacement of magnetic surfaces are recovered. To this end we use the equilibrium equations up to 10^{th} order. We will discuss the order-by-order solution of these equations presently, but it seems appropriate to make some preliminary observations which will allow us to anticipate the form of the lower orders of this solution.

Without a plasma we have the known vacuum field (Dobrott and Frieman, 1971). In appendix 1, we apply the present ordering to the exact solution in toroidal coordinates for the magnetic potential, and so obtain a very simple derivation of this result. Toroidal stellarator flux surfaces exist in the asymptotic sense (Kruskal, 1952) so that a function ψ exists which satisfies

$$\underset{\sim}{b} \cdot \nabla\psi = 0 \ , \qquad (6)$$

at any order of expansion. For the vacuum field the surface functions can be put in the form (Dobrott and Frieman, 1971) :

$$\psi(\rho,\ \theta,\ \bar{s}) = \psi_o(\rho) - \lambda \frac{3\alpha\rho^2}{\bar{p}^2} \psi_o'(\rho) \sin(3\theta + \bar{s})$$
$$- \lambda^2 \left(\frac{3\alpha\rho^2}{2\bar{p}^2}\right)^2 \rho^3 \frac{d}{d\rho}\left(\frac{1}{\rho}\frac{d\psi_o}{d\rho}\right) \cos(6\theta + 2\bar{s}) + 0(\lambda^3) \qquad (7)$$

where $\psi_o(\rho)$ is some function of radius. As the main effect of the plasma pressure and current on the flux surfaces is to displace them toroidally by small amounts of order ε, the above form is expected to hold true for the flux surfaces in the presence of plasma, although, in fact, the third order term would then differ significantly from the vacuum field term (see section 4). Any small, externally-applied field - for the purpose of recentring the plasma, say - will also leave the form of ψ unchanged below third order. Now we have noted already in Eq. (4) that h_c satisfies the equation of a surface function, so from Eq. (7) we can infer the structure of this part of the force-free current, up to 5^{th} order. Likewise, the plasma pressure must satisfy Eq. (6) so that its form can be anticipated up to 8^{th} order from Eq. (7).

The zeroth order field is of course constant and axial : $b^{(o)} = \hat{\xi}$. Following Dobrott and Frieman, there is no first order field and in second order we introduce the helical field

$$\underset{\sim}{b}^{(2)} = \frac{3\alpha\rho^2}{\bar{p}}(\hat{\rho} \cos(3\theta + \bar{s}) - \hat{\theta} \sin(3\theta + \bar{s})) \qquad (8)$$

given by the vacuum potential (see appendix 1). The strength of the external winding current determines the quantity α.

The plasma current first appears in next order, and to satisfy Eqs. (1) the third order current σ must be axial and independent of toroidal coordinate $\bar{s}$. In the absence of externally-applied shaping fields the lowest order flux surfaces are circular so that $\sigma \equiv \sigma(\rho)$, and so the third order field of the current is poloidal and given by

$$b_\theta^{(3)} = b(\rho)\,, \quad \text{where} \quad \frac{1}{\rho}\frac{d}{d\rho}(\rho b) = \sigma(\rho)\,, \quad \text{and} \quad b_\rho^{(3)} = 0\,, \tag{9}$$

as in cylindrical geometry. Since $\underset{\sim}{b} = \hat{\xi} + O(\lambda^2)$, the lowest order part of the force free current h_c is just $h_c^{(3)} = \sigma(\rho)$. There is a third order correction to the vacuum field axial component

$$b_\xi^{(3)} \doteq \rho\cos\theta - \alpha\rho^3 \sin(3\theta + \bar{s}) \tag{10}$$

which contains the leading order toroidal and helical modulations.

Proceeding now to fourth order, force balance (Eq. 1(i)) requires that $\hat{\underset{\sim}{J}}^{(4)}$ must have only an axial component, $\sigma^{(4)}$ which in order to satisfy the condition $\operatorname{div} \underset{\sim}{J} = 0$ must take the form

$$\sigma^{(4)} = -\frac{3\alpha\rho^2}{\bar{p}^2}\sigma'(\rho)\sin(3\theta + \bar{s}) + \bar{\sigma}^{(4)}(\rho,\theta) \tag{11}$$

(From a constraint equation obtained in higher order it will be seen that the axisymmetric term $\bar{\sigma}^{(4)}$ can be set equal to zero without loss of generality, and then since $h_c^{(4)} = \sigma^{(4)}$ we observe that Eq. (7) gives the above result directly.) We calculate the corresponding magnetic field from the 4th order parts of Eqs. 1(ii) and 1(iii)

$$\begin{aligned} \hat{\nabla}^{(0)}\cdot\underset{\sim}{b}^{(4)} &= \alpha\bar{p}\rho^3\cos(3\theta + \bar{s}) \\ \hat{\nabla}^{(0)}\times\underset{\sim}{b}^{(4)} &= \hat{\xi}\sigma^{(4)} \end{aligned} \tag{12}$$

The result may be written as

$$b_\rho^{(4)} = f^{(4)}(\rho)\cos(3\theta + \bar{s}) + \bar{b}_\rho^{(4)}(\rho,\theta)$$

$$b_\theta^{(4)} = g^{(4)}(\rho)\sin(3\theta + \bar{s}) + \bar{b}_\theta^{(4)}(\rho,\theta)$$

$$b_\xi^{(4)} = 0 . \tag{13}$$

Functions $f^{(4)}(\rho)$ and $g^{(4)}(\rho)$ are given in appendix 2 where the final expressions for the magnetic field are written in full, and $\bar{\underset{\sim}{b}}^{(4)}$ is the field due to $\bar{\sigma}^{(4)}$ alone.

In fifth order, force balance gives

$$\begin{aligned} \hat{J}_\rho^{(5)} &= \sigma(\rho)\, b_\rho^{(2)} = \frac{3\alpha\rho^2}{\bar{p}}\, \sigma(\rho) \cos(3\theta + \bar{s}) \\ \hat{J}_\theta^{(5)} &= \sigma(\rho)\, b_\theta^{(2)} = -\frac{3\alpha\rho^2}{\bar{p}}\, \sigma(\rho) \sin(3\theta + \bar{s}) \end{aligned} \tag{14}$$

These are just the force-free currents $h_c^{(3)}\underset{\sim}{b}^{(2)}$ due to the helical field. Using these currents in the transverse components of Eq. 1(ii) we find

$$b_\xi^{(5)} = \left\{ \frac{\bar{p}\rho g^{(4)}(\rho)}{3} + \frac{\alpha\rho^3\sigma(\rho)}{\bar{p}} \right\} \sin(3\theta + \bar{s}) \tag{15}$$

The fifth order axial current $\hat{J}_\xi^{(5)}$, which we denote by $\sigma^{(5)}$, is obtained in similar fashion to $\sigma^{(4)}$, using $\operatorname{div} \underset{\sim}{J} = 0$:

$$\begin{aligned} \sigma^{(5)} = \bar{\sigma}^{(5)}(\rho, \theta) &- \left(\frac{3\alpha\rho^2}{2\bar{p}^2}\right)^2 \rho \frac{d}{d\rho}\left(\frac{1}{\rho}\frac{d\sigma}{d\rho}\right) \cos(6\theta + 2\bar{s}) \\ &- \left(\frac{3\alpha\rho^2}{\bar{p}^2}\right) \left\{ \frac{\partial\bar{\sigma}^{(4)}}{\partial\rho} \sin(3\theta + \bar{s}) + \frac{1}{\rho}\frac{\partial\bar{\sigma}^{(4)}}{\partial\theta} \cos(3\theta + \bar{s}) \right\} \end{aligned} \tag{16}$$

At this stage we can derive a constraint on $\bar{\sigma}^{(4)}$ and its associated fourth order field by obtaining the seventh order of $\nabla . \underset{\sim}{J} = 0$ and of $\nabla \times (\underset{\sim}{J} \times \underset{\sim}{b}) = 0$. Averaging over $\bar{s}$ we find

$$b^*(\rho) \frac{\partial\bar{\sigma}^{(4)}}{\partial\theta} + \rho\sigma'(\rho)\bar{b}_\rho^{(4)} = 0 \tag{17}$$

where $b^*(\rho)$ is the effective mean poloidal field, given by

$$b^*(\rho) = b(\rho) - \frac{18\alpha^2\rho^3}{\overline{p}^3} \; .$$

Thus, as indicated earlier, it is consistent to set $\overline{\sigma}^{(4)} \equiv 0$ so that $\overline{\underset{\sim}{b}}^{(4)} \equiv 0$. This corresponds to the absence of axisymmetric shaping fields. Using the expression just obtained for $\sigma^{(5)}$ we obtain the fifth order poloidal fields in the usual way, the result being given in appendix 2, where $\overline{\sigma}^{(5)}$ and associated fields are set to zero. The consistency of this will be shown later.

It is in the 6^{th} order of expansion that the effects of pressure first appear. From the ξ-component of force balance in 7^{th} order we see that $\hat{P}^{(6)} \equiv \hat{P}^{(6)}(\rho, \theta)$ and as we insist that there are no surface-shaping fields present, $\hat{P}^{(6)} \equiv P(\rho)$. Thus the 6^{th} order force balance gives

$$\begin{aligned} \hat{J}_\rho^{(6)} &= -\frac{1}{2}\left(\frac{3\alpha\rho^2}{\overline{p}^2}\right)^2 \overline{p}\sigma'(\rho)\, \sin(6\theta + 2\overline{s}) \\ \hat{J}_\theta^{(6)} &= -\frac{db_\beta(\rho)}{d\rho} - \frac{1}{2}\left(\frac{3\alpha\rho^2}{\overline{p}^2}\right)^2 \overline{p}\sigma'(\rho)\, \cos(6\theta + 2\overline{s}) \end{aligned} \tag{18}$$

where we define

$$b_\beta(\rho) = -\int_0^\rho \left\{\frac{dP}{d\rho} + \sigma(\rho)\, b(\rho) + \frac{1}{2}\left(\frac{3\alpha\rho^2}{\overline{p}^2}\right)^2 \overline{p}\sigma'(\rho)\right\} d\rho$$

These correspond to the force-free currents $h_c \underset{\sim}{b}$ and the usual diamagnetic current. Using Eqs. (18) in Eq. 1(ii) we obtain $b_\xi^{(6)}$, given in appendix 2. It remains to find $J_\xi^{(6)} = \sigma^{(6)}$, for which purpose we must obtain first the pressure and poloidal currents in 7^{th} order.

