

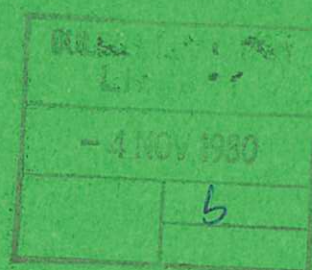


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METHODS FOR BOUNDARY-LAYERS
IN ARBITRARY GEOMETRIES

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CO-ORDINATE TRANSFORMATION METHODS FOR BOUNDARY-LAYERS IN ARBITRARY GEOMETRIES

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ABSTRACT

Co-ordinate transformation methods are developed so that the boundary surfaces of an arbitrary three-dimensional geometry can be represented accurately by suitably stretching or 'ironing-out' uneven surfaces to behave like planar ones. This change of co-ordinates converts the physical space into a transformed space which forms, in general, a non-orthogonal curvilinear system. The resulting Navier-Stokes equations now involve a few additional terms but the boundary conditions can be applied very simply and accurately. Another change of co-ordinates to a computational space is now effected in such a way that a uniform grid structure in the computational space corresponds to a variable grid structure in the transformed space. By choosing the second transformation with care we get a concentration of grids in the boundary layer without excessive numbers in the interior. This two-step technique, i.e. change of co-ordinates from the physical space to the transformed space and then to the computational space, is illustrated through its application to two uncoupled problems.

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1. INTRODUCTION

In thermohydraulic problems the features of interest are usually controlled by the geometry. The simplest finite difference approach with a uniform lattice of grid points is unsatisfactory on two counts, however. Firstly, grid points do not coincide with the boundaries unless the latter are of extremely simple shape. Secondly in order to resolve boundary layers large number of grid points are situated where not needed. We propose a different approach which consists of the following steps: (i) find and apply a mapping which regularizes the boundary (ii) find and apply a co-ordinate stretching which broadens boundary layers only (iii) discretize the transformed PDE on the now regularized mesh, and (iv) solve the resulting matrix equations. The co-ordinate transformations in (i) and (ii) need not be orthogonal but there must be one-to-one correspondence. Gal-Chen and Somerville [1] have examined mappings to regularize the boundary.

In this paper we illustrate the approach via two physical problems. In the first, stage (i) is not required since we consider turbulent flow in a straight channel, but a stretched co-ordinate is essential. In the second problem, slow flow through a pipe with undulating boundaries, stage (i) is required but the boundary layers already fill the pipe so that stage (ii) is not required. It is simpler to appreciate the benefits of the two stages in these uncoupled problems.

2. TRANSFORMATION FOR BOUNDARY LAYERS

We have examined various co-ordinate stretching transformations in use [2]. The transformation that we prefer (for two boundary layers at $z = 0$ and $z = 1$) is

$$\xi = \frac{c}{2} \left[1 + \frac{\tanh 2a_1 z}{\tanh 2a_1} + \frac{\tanh 2a_2(z-1)}{\tanh 2a_2} \right] + (1-c)z \quad (1)$$

where a_i 's are determined by the number of meshes within the boundary layer and c determines the number of nodes in the mainstream region. For the case $\delta_1 = 0.01$, $\delta_2 = 0.005$, four meshes in each boundary layer and with $c = 0.6$, the resulting grid patterns in both the physical and computational spaces are shown in Fig.1. Note that there are at least four meshes in each of the boundary layers. The uniform grid spacing in the transformed space corresponds to a continuously varying grid space in the physical space.

An application of the co-ordinate transformation methodology is demonstrated by considering turbulent flow in a straight channel driven by a constant pressure gradient. The normalized mean velocity $\beta (=U/\sqrt{\tau_0})$ satisfies [3]:

$$\frac{d}{d\eta} \left(\sigma \frac{d\beta}{d\eta} \right) + 1 = 0 \quad (2)$$

where $\eta = z/D$, $\sigma (=v_T/\sqrt{\tau_0}D)$ is the normalized viscosity, v_T is the turbulent viscosity, τ_0 is the shear stress at the wall ($\eta=0$) and D is the channel half-thickness. Owing to the symmetry we consider only

$0 \leq \eta \leq 1$ region. Analytical solutions of (2) exist for the constant stress layer ($0 \leq \eta \leq q$) where $\sigma = k\eta$, and for the central region ($q \leq \eta \leq 1$) $\sigma = kq$ where k is the Karman constant (≈ 0.41) experimentally).