From the ξ-component of force balance in 8^{th} order, we find

$$P^{(7)} = -\left(\frac{3\alpha\rho^2}{\overline{p}^2}\right)\frac{dP}{d\rho}\sin(3\theta + \overline{s}) + \overline{P}^{(7)}(\rho, \theta) \tag{19}$$

where the first term is clearly just the helical modulation indicated by Eq. (7) with $\psi_o(\rho) = P(\rho)$, and the second term is to be determined. Returning now to 7^{th} order the poloidal components of $\hat{\underset{\sim}{J}}^{(7)}$ can be found, and using div J = 0 we can set

$$\sigma^{(6)} = \overline{\sigma}^{(6)}(\rho, \theta) + \sigma_o^{(6)}(\rho) \cos\theta + \sigma_1^{(6)}(\rho) \sin(3\theta + \overline{s})$$

$$+ \sigma_2^{(6)}(\rho) \sin(9\theta + 3\overline{s})$$

$$- \frac{3\alpha\rho^2}{\overline{p}^2} \left\{ \sin(3\theta + \overline{s}) \frac{\partial\overline{\sigma}^{(5)}}{\partial\rho} + \cos(3\theta + \overline{s}) \frac{1}{\rho}\frac{\partial\overline{\sigma}^{(5)}}{\partial\theta} \right\} \quad (20)$$

where $\sigma_1^{(6)}(\rho)$ and $\sigma_2^{(6)}(\rho)$ are defined in Appendix 2. For convenience the cos θ component of the as yet undetermined axisymmetric part is written separately. By using Eq. (20) in the poloidal 8^{th} order force balance and making use of div J = 0, we obtain the constraint equation

$$b^*(\rho) \frac{\partial\overline{\sigma}^{(5)}}{\partial\theta} + \rho\overline{b}_\rho^{(5)}\sigma'(\rho) = 0 ,$$

after averaging over $\overline{s}$. This is the same condition as was found for $\overline{\sigma}^{(4)}$ and by the same argument we set $\overline{\sigma}^{(5)} \equiv 0$ without loss of generality.

Solving for the sixth order fields using the form given by Eq. (20) for $\sigma^{(6)}$, we find

$$b_\rho^{(6)} = A(\rho) \sin\theta + f_1^{(6)}(\rho) \cos(3\theta + \bar{s}) + f_2^{(6)}(\rho) \cos(9\theta + 3\bar{s}) + \bar{b}_\rho^{(6)}(\rho,\theta)$$

$$b_\theta^{(6)} = B(\rho) \cos\theta + g_1^{(6)}(\rho) \sin(3\theta + \bar{s}) + g_2^{(6)}(\rho) \sin(9\theta + 3\bar{s}) + \bar{b}_\theta^{(6)}(\rho,\theta)$$

(21)

where the functions $f_{1,2}^{(6)}$ and $g_{1,2}^{(6)}$ are given in appendix 2, and where

$$\frac{1}{\rho}\frac{d}{d\rho}(\rho B) - \frac{A(\rho)}{\rho} = \sigma_o^{(6)}(\rho) + \rho\sigma(\rho)$$

$$\frac{1}{\rho}\frac{d}{d\rho}(\rho A) - \frac{B(\rho)}{\rho} = - b(\rho) \quad (22)$$

As $\underset{\sim}{\bar{b}}^{(6)}(\rho, \theta)$ is the field due to $\bar{\sigma}^{(6)}(\rho, \theta)$, $\bar{b}_\rho^{(6)}$ has no $\sin\theta$ component and $\bar{b}_\theta^{(6)}$ has no $\cos\theta$ component.

To complete the solution for $\underset{\sim}{b}$ to sixth order we need to find $\sigma_o^{(6)}(\rho)$ and $\bar{\sigma}^{(6)}(\rho, \theta)$. The first step towards this end is to obtain the form of $P^{(8)}$, by considering the ξ component of 9^{th} order pressure balance. In this way we find

$$P^{(8)} = \bar{P}^{(8)}(\rho, \theta) - \left(\frac{3\alpha\rho^2}{2\bar{p}^2}\right)^2 \rho \frac{d}{d\rho}\left(\frac{1}{\rho}\frac{dP}{d\rho}\right)\cos(6\theta + 2\bar{s})$$
$$- \frac{3\alpha\rho^2}{\bar{p}^2}\left[\frac{\partial\bar{P}^{(7)}}{\partial\rho}\sin(3\theta + \bar{s}) + \frac{1}{\rho}\frac{\partial\bar{P}^{(7)}}{\partial\theta}\cos(3\theta + \bar{s})\right] \quad (23)$$

Next we obtain the poloidal components of $\hat{\underset{\sim}{J}}^{(8)}$ from the force balance equation, and so after $\bar{s}$-averaging the ξ component of the same equation in 10^{th} order, we find that

$$\frac{b^*(\rho)}{\rho}\frac{\partial\bar{P}^{(7)}}{\partial\theta} = 0 \quad (24)$$

Clearly we can set $\bar{P}^{(7)} \equiv 0$ without loss of generality as anticipated from Eq. (7). After obtaining $\hat{J}_\xi^{(7)}$ from the 8^{th} order of div J = 0 (apart from an undetermined axisymmetric part), we form the poloidal components of force balance in 9^{th} order. Applying to these the condition $\hat{\xi}.\nabla \times \nabla p = 0$ and averaging the result over $\bar{s}$, we obtain, after some algebraic reduction, the final form

$$V(\rho)\sin\theta + W(\rho, \theta) = 0 \quad (25)$$

where

$$W(\rho, \theta) = -\left[b^*(\rho)\frac{\partial\bar{\sigma}^{(6)}}{\partial\theta} + \rho\sigma'(\rho)\,\bar{b}_\rho^{(6)}\right],$$

which by virtue of the definitions earlier, has no $\sin\theta$ fourier component.

Thus the terms of Eq. (25) vanish separately, so that firstly we may set $\overline{\sigma}^{(6)} \equiv 0$ without loss of generality. From the calculated form of $V(\rho)$ we get the condition which determines $\sigma_o^{(6)}(\rho)$:

$$- b^*(\rho)\, \sigma_o^{(6)}(\rho) + \frac{d}{d\rho}(\rho\sigma A) + \rho(2P' + b\sigma) - \sigma B + \frac{45\alpha^2}{4\overline{p}^3} \rho^5\sigma'(\rho) = 0 \qquad (26)$$

Combining Eqs. (22) and (26) we find

$$\frac{b^*(\rho)}{\rho^2} \frac{d}{d\rho}\left(\rho^3 \frac{dA}{d\rho}\right) - \rho\sigma'(\rho)\, A(\rho) = 2\rho P'(\rho) - b(\rho)b^*(\rho) + \frac{45\alpha^2}{4\overline{p}^3} \rho^5\sigma'(\rho) \qquad (27)$$

With this last equation, given suitable boundary conditions the determination of $\underset{\sim}{b}$ to sixth order in λ is complete and the result is summarised in appendix 2. If we set $A(\rho) = b^*(\rho)\, \Delta(\rho)/\rho$ then Eq. (27) may be rewritten in the form

$$b^{*2}\Delta'' + b^*\left(2b^{*\prime} + \frac{1}{\rho} b^*\right)\Delta' + b^*\Delta \frac{d}{d\rho}\left(\frac{1}{\rho}\frac{d}{d\rho}(\rho b_o(\rho))\right)$$

$$= 2\rho P' - b^* b + \frac{45\alpha^2}{4\overline{p}^3} \rho^5 \sigma'(\rho) \qquad (28)$$

where $b_o = b^* - b = - 18\alpha^2\rho^3/\overline{p}^3$. In the special case of a tokamak $\alpha = 0$, so $b^* = b$ and Eq. (28) is just the well known toroidal shift equation (Greene, Johnson and Weimer, 1971) Δ being the major radius displacement of the flux surface with mean radius ρ. In the following section we shall see that in a slightly modified form $\Delta(\rho)$ retains its interpretation as a flux surface displacement. Furthermore, we shall find it possible to give a simple explanation for the form of Eq. (26).

4. Calculation of Flux Surfaces

In order to obtain the equation describing the magnetic surfaces we solve Eq. (6) for ψ up to third order, using the fields calculated in the last section, and following the method employed by Dobrott and

Frieman (1971). In leading order (1^{st}) Eq. (6) yields the condition $\frac{\partial\psi^{(o)}}{\partial \bar{s}} = 0$, and in second order gives

$$\underset{\sim}{b}^{(2)} \cdot \hat{\nabla}^{(o)} \psi^{(o)} + \bar{p} \frac{\partial \psi^{(1)}}{\partial \bar{s}} = 0$$

which determines $\psi^{(1)}$ in terms of $\psi^{(o)}$, apart from an axisymmetric term $\overline{\psi}^{(1)}$. Going to third order and first of all integrating over one complete period in $\bar{s}$, we obtain a constraint on $\psi^{(o)}$ to ensure the absence of any secular term:

$$\frac{b^*(\rho)}{\rho} \frac{\partial \psi^{(o)}}{\partial \theta} = 0 .$$

Thus, $\psi^{(o)} \equiv \psi_o(\rho)$, and although we could choose a particular functional form for ψ_o at this stage we prefer to retain the general form. Apart from an axisymmetric term, $\psi^{(2)}$ is determined at this order. To ensure the absence of secular terms in fourth order it is found that $\psi^{(1)}$ must satisfy the above constraint as well, so that without loss of generality we can set $\overline{\psi}^{(1)} \equiv 0$. An axisymmetric term in $\psi^{(3)}$ as well as that in $\psi^{(2)}$ remains undetermined at this stage. Proceeding in the same fashion, however, we find that $\overline{\psi}^{(2)} \equiv 0$, and in sixth order the secular equation reduces to

$$\frac{b^*(\rho)}{\rho} \frac{\partial \overline{\psi}^{(3)}}{\partial \theta} = - \psi_o'(\rho) \left\{ A(\rho) + \left(\frac{45\alpha^2\rho^4}{4\bar{p}^3} \right) \right\} \sin\theta \qquad (29)$$

Thus, the flux surfaces up to third order are given by

$$\psi = \psi_o(\rho) - \lambda \frac{3\alpha\rho^2}{\bar{p}^2} \psi_o'(\rho) \sin(3\theta + \bar{s}) - \lambda^2 \left(\frac{3\alpha\rho^2}{2\bar{p}^2} \right)^2 \rho \frac{d}{d\rho} \left(\frac{1}{\rho} \frac{d\psi_o}{d\rho} \right) \cos(6\theta + 2\bar{s})$$

$$+ \lambda^3 \left\{ \frac{\rho\psi_o'(\rho)}{b^*(\rho)} \left(A(\rho) + \frac{45\alpha^2}{4\bar{p}^3} \rho^4 \right) \cos\theta + \pi_1(\rho) \sin(3\theta + \bar{s}) + \pi_2(\rho) \sin(9\theta + 3\bar{s}) \right\} \qquad (30)$$

where $\pi_1(\rho)$ and $\pi_2(\rho)$ are defined in Appendix 2.

As anticipated in section 3, the plasma current and pressure leave ψ unaltered from its vacuum field form below third order.

Using the known expression for $\psi^{(3)}$ now we can see how the terms of Eq. (26) arise for the sixth order current $\sigma_o^{(6)}(\rho)$. Firstly by Eq. (5), $h_p^{(6)} \equiv h_p^{(6)}(\rho, \theta)$ and

$$\frac{b^*(\rho)}{\rho} \frac{\partial h_p^{(6)}}{\partial \theta} = -\ 2P'(\rho) \sin \theta \tag{31}$$

where Eq. (31) results from the elimination of terms secular in $\bar{s}$ at ninth order. Thus,

$$h_p^{(6)} = 2\rho P'(\rho) \cos \theta / b^*(\rho) \tag{32}$$

which is the so called Pfirsch-Schlüter current. The remaining force-free part of $\hat{J}_\xi^{(6)}$ is $(h_c^{(6)} + (gb_\xi)^{(3)} h_c^{(3)})$, and the axisymmetric component of this is just $\bar{h}_c^{(6)}$. By virtue of Eq. (4) and the fact that $h_c^{(3)} = \sigma(\rho)$ we see from Eq. (30) that

$$\bar{h}_c^{(6)} = \frac{\rho\sigma'(\rho)}{b^*(\rho)} \left(A(\rho) + \frac{45\alpha^2\rho^4}{4\bar{p}^3}\right) \cos \theta \tag{33}$$

and so combining (32) and (33), we have

$$\sigma_o^{(6)}(\rho) = \rho\left\{2P'(\rho) + \sigma'(\rho) \left[A(\rho) + \frac{45\alpha^2\rho^4}{4\bar{p}^3}\right]\right\} / b^*(\rho)$$

which is equivalent to Eq. (26).

The form of the magnetic surfaces is now clear: the vacuum surfaces are subjected to an additional third order toroidal shift due to the plasma pressure and current, and also to additional helical distortions at third order and higher which are associated with the plasma currents. Toroidicity, in fact, also produces $(2\theta + \bar{s})$ and $(4\theta + \bar{s})$ "sidebands" of the stellarator field which are recovered at fourth order in ψ.

To calculate the magnetic axis position, we note first of all that $\psi_o(\rho)$ must tend to zero at least as rapidly as $b^*(\rho)$, otherwise $\bar{\psi}^{(3)}$ cannot be defined at $\rho = 0$. Then, defining $f = \int_0^\rho b^*(\rho)d\rho$, we see that $\psi_o'(f) = \psi_o'(\rho)/b^*(\rho)$ remains finite as $\rho \to 0$, and neglecting the helical terms in Eq. (30) we have

$$\psi = \psi_o \left[f + \lambda^3 \rho \left(A(\rho) + \frac{45\alpha^2}{4\bar{p}^3} \rho^4 \right) \cos\theta \right] + O(\lambda^4)$$

Since $\nabla\psi$ vanishes at the magnetic axis, the coordinates of this point are given to sufficient accuracy by $\theta = \pi$, and $\rho = \rho_x$, where

$$b^*(\rho_x) = \lambda^3 \frac{d}{d\rho} (\rho\, A(\rho)) \big|_{\rho = \rho_x} \tag{34}$$

The consistency of the approximations used in obtaining Eq. (34) is verified by substitution of the result in the full expression for $\nabla\psi$. In terms of the quantity $\Delta(\rho)$ which satisfies Eq. (28), we can rewrite Eq. (34) in the form

$$b^*(\rho_x - \lambda^3 \Delta(\rho_x)) = 0\,,$$

provided $\Delta(0) \sim O(1)$, and solution in leading order is then $\rho_x = \lambda^3 \Delta(0)$. Although this result covers a wide range of conditions, including the tokamak special case $(\alpha = 0)$, it is less general than Eq. (34) since $\Delta(0)$ is assumed to be of order unity, which, for example, is not appropriate to currentless stellarators if $A(0)$ is non-zero: in fact, the axis shift in the latter case is $\lambda \left| \frac{A_o \bar{p}^3}{18\alpha^2} \right|^{1/3}$ to leading order in λ. Thus, in general the magnetic axis can take up widely different positions, depending on the relative strengths of the rotational transform due to the current and that due to the vacuum field.

Sufficiently far from the coordinate axis, we can express the radius of a given flux surface $\rho(\psi, \theta, \bar{s})$ as a single-valued function which at leading order in λ depends on ψ alone. Then a straightforward inversion of Eq. (30) yields

$$\rho = \rho_o(\psi) + \lambda \frac{3\alpha\rho_o^2}{\bar{p}^2} \sin(3\theta + \bar{s}) + \lambda^2 \frac{45}{2} \frac{\alpha^2\rho_o^3}{\bar{p}^4} \sin^2(3\theta + \bar{s})$$

$$+ \lambda^3 \Bigg[\frac{24\alpha}{\bar{p}^2} \left(\frac{3\alpha\rho_o^2}{2\bar{p}^2}\right)^2 \sin^3(3\theta + \bar{s}) + \left(\frac{f^{(4)}(\rho_o)}{\bar{p}} - \frac{9\alpha\rho_o b(\rho_o)}{\bar{p}^3}\right) \sin(3\theta + \bar{s})$$

$$- \frac{\rho_o \cos\theta}{b^*(\rho_o)} \left\{ A(\rho_o) + \frac{45\alpha^2\rho_o^4}{4\bar{p}^3} \right\} \Bigg] + O(\lambda^4) . \qquad (35)$$

Thus, when use of the variable $\Delta(\rho)$ is appropriate, the outward axis shift of a surface with lowest order radius ρ_o may be written as $\lambda^3\left\{\Delta(\rho_o) + \frac{45\alpha^2\rho_o^5}{4\bar{p}^3 b^*(\rho_o)}\right\}$, so that $\Delta(\rho_o)$ is indeed the displacement function for tokamaks and, modified by a small extra term, remains so for stellarators, which in this case can be referred to as "tokamak-like". Although Eq. (35) does not apply close to the coordinate axis in general, a simple change of variables is possible for tokamak-like configurations which relocates the coordinate axis on the magnetic axis at $\rho_x = \lambda^3\Delta(0)$. In terms of the new coordinates, Eq. (35) now correctly describes all the flux surfaces provided $\Delta(\rho)$ is replaced by $(\Delta(\rho) - \Delta(0))$.

Finally in this section we note that the rotational transform on a given surface can be calculated, using Eq. (35), in the same way as for the vacuum field (Dobrott and Frieman, 1971)

$$\iota = \frac{1}{2\pi} \frac{d\chi}{d\phi}$$

where χ and ϕ respectively measure the poloidal and toroidal magnetic fluxes. We obtain

$$i(\rho_o) = \frac{b(\rho_o)}{\rho_o} - \left(\frac{18\alpha^2\rho_o^2}{\bar{p}^3}\right) = \frac{b^*(\rho_o)}{\rho_o} \quad ,$$

a result which may also be derived from the fields up to third order using the method of averaging (Morozov and Solovev, 1966). Contributions from the current and from the vacuum field are additive, as usual in stellarators with many helical periods.

5. Vertical Field Calculation

In order to maintain a given equilibrium configuration - say one in which the magnetic axis position is specified - an externally applied vertical magnetic field is necessary in general. Of course, in contrast to tokamaks, stellarator equilibria may exist without a vertical field as we shall see, but in this case the axis displacement may be considerable under the influence of plasma current and pressure, and in addition the cross-sectional form of the surfaces is sensitive to the effects of finite pressure (Greene and Johnson, 1961).

Thus, in determining the boundary condition for Eq. (27), we shall not make any particular assumption about the size of the vertical field, but rather we shall obtain the relationship between this quantity and the function $A(\rho)$.

To this end we make use of the fact that within any given flux surface the equilibrium is unchanged if that surface is replaced by a coincident, perfectly conducting shell carrying a surface current $\underset{\sim}{I}$ given by

$$\underset{\sim}{I} = \frac{\underset{\sim}{B} \times \hat{n}}{\mu_o} \qquad (36)$$

where $\underset{\sim}{B}$ is the equilibrium magnetic field at the chosen surface, and $\hat{n}$ is a unit normal directed outwards. This is sometimes referred to as the

"Virtual Casing Principle" (Shafranov and Zakharov, 1972) and it allows us to calculate all the external fields necessary for maintaining a given equilibrium within the chosen surface. Thus, regarding $A(\rho)$ as known we can select any (closed) flux surface outside the plasma as the virtual casing, and calculate the casing current I from Eq. (36) using the known field $\underset{\sim}{B}$. For the purpose of determining the external vertical field it is sufficient to have $\underset{\sim}{B}$ up to sixth order and ψ up to third.

We can use Eq. (35) to describe the virtual casing, so that the unit normal $\hat{n}$ may be calculated from $\hat{n} = \nabla\psi/|\nabla\psi|$. To the order in λ required, we obtain

$$n_\rho = 1 - \lambda^2 \frac{81\alpha^2}{2\bar{p}^4} \rho_b^2 \cos^2(3\theta + \bar{s}) - \lambda^3 \frac{243\alpha^3\rho_b^3}{\bar{p}^6} \sin(3\theta + \bar{s})\left\{1 + \frac{7}{2}\sin^2(3\theta + \bar{s})\right\} + O(\lambda^4)$$

$$n_\theta = -\lambda \frac{9\alpha\rho_b}{\bar{p}^2} \cos(3\theta + \bar{s}) - \lambda^2 \frac{54\alpha^2\rho_b^2}{\bar{p}^4} \sin(6\theta + 2\bar{s}) + O(\lambda^3)$$

and

$$n_\xi = -\lambda^2 \frac{3\alpha\rho_b^2}{\bar{p}} \cos(3\theta + \bar{s}) + O(\lambda^3) , \qquad (37)$$

ρ_b being the lowest order radius of the boundary.

The vacuum field $\underset{\sim}{b}_I B_o$ will be determined by solving the magnetostatic equations order by order in both interior (i) and exterior (e) domains. At the virtual casing surface $\underset{\sim}{b}_I$ must satisfy the matching conditions

$$[\underset{\sim}{b}_I \cdot \hat{n}] = 0 \qquad \text{(i)}$$

and (38)

$$[\underset{\sim}{b}_I \times \hat{n}] = -\underset{\sim}{i} \qquad \text{(ii)}$$

where $\underset{\sim}{i}$ denotes the normalised casing current given, from Eq. (36), by

$$\underset{\sim}{i} = \underset{\sim}{b} \times \hat{n}, \tag{39}$$

and where $[x] = x_e - x_i$ denotes the discontinuity of a quantity at the casing. The condition of regularity in the interior provides a further restriction on the solutions. Likewise, the vanishing of exterior fields at large distance from the shell restricts the solution form for b_{Ie}; however to apply this condition to the expanded form of b_{Ie} we need to obtain first the appropriate form of exact solution in toroidal coordinates, and then to transform coordinates in these solutions to the quasi-cylindrical $(\rho, \theta, \bar{s})$ system in the limit $\rho \ll 1/\varepsilon$, a procedure which was used for axisymmetric fields by Greene et al. (1971).