Equation (2) can also be solved numerically. For the case of the bulk Reynolds number of 10^4 , it appears that a very fine grid in the physical space (η - co-ordinates) would be needed. Our experience, however, shows that the resulting solution is far from the actual one even with 200 meshes and that increasing the number of meshes beyond that introduces compounding of round-off and truncation errors. Alternately, (2) can be transformed by using (1) which for the half-region becomes

$$\zeta = (1-c)\eta + c \tanh a\eta \quad (3)$$

A close to analytical solution is obtained with as few as 40-50 meshes and $c = 0.2$. Note that the value of c is related to the desired number of meshes within the boundary layer and the total number of meshes. For the case under discussion, Reference 2 gives $n/N = 0.42 c$. Thus, in order to have a minimum of four meshes within the boundary layer (10^{-5}), one needs to have a total number of 50 meshes. Any increase in the total number of meshes should be compensated by appropriate adjustments in either n or c .

3. BOUNDARY REGULARIZATION

For the mapping which regularizes the boundary surface we find it convenient to go to the metric tensor formulation as used by Gal-Chen and Sommerville [1]. The detailed equations for a three-dimensional system bounded by irregular surfaces on its boundaries are noted elsewhere [4]. We illustrate the technique here by using a two dimensional system with irregular boundaries as shown in Fig.2. The upper and lower boundaries are represented by $x^3 = \psi(x^1)$ and $x^3 = \phi(x^1)$ curves, respectively. The interior region can be transformed to a regular region by using the following transformation

$$\bar{x}^1 = x^1/L \quad ; \quad \bar{x}^3 = [x^3 - \phi(x^1)] / [\psi(x^1) - \phi(x^1)] \quad (4)$$

The physical space $\{0 \leq x^1 \leq L; 0 \leq \phi(x^1) \leq x^3 \leq \psi(x^1)\}$ is now mapped in a square region $\{0 \leq \bar{x}^1 \leq 1; 0 \leq \bar{x}^3 \leq 1\}$. The transformed momentum equation for steady-state flow with negligible inertia becomes

$$g^{ij} \frac{\partial p}{\partial \bar{x}^j} = J^{-1} \frac{\partial}{\partial \bar{x}^j} (J \lambda_\mu \bar{e}^{ij}) + \left\{ m^i \quad n \right\} \lambda_\mu \bar{e}^{mn} \quad (5)$$

where

$$\bar{e}^{ij} = \frac{1}{2} \left(g^{in} \frac{\partial \bar{u}^l}{\partial \bar{x}^n} + g^{in} \frac{\partial \bar{u}^i}{\partial \bar{x}^n} - \frac{\partial g^{ij}}{\partial \bar{x}^n} \bar{u}^n \right)$$

$$g^{ij} = \begin{pmatrix} L^{-2} & 0 & \gamma_1 \\ 0 & 1 & 0 \\ \gamma_1 & 0 & \gamma_2 \end{pmatrix}$$

$J = L(\psi - \phi)$, $\gamma_1 = [(\bar{x}^3 - 1)\phi_1 - \bar{x}^3\psi_1]/[L(\psi - \phi)]$, $\gamma_2 = [1 + \{(\bar{x}^3 - 1)\phi_1 - \bar{x}^3\psi_1\}^2]/(\psi - \phi)^2$
 $\phi_1 = \partial\phi/\partial x^1$, $\psi_1 = \partial\psi/\partial x^1$ and the only non-vanishing Christoffel symbols are

$$\left\{ \begin{matrix} 3 \\ 1 \end{matrix} \right\} = L^2 [\phi_{11} + (\psi_{11} - \phi_{11})\bar{x}^3] / (\psi - \phi)$$

$$\left\{ \begin{matrix} 3 \\ 3 \end{matrix} \right\} = \left\{ \begin{matrix} 3 \\ 1 \end{matrix} \right\} = L(\psi_1 - \phi_1) / (\psi - \phi)$$

where $\phi_{11} = \partial\phi_1/\partial x^1$, $\psi_{11} = \partial\psi_1/\partial x^1$. In these relations a bar over the quantities mean that these are in the transformed co-ordinates. When (5) is combined with the equation of continuity

$$\frac{1}{J} \frac{\partial}{\partial \bar{x}^i} (J \bar{u}^i) = 0 \quad (6)$$

a consistent set of solutions results.