Up to second order, the fields $\underset{\sim}{b}_I$ and $\underset{\sim}{b}$ must coincide in the interior since the effects of currents in the plasma do not show below third order. In the exterior region, $\underset{\sim}{b}_I$ vanishes identically up to second order, since the casing is a perfect conductor. Similarly, $b^{(3)}_{I\xi} \equiv b^{(3)}_{\xi}$ in the interior, and is zero outside, since $b^{(3)}_{\xi}$ is a vacuum field term. The third order exterior poloidal field of the casing current does not vanish however, since it must exactly cancel the field generated by the plasma current. Thus, we have

$$i^{(3)} = -\hat{\xi} b(\rho_b)$$

and

$$\begin{aligned} b^{(3)}_{I\theta} &= 0 && \text{in the interior} \\ &= -b(\rho_b)\rho_b/\rho && \text{in the exterior} \end{aligned} \tag{40}$$

In fourth order the matching conditions (38) may be reduced to

$$\left[b^{(4)}_{I\rho}\right] = -f^{(4)}(\rho_b)\cos(3\theta + \bar{s})$$

$$\left[b_{I\theta}^{(4)}\right] = -\left[g^{(4)}(\rho_b) + \frac{3\alpha\rho_b^2\sigma(\rho_b)}{\bar{p}^2}\right]\sin(3\theta + \bar{s}) .$$

Solving the magnetostatic equations at this order we find that $b_I^{(4)}$ can be written in the form

$$b_I^{(4)} = (A_i^{(4)}\rho^2 + \frac{5}{16}\alpha\bar{p}\rho^4)\hat{\rho}\cos(3\theta + \bar{s}) - \left(A_i^{(4)}\rho^2 + \frac{3}{16}\alpha\bar{p}\rho^4\right)\hat{\theta}\sin(3\theta + \bar{s}) \quad (41)$$

in the interior, $A_i^{(4)}$ being a constant determined by the matching conditions.

The fifth order poloidal fields are found to be expressible in the form

$$b_I^{(5)} = \hat{\rho}\left\{\frac{\alpha\rho^3}{\bar{p}}\cos(2\theta + \bar{s}) + A_i^{(5)}\rho^5\sin(6\theta + 2\bar{s})\right\}$$

$$+ \hat{\theta}\left\{-\frac{\alpha\rho^3}{2\bar{p}}\sin(2\theta + \bar{s}) + A_i^{(5)}\rho^5\cos(6\theta + 2\bar{s})\right\} \quad (42)$$

in the interior, where $A_i^{(5)}$ satisfies the appropriate matching conditions.

The expression for the vertical field is obtained from the sixth order casing current. After averaging the second matching condition, Eq. 38(ii), over $\bar{s}$ we find

$$\left[\left\langle b_{I\theta}^{(6)}\right\rangle\right] = \cos\theta\left\{-B(\rho_b) + \rho_b\sigma_b(\rho_b)\left[A(\rho_b) + \frac{45\alpha^2}{4\bar{p}^3}\rho_b^4\right]\Big/ b^*(\rho_b)\right\} \quad (43)$$

where use is made of Eqs. (40) - (42) and the known lower order vacuum field form of b_I, and $B(\rho)$, $A(\rho)$ are the functions appearing in Eq. (21). The $\bar{s}$-average of a quantity x is denoted by $\langle x \rangle$.

Similarly, Eq. 38(i) after averaging yields

$$\left[\left\langle b_{I\rho}^{(6)}\right\rangle\right] = -A(\rho_b)\sin\theta \quad (44)$$

At this order, the magnetostatic equations for the average poloidal field are just

$$\hat{\xi}\cdot\hat{\nabla}^{(0)} \times \langle b_I^{(6)} \rangle = 0$$

and

$$\hat{\nabla}^{(0)}\cdot\langle b_I^{(6)} \rangle = - b_{I\theta}^{(3)} \sin\theta$$

Dropping the average symbol, these may be solved to give, in the interior,

$$b_{I\rho}^{(6)} = A_i^{(6)} \sin\theta$$

$$b_{I\theta}^{(6)} = A_i^{(6)} \cos\theta \tag{45}$$

and in the exterior domain

$$b_{I\rho}^{(6)} = \left\{A_e^{(6)} - B_e^{(6)}/\rho^2 + \frac{1}{2}\rho_b b(\rho_b)(1 + \ell n\rho)\right\} \sin\theta$$

$$b_{I\theta}^{(6)} = \left\{A_e^{(6)} + B_e^{(6)}/\rho^2 + \frac{1}{2}\rho_b b(\rho_b)\ \ell n\rho\right\} \cos\theta\ . \tag{46}$$

Thus, the applied vertical field which we seek is given by $b_I^{(6)}$ in the interior, its strength being $A_i^{(6)}$. The jump conditions Eqs. (43) and (44) determine only the difference $A_e^{(6)} - A_i^{(6)}$, there being three unknown quantities in Eqs. (45) and (46). To obtain a third relation, we must recall that the quasi-cylindrical coordinates $(\rho, \theta, \bar{s})$ do not form a true coordinate system, as they apply only within the domain $\rho < 1/\varepsilon$. Thus, the boundary condition at infinity cannot be applied to Eqs. (46) directly, as is obvious from the presence of logarithmic terms, and must instead be deduced in the manner described earlier from the exact solution form in toroidal coordinates. As discussed in appendix 2, the axisymmetric part of the casing current produces a field which can be represented by a stream function χ such that $b_I = \hat{\nabla}\chi \times \hat{\xi}/g^2$. From the exact form of general solution for χ in the exterior domain, with the correct behaviour at infinity, we find that the expansion of χ for $\rho\varepsilon \ll 1$ must take the corresponding form

$$\chi = \varepsilon K_o \left\{ \left(2 + \ln \frac{\rho\varepsilon}{8}\right) - \varepsilon \cos\theta \left(\frac{1}{2}\rho\left(1 + \ln\frac{\rho\varepsilon}{8}\right) + \frac{K_1}{\rho}\right)\right\} + O(\varepsilon^3) \tag{47}$$

where K_o and K_1 are constants.

Identifying the $O(\varepsilon)$ part of the field given by Eq. (47) with $b_I^{(3)}$, we see that

$$K_o = -\rho_b b(\rho_b) .$$

A similar comparison at $O(\varepsilon^2)$ then shows that

$$A_e^{(6)} = \frac{1}{2}\rho_b b(\rho_b) \ln\frac{\varepsilon}{8} .$$

and so from Eqs. (43) and (44) we find $A_i^{(6)}$ and $B_e^{(6)}$. The externally applied vertical field is thus given by

$$B_v = \frac{\varepsilon^2 B_o}{2}\left\{A(\rho) + \frac{d}{d\rho}(\rho A(\rho)) + \rho b(\rho)\left(\frac{3}{2} + \ln\frac{\varepsilon\rho}{8}\right) - \frac{\rho\sigma(\rho)}{b^*(\rho)}\left[A(\rho) + \frac{45\alpha^2\rho^4}{4\bar{p}^3}\right]\right\}_{\rho=\rho_b} \tag{48}$$

This completes our determination of the plasma equilibrium to sixth order of λ in the magnetic field, by establishing the boundary condition which, together with regularity at $\rho = 0$, fixes the solution of Eq. (27) for $A(\rho)$.

6. Specific Equilibria

In a few, particularly simple cases, $A(\rho)$ may be found by analytical solution of Eq. (27). Firstly we note that in the absence of a stellarator field, Eq. (48) may be written in the form

$$B_v = \frac{\varepsilon^2 B_o}{2} b(\rho_a)\left\{\Delta'(\rho_a) + \rho_a\left(\frac{3}{2} + \ln\frac{a}{8R_o}\right)\right\}$$

where ρ_a is the radius of the plasma boundary. Allowing for differences in the sign convention for Δ, this agrees with the result of Greene et al. (1971) for tokamaks, and it is to be noted that only Δ' is determined: clearly any $O(1)$ constant term added to $\Delta(\rho)$ corresponds to a mere change of coordinate origin. Equation (28) possesses a well-known integral in this case

$$\rho b^2 \Delta'(\rho) = \int_0^\rho \rho\{2\rho P' - b^2\} d\rho$$

showing that for equilibrium, B_v must take on a specific non-zero value, determined at a given value of aspect ratio by the plasma beta and rotational transform profiles: a given fractional change in B_v will in general induce a shift of the plasma column by a similar amount in major radius.

In stellarators, however, the helical field can provide the necessary force to maintain positional stability so that equilibria with given pressure and rotational transform profiles exist for a range of different vertical field values, including zero field, at essentially the same value of plasma column aspect-ratio. The main effect of changing the vertical field in this case is to alter the internal disposition of the flux surfaces. As a specific example, consider a current-free stellarator. Equation (27) is easily integrated, yielding

$$\rho^3 \frac{dA(\rho)}{d\rho} = - \frac{\bar{p}^3}{9\alpha^2} (P(\rho) - P_o)$$

where $P_o = P(0)$. Regularity of A at the origin constrains the form of P so that $P' \sim \rho^3$ for small ρ. This is a consequence of the fact that $P \equiv P(f)$ where, as in section 4, $f = \int b^* d\rho$ and $dP/df \sim O(1)$ for the most general profile consistent with equilibrium. Then choosing

$$P(\rho) = P_o \left(1 - \left(\frac{\rho}{\rho_a}\right)^4\right)$$

we obtain

$$A(\rho) = A_o - \frac{P_o}{i_a}\left(\frac{\rho}{\rho_a}\right)^2$$

for $0 \leqslant \rho \leqslant \rho_a$, where ρ_a is the normalised radius of the plasma boundary, and $i_a = i(\rho_a)$ is the corresponding rotational transform. The constant A_o is determined in terms of the applied vertical field by Eq. (48) :

$$A_o = \frac{2\,P_o}{i_a} + \frac{B_v}{\varepsilon^2 B_o} \tag{49}$$

In general, Eq. (34) determines a single magnetic axis position, given approximately (for $A_o \sim O(1)$) by

$$\frac{\rho_x}{\rho_a} = \lambda \left\{\frac{A_o}{\rho_a\, i_a}\right\}^{1/3} - \lambda^3 \left\{\frac{P_o}{i_a^{\,2}\, \rho_a}\right\}$$

while the plasma boundary displacement $\delta\rho_a$, found from Eq. (35), is

$$\delta\rho_a = \lambda^3 \left[\frac{A_o}{i_a} - \frac{P_o}{i_a^2} - \frac{5}{8}\,\rho_a^{\,2}\right].$$

both ρ_x and $\delta\rho_a$ being measured positive along $\theta = \pi$ (outwards).

Thus, surfaces close to the magnetic axis are displaced to a greater degree by a given change in the applied vertical field than are the surfaces near the plasma boundary. An interesting special case of this equilibrium has been studied previously by Yurchenko (1968) who used a Mercier expansion about the vacuum magnetic axis. The plasma boundary was assumed to be coincident with a vacuum field flux surface, which corresponds to setting $\delta\rho_a = -\frac{5}{8}\lambda^3\rho_a^2$. With the value of A_o determined in this manner, Yurchenko's result for the axis shift is recovered from Eq. (34), and Eq. (49) shows that the vertical field is in fact $B_v = -\varepsilon^2 B_o P_o / i_a$. The appearance at the plasma edge of a stagnation point in the flux function places an upper limit on the range of pressures whose confinement this model can describe adequately. The limiting value can be estimated at $P_o \le \frac{i_a^2}{2\varepsilon}$, that is $\beta = \frac{2\mu_o p}{B_o^2} \le i_a^2 \varepsilon$ (which is in any case out with the formal ordering).

In most situations of practical interest the measured profiles of pressure and plasma current do not permit Eq. (27) to be solved analytically, however its numerical solution for given boundary data is very simple. We illustrate the use of the above analysis in calculating specific equilibria by presenting some results obtained for typical equilibria in CLEO stellarator (Atkinson et al. 1976). In this device, consisting of a toroidal chamber of 90cm major radius and liner minor radius 14cm, hydrogen plasma with $T_e \sim 200$eV, and $T_i \sim 100$eV can be readily produced and sustained with an ohmic heating current of several kilo-amperes, at an electron density in the range $10^{13} - 10^{14}/\text{cm}^3$. The main device characteristics used for calculations are listed in table I, along with the relevant dimensionless parameters. The vacuum rotational transform at the plasma edge, i_a, is 0·4 for the stated winding current, and field-line following calculations (P.C. Johnson, 1978) have shown that the separatrix lies outside the vacuum vessel, in agreement with a predicted separatrix radius of 16 cm for a straight $\ell = 3$ stellarator field (Morozov and Solovev, 1966) with the same characteristics. A mean plasma pressure profile shown in fig. 2 (lower curve) is determined from measured electron and ion temperatures and electron density in a discharge with plasma parameters as listed in table II. The current density profile, determined from the temperature profile assuming Spitzer resistivity, is given by the upper curve in fig. 2. The shift function $A(\rho)$ is determined at points within the plasma by a numerical shooting procedure using trial values of A_o until the boundary condition Eq. (48) is satisfied for a given value of B_v. Then the flux function ψ may be calculated from Eq. (30) and the level contours plotted in any chosen minor cross-section. For simplicity we choose $\psi_o(\rho) = \frac{1}{2} b'(o)\rho^2 + \frac{1}{4} i_a \rho_a^{-2} \rho^4$. In practice, it is found that extending Eq. (30) by including terms at fourth order in λ does not significantly change the surfaces near the axis, so third order accuracy for these is sufficient. However the influence of toroidicity on the positions of the vacuum field separatrices is quite marked and in the present case it proves necessary to retain the fourth order term so that λ yields a minimum separatrix radius greater than the liner radius (14cm.) In addition it proves convenient to re-express $\psi(\rho, \theta, \bar{s})$ in the form

$$\psi = \frac{b'(o)}{2} \psi_L + \frac{i_a \rho_a^{-2}}{4} \psi_L^2 + \lambda^3 \rho \frac{\psi_o'(\rho)}{b^*(\rho)} A(\rho) \cos\theta + O(\lambda^4)$$

where ψ_L is given to third order by Eq. (30) with $\psi_o = \rho^2$, and with $A(\rho)$ replaced by 0. In this way, the flux surface configuration shown in fig. 3 was obtained for the vacuum field with table I parameters. The inner circle (dashed) indicates the position of the stainless steel liner, and the location of the stellarator field coils in this cross-section $(\bar{s} = \pi/2)$ is also shown. For comparison, the third order 'boundary' flux surface, given by Eq. (35), is included (also dashed). Now using the measured profiles of fig. 2, the equilibrium surfaces in fig. 4 were obtained for the case where, as in the experiment, $B_v = 0$. The axis shift is found to be 2·5 cm and it is seen that there is considerable distortion of the flux surfaces near the magnetic axis. The application of a +68 gauss vertical field with the same profiles re-centres the magnetic axis, and, as shown in fig. 5, considerably reduces the surface distortions. This property of an appropriately chosen vertical field has been noted previously by Bykov et al. (1977).

Direct experimental confirmation of the above value of B_v for re-centring is not possible, but satisfactory agreement has been found between the measured and calculated ratio of signals in two soft X-ray detectors, angled to receive emissions along chords intersecting the equatorial plane at approximately ± 4cm on either side of the geometric axis. The detector characteristics are such that the signals are proportional to P_e^2 integrated along the lines of sight, where P_e denotes electron pressure. Figure 6 shows the plasma-detector configuration for the profiles of fig. 2, with an applied field of 32 gauss, at which value the detector signals as measured by $\int P^2 d\ell$, become equal. Note that because the surfaces are asymmetrical this balance occurs when the plasma is still off-centred. In general, of course, the ratio of P_e to P will vary over the cross-section but we shall assume for simplicity that ion-electron equilibration is sufficiently effective for such variation to be neglected. By changing the normalising value of pressure at the geometric axis whilst retaining the same profiles and vertical field, we obtain the variation of signal ratio with β. This is shown in fig. 7, for a fixed vertical field of 33 gauss, with corresponding data points taken from CLEO (shots 11757 - 11771). The plasma current was 8 kA, $B_\phi = 18{\cdot}6$ kG, and $I_L = 102$ kA. As the pressure is varied experimentally by raising or lowering the plasma density at nearly constant electron temperature, there is a systematic narrowing of the pressure profiles as β falls. This is not represented in our calculations and probably accounts for at least part of the divergence in the ratios at low β, where in addition, species equilibration is poor and the errors in the measured signals are larger. Otherwise the agreement is generally good and may be taken as an indication of the extent to which our calculation sucessfully represents the stellarator flux surface structure, at least in the regions of higher pressure within the plasma.

Finally, we examine the effect of increasing the plasma pressure, retaining the same profiles and with remaining parameters as in tables I and II. The value of β is set equal to 1% on the axis and without a vertical field the equilibrium shown in figure 8 is obtained. Not only is the plasma column shifted outwards but the volume of confined plasma has been reduced due to the destruction of surfaces. The application of a vertical field of 150 gauss has the effect shown in figure 9 of reducing the axis shift and the surface distortions as well as increasing the volume of confined plasma. Thus, while it is possible to obtain equilibria without a vertical field, at higher plasma pressures especially the application of such a field, suitably chosen in strength, results in a marked improvement in the equilibrium configuration.

7. Conclusions

By solving the equilibrium equations in an ordering based on the Dobrott-Frieman scheme for $\ell = 3$ toroidal stellarators, we have obtained the magnetic fields and flux surfaces for a current-carrying, low-beta plasma with arbitrary pressure and current density profiles. The flux surfaces are circular in lowest order with helical distortions and toroidal displacements appearing as higher order corrections. The equation for the toroidal shift has been derived and the boundary value relation between this quantity and an applied vertical field was found.

Tokamak equilibria and a class of analytically soluble, current-free stellarator equilibria are recovered as special cases, and numerical results for CLEO stellarator are presented. The role of a vertical field in reducing distortions is shown, and results of calculation are shown to compare acceptably with measurements of plasma position. Finally we draw attention to the increasing importance of a suitably chosen vertical field at high plasma pressure in stellarators like CLEO, not withstanding their ability to sustain equilibrium without a vertical field.

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TABLE I

Major radius, $R_o = 0{\cdot}90$ m

Liner minor radius $= 0{\cdot}14$ m

Radius of stellarator ℓ-windings, $a = 0{\cdot}175$ m

Number of helical field periods, $p = 7$

ℓ-winding current $= 102$ kA

Toroidal field at minor axis, $B_o = 1{\cdot}84$ Tesla

Dimensionless parameters:

$\varepsilon = a/R_o = 0{\cdot}194$	$\bar{p} = 2{\cdot}35$
$\lambda = 0{\cdot}58$	$\alpha = 0{\cdot}887$

TABLE II

Central electron temperature,	T_e	= 176 eV
ion temperature ,	T_i	= 130 eV
electron density		= $4{\cdot}07 \times 10^{19}/m^3$
Plasma current		= 6·5 kA
β (on axis)		= 0·15%

Appendix 1

Vacuum Fields

The general solution of Laplace's equation for a potential field may be represented in terms of toroidal harmonics (Morse and Feshbach,1953) which are particularly well suited to problems where the boundary conditions are applied at toroidal surfaces. Solutions in this form are expressed in terms of orthogonal, toroidal coordinates $\{\eta, \tau, \xi\}$ defined about a ring of radius R_o, and related to the usual cylindrical polar coordinates (R, ϕ, z) measured about the axis of the ring, by

$$R = \frac{R_o \sinh\eta}{\cosh\eta - \cos\tau}, \qquad \phi = \xi \qquad \text{and} \qquad z = \frac{R_o \sin\tau}{\cosh\eta - \cos\tau}$$

Thus, the vacuum field of the stellarator can be represented by a potential V in the general form

$$V = B_o R_o \xi + (\cosh\eta - \cos\tau)^{\frac{1}{2}} \sum_{m,n=0}^{\infty} \left\{ A_{m,n} P_{n-\frac{1}{2}}^{(m)} (\cosh\eta) + \right.$$

$$\left. + B_{m,n} Q_{n-\frac{1}{2}}^{(m)} (\cosh\eta) \right\} e^{i(m\xi - n\tau)} \qquad \text{(A1.1)}$$

where $P_{n-\frac{1}{2}}^{(m)}$ and $Q_{n-\frac{1}{2}}^{(m)}$ are the associated Legendre functions of half-integral order.

For solutions which remain bounded at the origin the coefficients $A_{m,n}$ must vanish, and as we are interested in representing a stellarator field which has exactly p periods round the torus, the solution form appropriate to the present discussion is

$$V = B_o R_o \xi + (\cosh\eta - \cos\tau)^{\frac{1}{2}} \sum_{n} B_{p,n} Q_{n-\frac{1}{2}}^{(p)} (\cosh\eta) \cos(n\tau - p\xi) \qquad \text{(A1.2)}$$

To recover the large aspect ratio limit we revert to quasi-cylindrical coordinates (ρ, θ, ξ) and expand A1.2 in powers of ε. Using the 'optimal' ordering of Dobrott and Frieman (1971) for p we obtain, for one harmonic

$$V_n^p = B_{p,n}(\varepsilon\rho)^n \left\{\cos(n\theta + \bar{s}) + \frac{\bar{p}^2 \varepsilon^{2/3} \rho^2}{4(n+1)} \cos(n\theta + \bar{s})\right.$$

$$+ \varepsilon\rho\left(\frac{2n+1}{4}\right) \cos([n+1]\theta + \bar{s}) + \frac{\varepsilon\rho}{4} \cos([n-1]\theta + \bar{s}) + \frac{\bar{p}^4 \varepsilon^{4/3} \rho^4}{32(n+1)(n+2)} \cos(n\theta + \bar{s})$$

$$\left. + \frac{\bar{p}^2 \varepsilon^{5/3} \rho^3}{16(n+1)} \{(2n+3) \cos([n+1]\theta + \bar{s}) + 3\cos([n-1]\theta + \bar{s})\}\right\} + 0(\varepsilon^{n+2}) \tag{A1.3}$$

If we instead had taken $p \sim \varepsilon^{-1}$, as in the old stellarator ordering (Greene & Johnson, 1961), then $p^n(\cosh\eta)^{-n} \sim 0(1)$ since $e^{-\eta} = \frac{1}{2}\varepsilon\rho + 0(\varepsilon^2)$ and the small aspect ratio expansion would have produced an infinite series in powers of ρ all of equal order in ε, and giving rise to Bessel functions.