Note that this transformation preserves the linearity of the equations in the dependent variables p and u^1 , but introduce a couple of spatially dependent coefficients. The above equations can also be used when the geometry under consideration is simplified by taking either $\phi(x^2)$ or $\psi(x^1)$ as constant. In the event of both (lower and upper) curves being a straight line, the above set of equations reduce to the more familiar ones as the transformation becomes an identity one.

Figure 2 shows the grid structure for a wavy shaped capillary tube in both physical and computational spaces. The upper and lower boundaries are seen to be transformed into straight lines. In other words, the irregular boundaries are represented exactly without having to approximate them to fit into the grid pattern used elsewhere. The computational grids become squares or rectangles depending upon whether the mesh sizes are the same or different. Another point worth emphasizing is that the restricted regions, such as the neck in Fig.2, are automatically represented by finer meshes which is very desirable.

CONCLUSION

The use of co-ordinate transformations to regularise the boundaries and to stretch boundary layers has been demonstrated. The additional complexity of the PDE appears to be satisfactorily offset by the ease with which the boundary conditions can be applied, and the relatively small number of grid points required.

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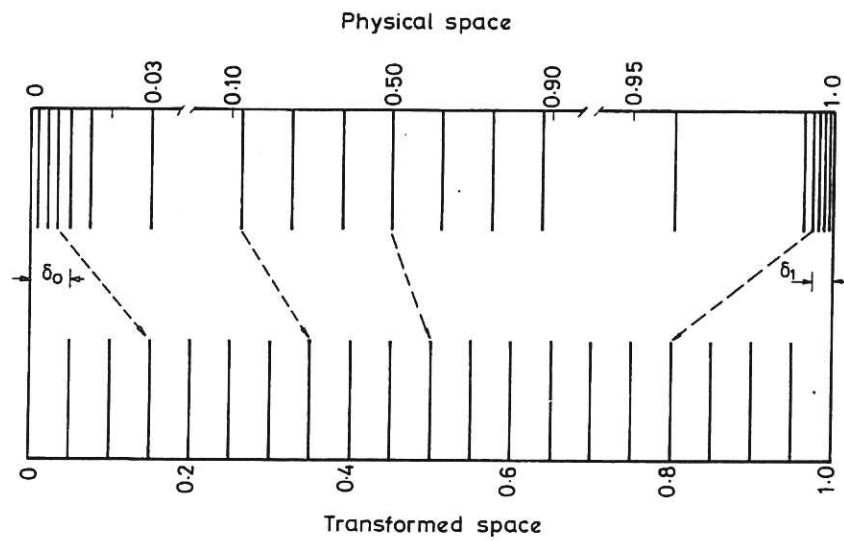


Fig.1 Physical and transformed grid structures (note the change in scale in the physical co-ordinate).

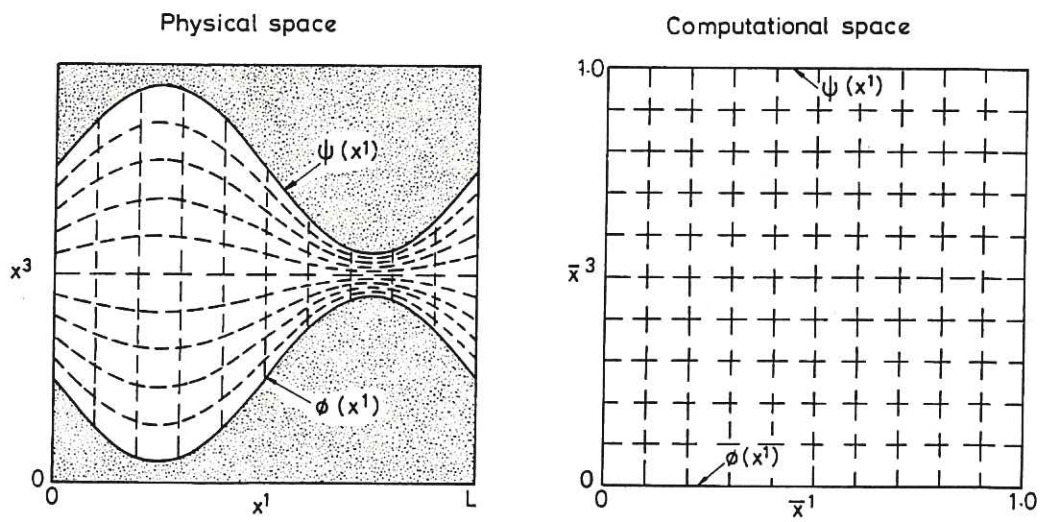


Fig.2 The mapping of physical space into computational space for a wavy capillary.