By choosing

$$B_{p,n} = 0 \qquad n = 0,\ 1 \text{ and } 2, \quad \text{and}$$

$$\varepsilon^3 B_{p,n} = A_n\ \varepsilon^{2/3}\left(\frac{\alpha B_o a}{\bar{p}}\right), \quad \text{where } A_3 = 1 \text{ and } A_n \sim 0(1) \text{ for}$$

$n \geqslant 4$, we obtain a potential in which the $\ell = 3$ helical component is dominant, and which may be written as

$$V = \frac{B_o a}{\bar{p}} \left\{ \frac{\bar{s}}{\varepsilon^{1/3}} + \varepsilon^{2/3} \alpha\rho^3 \cos(3\theta + \bar{s}) + \varepsilon V^{(3)} + \varepsilon^{4/3} V^{(4)} \ldots \right\} \tag{A1.4}$$

We fix the higher order coefficients A_{n+3} by requiring that the associated terms represent only toroidal corrections to the lowest order helical field. Thus, A_4 is determined by noting that $V^{(3)} = 0$ with the chosen form of coefficients $B_{p,n}$, so that

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial V^{(5)}}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V^{(5)}}{\partial\theta^2} + \left(\frac{\partial V^{(2)}}{\partial\rho}\cos\theta - \frac{\partial V^{(2)}}{\partial\theta}\sin\theta\right) = 0 \qquad \text{(A1.5)}$$

From the expression for $V^{(2)}$ in A1.4 it is clear that $V^{(5)}$ has no component proportional to $\cos(4\theta + \bar{s})$, apart from a possible homogeneous term in the solution of A1.5. Since $V^{(5)}$ is a correction term to $V^{(2)}$ however, all such homogeneous solutions are discarded and to eliminate terms proportional to $\cos(4\theta + \bar{s})$ in $V^{(5)}$, we choose $A_4 = -7/4$. Up to seventh order in $\varepsilon^{1/3}$ therefore, we can write V in the form

$$V = \frac{B_o a}{\bar{p}}\left\{\frac{\bar{s}}{\varepsilon^{1/3}} + \varepsilon^{2/3}\alpha\rho^3\cos(3\theta + \bar{s}) + \varepsilon^{4/3}\frac{\alpha\bar{p}^2}{16}\rho^5\cos(3\theta + \bar{s})\right.$$

$$+ \varepsilon^{5/3}\frac{\alpha\rho^4}{4}\cos(2\theta + \bar{s}) + \varepsilon^2\frac{\alpha\bar{p}^4}{640}\rho^7\cos(3\theta + \bar{s})$$

$$\left. + \varepsilon^{7/3}\frac{\alpha\bar{p}^2\rho^6}{64}\{3\cos(2\theta + \bar{s}) + \frac{17}{5}\cos(4\theta + \bar{s})\}\right\} \qquad \text{(A1.6)}$$

which is the vacuum field given by Dobrott and Frieman (1971).

Toroidal coordinates are also useful in establishing the appropriate boundary condition at large distance from the sources of current, when the field solution is expressed in quasi-cylindrical coordinates. For the case of an axisymmetric poloidal field such as that discussed in section 5, it is convenient to make use of a poloidal flux function χ, in terms of which fields are given by

$$\underset{\sim}{B} = B_o(\hat{\nabla}\chi \times \hat{\xi})/g^2$$

For vacuum fields, χ must satisfy

$$\frac{\partial}{\partial\eta}\left(\frac{1}{R}\frac{\partial\chi}{\partial\eta}\right) + \frac{\partial}{\partial\tau}\left(\frac{1}{R}\frac{\partial\chi}{\partial\tau}\right) = 0 \qquad \text{(A1.7)}$$

The solutions of this equation can be expressed in the form (Shafranov 1960)

$$\chi = F(\eta, \tau) \Big/ \sqrt{\cosh\eta - \cos\tau} \qquad \text{(A1.8)}$$

where

$$F(\eta, \tau) = \sum_{n=0}^{\infty} \left\{ A_n \frac{d}{d\eta} P_{n-\frac{1}{2}}(\cosh\eta) + B_n \frac{d}{d\eta} Q_{n-\frac{1}{2}}(\cosh\eta) \right\} \times \sinh\eta \; e^{in\tau}$$

and for a bounded solution at infinity $(\eta \to 0)$ we have $B_n = 0$. Taking the large aspect ratio limit in A1.8 and expanding in powers of ε we recover the appropriate form of solution for representing the axisymmetric fields in Eqs. (46), by choosing

$$A_n = \varepsilon^{2n+1} K_n \pi \sqrt{2} \, \{1 + O(\varepsilon)\} .$$

Thus,

$$\chi = \varepsilon K_o \left\{ \left(2 + \ell n \frac{\rho\varepsilon}{2}\right) - \varepsilon \cos\theta \left[\frac{1}{2}\rho\left(1 + \ell n \frac{\rho\varepsilon}{8}\right) + \frac{K_1}{\rho}\right]\right\} + O(\varepsilon^3) \qquad \text{(A1.9)}$$

and from this the correct choice of coefficients subject to the boundary condition at infinity may be made in Eqs. (46).

Appendix 2

The equilibrium fields

We give here the full expressions for the fields up to sixth order which are derived in section 3,

$$b_\rho = \lambda^2 \frac{3\alpha\rho^2}{\bar{p}} \cos(3\theta + \bar{s}) + \lambda^4 f^{(4)}(\rho)\cos(3\theta + \bar{s})$$
$$+ \lambda^5 \left\{ \frac{\alpha\rho^3}{\bar{p}} \cos(2\theta + \bar{s}) + f^{(5)}(\rho)\sin(6\theta + 2\bar{s}) \right\}$$
$$+ \lambda^6 \left\{ A(\rho) \sin\theta + f_1^{(6)} \cos(3\theta + \bar{s}) + f_2^{(6)}(\rho)\cos(9\theta + 3\bar{s}) \right\}$$

$$b_\theta = -\lambda^2 \frac{3\alpha\rho^2}{\bar{p}} \sin(3\theta + \bar{s}) + \lambda^3 b(\rho) + \lambda^4 g^{(4)}(\rho)\sin(3\theta + \bar{s})$$
$$+ \lambda^5 \left\{ -\frac{\alpha\rho^3}{2\bar{p}} \sin(2\theta + \bar{s}) + g^{(5)}(\rho) \cos(6\theta + 2\bar{s}) \right\}$$
$$+ \lambda^6 \left\{ B(\rho) \cos\theta + g_1^{(6)}(\rho) \sin(3\theta + \bar{s}) + g_2^{(6)}(\rho) \sin(9\theta + 3\bar{s}) \right\}$$

$$b_\xi = 1 + \lambda^3 \{\rho \cos\theta - \alpha\rho^3 \sin(3\theta + \bar{s})\}$$
$$+ \lambda^5 \left\{ \frac{\bar{p}\rho}{3} g^{(4)}(\rho) + \frac{\alpha\rho^3\sigma(\rho)}{\bar{p}} \right\} \sin(3\theta + \bar{s})$$
$$+ \lambda^6 \left\{ \left[2\bar{p} g^{(5)}(\rho) + \left(\frac{3\alpha\rho^2}{\bar{p}}\right)^2 \frac{\sigma'(\rho)}{2\bar{p}} \right] \frac{\rho}{6} \cos(6\theta + 2\bar{s}) - \frac{\alpha\rho^4}{2} \sin(4\theta + \bar{s}) \right.$$
$$\left. - \frac{3\alpha\rho^4}{4} \sin(2\theta + \bar{s}) + \rho^2 \cos^2\theta + b_\beta(\rho) \right\} \tag{A2.1}$$

Recalling that $b(\rho)$ is the field of the third order plasma current, we have

$$\frac{1}{\rho} \frac{d}{d\rho} (\rho b) = \sigma(\rho) . \tag{A2.2}$$

We have defined

$$f^{(4)}(\rho) = \frac{5}{16}\alpha\bar{p}\rho^4 + u(\rho)$$

$$g^{(4)}(\rho) = -\frac{3}{16}\alpha\bar{p}\rho^4 - \frac{1}{3}\frac{d}{d\rho}(\rho u(\rho)) \tag{A2.3}$$

where

$$-\frac{1}{3\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}(\rho u)\right) + \frac{3u}{\rho} = -\frac{3\alpha\rho^2\sigma'(\rho)}{\bar{p}^2} \quad . \tag{A2.4}$$

Similarly,

$$f^{(5)}(\rho) = V(\rho)$$

$$g^{(5)}(\rho) = \frac{1}{6}\frac{d}{d\rho}(\rho V(\rho)) \tag{A2.5}$$

where

$$\frac{1}{6}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}(\rho V)\right) - 6V = -\left(\frac{3\alpha}{2\bar{p}^2}\right)^2\rho^6\frac{d}{d\rho}\left(\frac{\sigma'(\rho)}{\rho}\right) \tag{A2.6}$$

$A(\rho)$ and $B(\rho)$ have been discussed already.

Next, the coefficients of the helical terms appearing in the sixth order axial current, Eq. (20), are defined.

$$\sigma_1^{(6)} = +\frac{3\alpha}{2\bar{p}^2}\frac{1}{\rho^4}\frac{d}{d\rho}\left\{\left(\frac{3\alpha\rho^2}{2\bar{p}^2}\right)^2\rho^7\frac{d}{d\rho}\left(\frac{\sigma'(\rho)}{\rho}\right)\right\} - \alpha\rho^3\sigma(\rho)$$

$$+\frac{1}{\bar{p}}\left\{\frac{9\alpha\rho b(\rho)}{\bar{p}} - f^{(4)}(\rho)\right\}\sigma'(\rho)$$

$$\sigma_2^{(6)} = +9\rho^2\left(\frac{\alpha\rho^2}{2\bar{p}^2}\right)^3\frac{d}{d\rho}\left(\frac{1}{\rho}\frac{d}{d\rho}\left(\frac{\sigma'(\rho)}{\rho}\right)\right)$$

Then $f_1^{(6)}$ and $g_1^{(6)}$ are given by

$$f_1^{(6)}(\rho) = \frac{7\alpha\bar{p}^3\rho^6}{640} + h(\rho)$$

$$g_1^{(6)}(\rho) = \frac{-3\alpha\bar{p}^3\rho^6}{640} + k(\rho)$$

where

$$\frac{1}{\rho}\frac{d}{d\rho}(\rho k(\rho)) + \frac{3h(\rho)}{\rho} = \sigma_1^{(6)}(\rho)$$

$$\frac{1}{\rho}\frac{d}{d\rho}(\rho h(\rho)) + \frac{3k(\rho)}{\rho} = -\bar{p}\rho\left\{\frac{\alpha\rho^2\sigma(\rho)}{\bar{p}} - \frac{\bar{p}}{9}\frac{d}{d\rho}(\rho u)\right\}$$

Also,

$$f_2^{(6)}(\rho) = w(\rho)$$

$$g_2^{(6)}(\rho) = -\frac{1}{9}\frac{d}{d\rho}(\rho\, w(\rho))$$

where

$$\frac{-1}{9\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}(\rho w)\right) + \frac{9}{\rho}w = \sigma_2^{(6)}(\rho)$$

Finally, we recall that

$$\frac{db_\beta(\rho)}{d\rho} = -\left\{\frac{dP}{d\rho} + b(\rho)\sigma(\rho)\right\} - \frac{\sigma'(\rho)}{2\bar{p}}\left(\frac{3\alpha\rho^2}{\bar{p}}\right)^2$$

In section 4 we introduced functions $\pi_1(\rho)$ and $\pi_2(\rho)$ in the expression for the flux function, Eq. (30). These are given by

$$\pi_1(\rho) = \left\{\frac{9\alpha\rho b(\rho)}{\bar{p}^3} - \frac{f^{(4)}(\rho)}{\bar{p}}\right\}\frac{d\psi_o}{d\rho} + \frac{3\alpha\rho^2}{2\bar{p}^2}\left\{6\left(\frac{3\alpha\rho^2}{2\bar{p}^2}\right)^2\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d\psi_o}{d\rho}\right]\right.$$

$$\left. + \frac{d}{d\rho}\left[\left(\frac{3\alpha\rho^2}{2\bar{p}^2}\right)^2\rho\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d\psi_o}{d\rho}\right]\right]\right\}$$

and

$$\pi_2(\rho) = \frac{3\alpha\rho^2}{2\bar{p}^2}\left\{\frac{1}{3}\,\frac{d}{d\rho}\left[\left(\frac{3\alpha\rho^2}{2\bar{p}^2}\right)^2 \rho\,\frac{d}{d\rho}\left[\frac{1}{\rho}\,\frac{d\psi_o}{d\rho}\right]\right] - 2\left(\frac{3\alpha\rho^2}{2\bar{p}^2}\right)^2 \frac{d}{d\rho}\left[\frac{1}{\rho}\,\frac{d\psi_o}{d\rho}\right]\right\} .$$

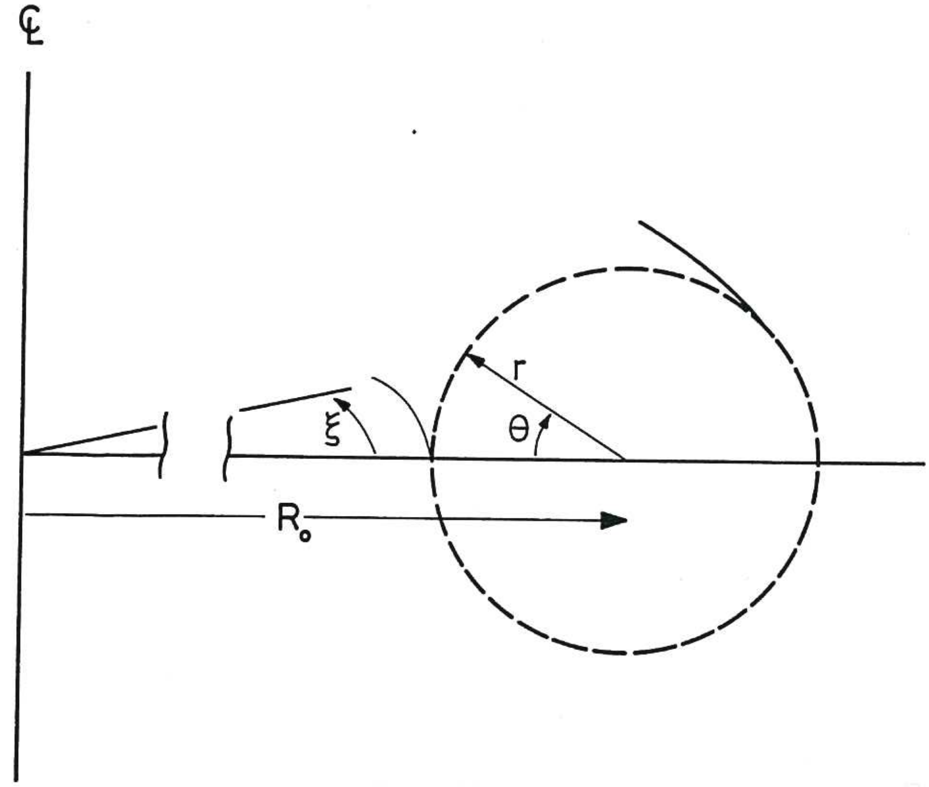

Fig.1 Coordinate geometry.

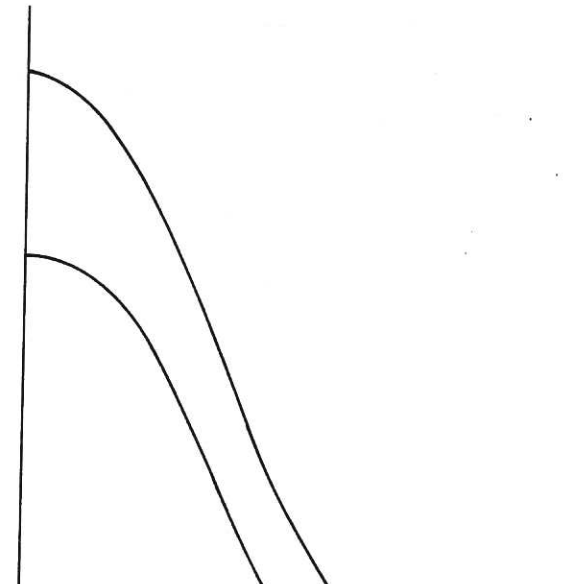

Fig.2 Normalised radial profiles of plasma pressure (lower curve) and current density.

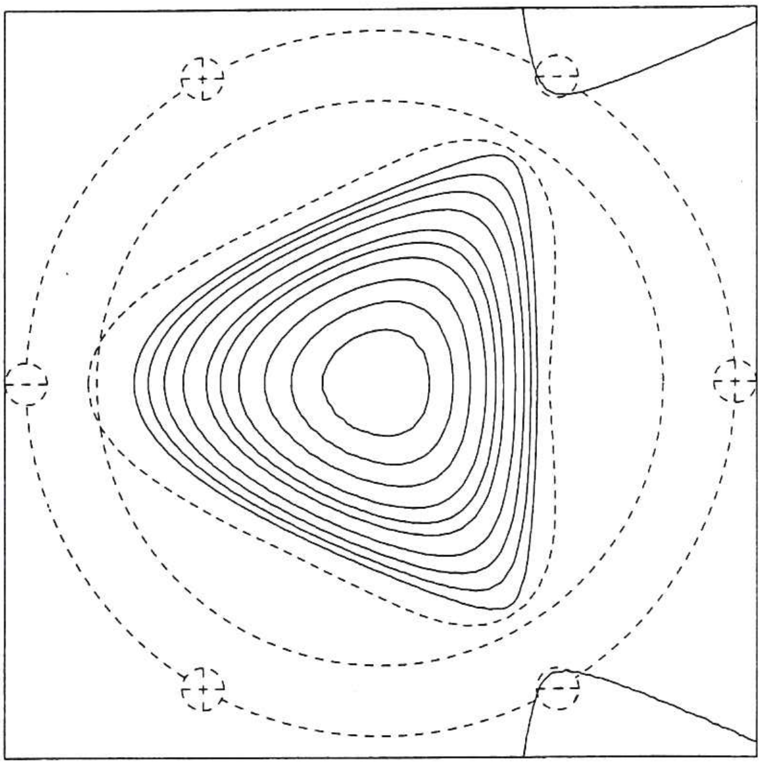

Fig.3 Flux surfaces of the vacuum magnetic field for the CLEO parameters given in Table I. The location of the stellarator windings is indicated (dashed) together with the direction of current flow (+ into the page), and the vacuum vessel position is marked by the inner circle (also dashed). A third order flux surface (dashed line) is included for comparison. At the cross-section shown, $\bar{s} = \pi/2$, the major axis lying to the left.

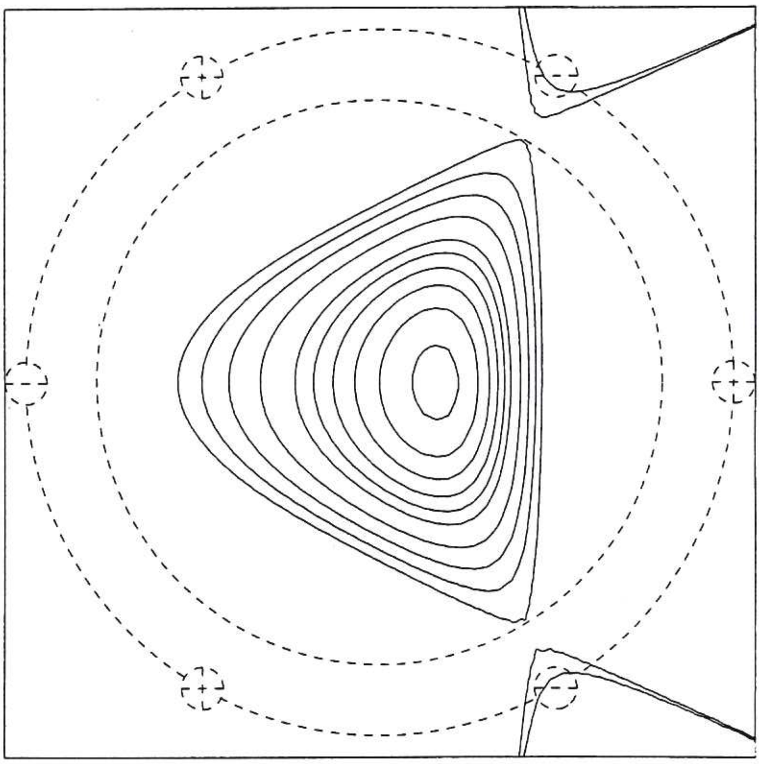

Fig.4 Equilibrium flux surfaces for the plasma parameters of Table II, with the profiles of Fig.2 and the vacuum field as for Fig.3, that is $B_V = 0$.

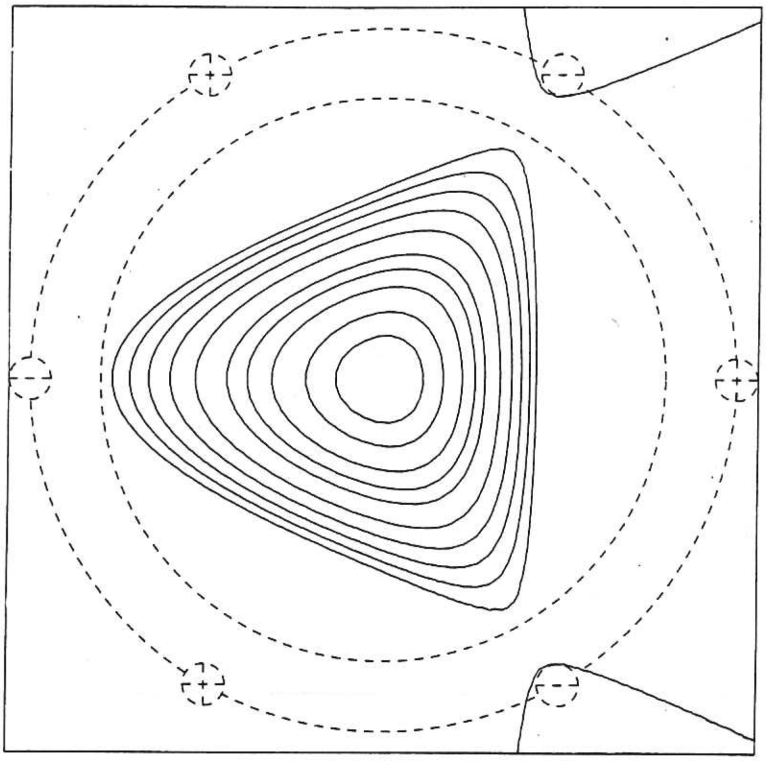

Fig.5 Flux surfaces as for Fig.4, but with B_V = 68·4 gauss.

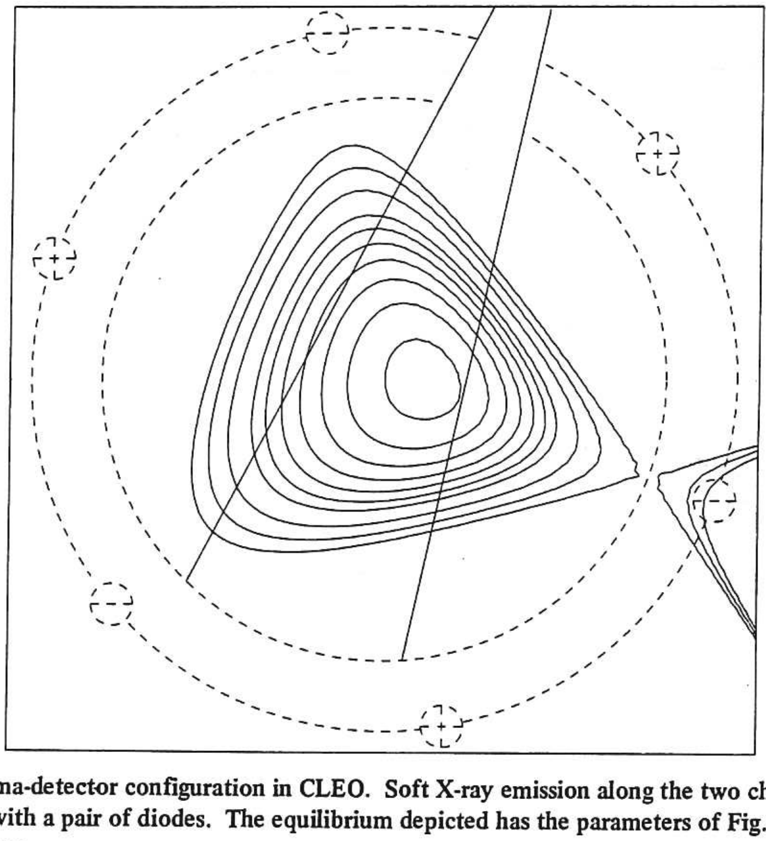

Fig.6 Plasma-detector configuration in CLEO. Soft X-ray emission along the two chords shown is measured with a pair of diodes. The equilibrium depicted has the parameters of Fig.4, but with B_V = 32 gauss.

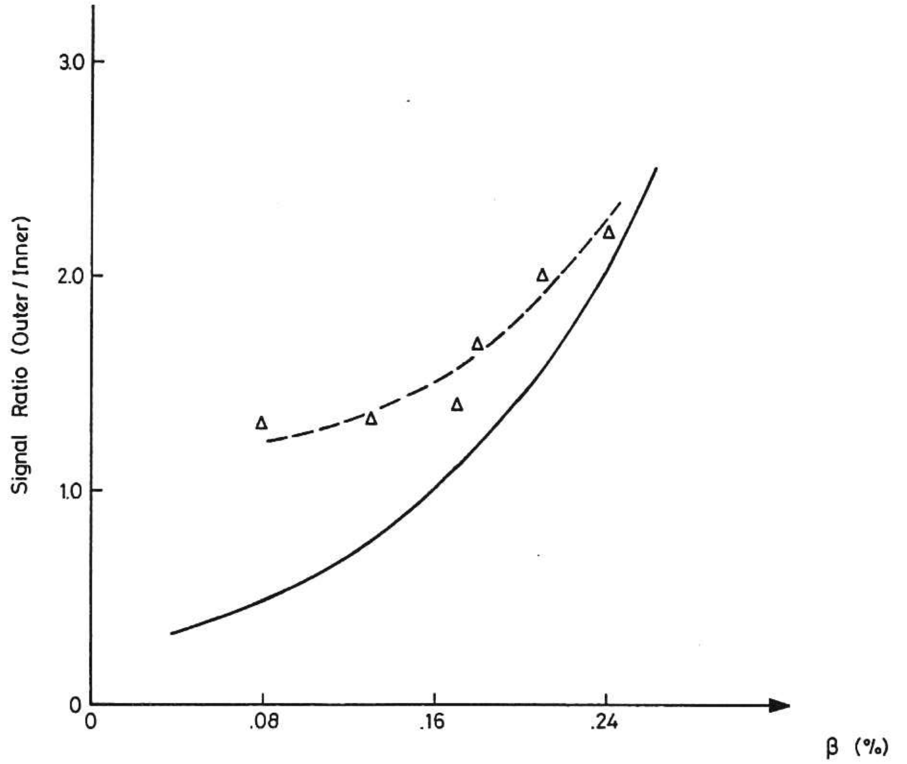

Fig.7 Variation of X-ray diode signal ratio vs plasma β, at a fixed value of B_V = 33 gauss. At the higher values of β where species equilibration is better and the signals are stronger, the calculated (solid line) and measured (triangles) ratios are in quite good agreement.

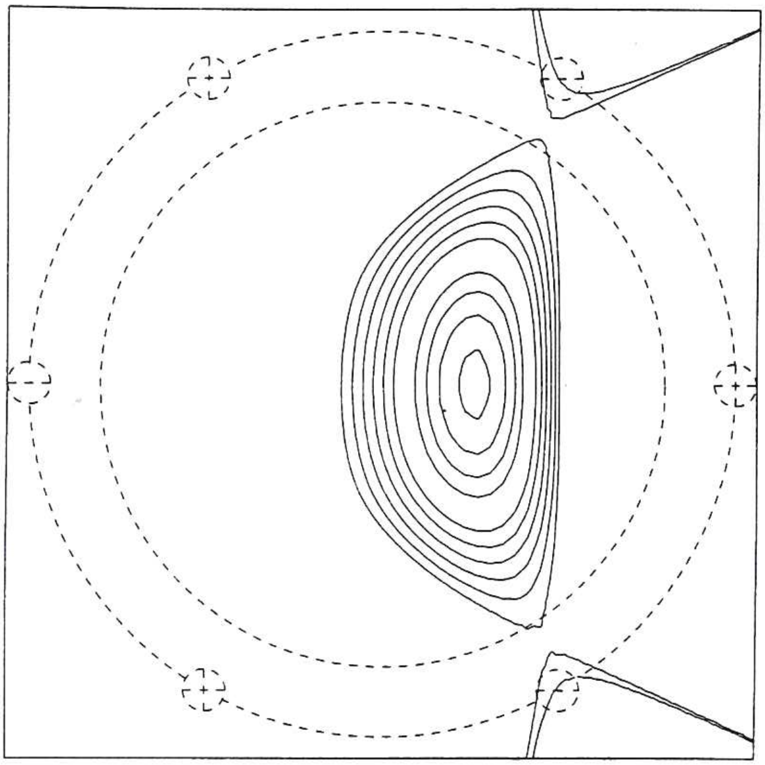

Fig.8 Flux surfaces as for Fig.4 but with β = 1% on the axis.

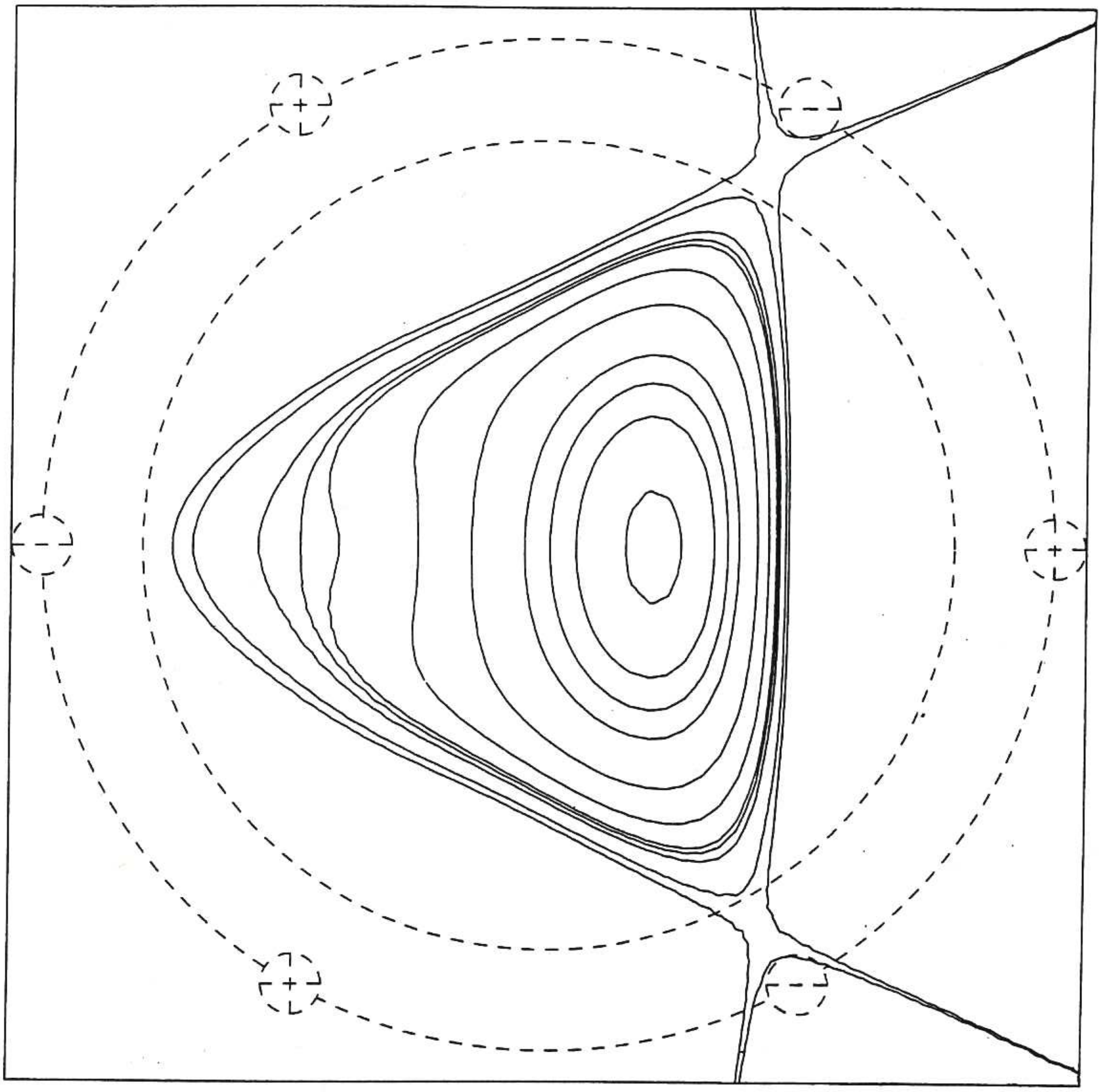

Fig.9 Flux surfaces for the equilibrium profiles of Fig.8, but with B_V = 150 gauss.