CLM-P618

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ABSTRACT. It is pointed out that the range of mode numbers n for which high-n, ballooning-mode, theory is valid is more restricted when the shear is weak. A new theory valid for weak shear and intermediate mode number is outlined. Combined with the standard theory this analysis shows that the growth rate of high-n instabilities is linear in $1/n$ at large n but in equilibria with weak shear there may be intermediate values of n for which the growth rate is an oscillatory function of $1/n$, with amplitude and period proportional to $1/n^2$.

(Submitted for publication in Nuclear Fusion)

September 1980

1. INTRODUCTION

The theory of high toroidal mode number (ballooning) instabilities in axisymmetric toroidal plasmas [1-3] was developed to complement the 2-D computer codes used for low mode number instabilities. In many equilibria the predictions of ballooning theory [3-5] for large mode number n link smoothly with the results of 2-D codes at values of $n \sim 4\text{-}5$, so providing a comprehensive analysis of mhd stability. Indeed it is claimed [6] that ballooning theory is itself accurate down to small values of n. On the other hand [7] doubt has been cast on the accuracy of ballooning theory at much larger values of n when ν' (an important parameter of ballooning theory related to local shear) is vanishingly small.

In this note we (i) point out that the validity of ballooning theory may indeed be restricted if the shear is very weak - but not for the reason referred to in Ref. [7], (ii) describe a modified theory valid for weak shear and moderate n, and (iii) combine this with standard ballooning theory to describe the qualitative behaviour of growth rates of high mode number instabilities as a function of mode number in weak shear systems.

We shall follow closely the derivation of ballooning theory given by Connor, Hastie and Taylor (CHT) in Refs. [1] and [2]. [An alternative derivation [3] given by Chance et al based on a WKB formalism is simpler but neglects some $1/n$ terms. These do not affect standard ballooning theory but can be significant in the modified theory discussed in section 3.]

2. STANDARD BALLOONING THEORY

In CHT one introduces a transformation

$$X(\theta) = \sum_{m=-\infty}^{\infty} e^{-im\theta} \int_{-\infty}^{\infty} d\eta \, e^{im\eta} \hat{X}(\eta) \tag{1}$$

from the poloidal angle $\theta \, (0 < \theta \leqslant 2\pi)$ to an extended, infinite domain $\eta \, (-\infty < \eta < \infty)$. Then the frequency Ω of a toroidal mode $\sim \exp in\zeta$ is determined from the eigenvalue $\omega^2(\psi, y_o)$ of an ordinary differential equation in the infinite domain [1,2], with boundary conditions at infinity. In the notation of CHT this ordinary differential equation is

$$\begin{aligned} &\frac{1}{J}\frac{\partial}{\partial y}\left\{\frac{1}{JR^2B_\chi^2}\left[1 + \left(\frac{R^2B_\chi^2}{B}\int_{y_o}^{y} \nu' dy\right)^2\right]\frac{\partial F}{\partial y}\right\} \\ &+ \frac{2Fp'}{B^2}\left[\frac{\partial}{\partial\psi}\left(p + \frac{1}{2}B^2\right) - \frac{I}{B^2}\left(\int_{y_o}^{y} \nu' dy\right)\frac{1}{J}\frac{\partial}{\partial y}\left(\frac{1}{2}B^2\right)\right] \\ &+ \frac{\omega^2(\psi, y_o)}{R^2B_\chi^2}\left[1 + \left(\frac{R^2B_\chi^2}{B}\int_{y_o}^{y} \nu' dy\right)^2\right] F = 0 \end{aligned} \tag{2}$$

and the frequency Ω is then given as

$$\Omega^2 = \omega_o^2 + \frac{1}{2n|\nu'(\psi, y_o)|}\left\{\frac{\partial^2\omega^2}{\partial\psi^2}\frac{\partial^2\omega^2}{\partial y_o^2}\right\}^{\frac{1}{2}} \tag{3}$$

where $\omega_o^2 = \min \omega^2(\psi, y_o)$. As we have noted, ν' is related to the local shear; the global shear is $q' = \frac{1}{2\pi}\oint \nu' d\chi$.

This theory is formally an expansion in powers of $1/n$, representing the ratio of perturbation scale length transverse to $\underline{B}$ to equilibrium scale lengths. It is assumed that all equilibrium lengths are $O(1)$; indeed it is the great strength of high-n theory that it is not dependent on large aspect ratio, weak shear, or low-β.

However when all equilibrium quantities are not $O(1)$, modification of the high-n theory may be necessary. One example [8] arises when the scale length of the current density is exceptionally short and results in a modification of the $1/n$ term in Eq. (3). In this note we consider the important case of exceptionally weak shear.

It may appear from the form of Eq. (3) that if $\nu'(y_o)$ is small the second term becomes large. This in turn suggests that one would require $n\nu' \gg 1$ for ballooning theory to be valid, as suggested in Ref. [7]. However, an examination of the differential Eq. (2) for $\omega^2(\psi, y_o)$ shows that if $\nu' \to 0$ then so does $(\partial^2\omega^2/\partial y_o^{\,2})$ and that the ratio $(1/\nu')\left(\dfrac{\partial^2\omega^2}{\partial y_o^{\,2}}\right)^{\frac{1}{2}}$ remains finite.

Nevertheless it is clear that ballooning theory must break down if the magnetic shear tends to zero, since the basis of the theory is that ballooning modes may be constructed from the overlap of many localised Fourier modes centred on adjacent rational surfaces [2,3,9,10]. When the spacing of the rational surfaces becomes too large this construction is no longer valid. In terms of the total poloidal flux ψ the typical width of an unstable ballooning mode is $\psi/n^{\frac{1}{2}}$ while the separation of rational surfaces is $1/nq'(\psi)$. Thus we expect ballooning theory to require $n(\psi q')^2 \gg 1$.

Mathematically this requirement can be seen most easily if the transformation of Eq. (1) is substituted directly into the energy integral $\delta W(\underset{\sim}{\xi}, \underset{\sim}{\xi})$. Then the resulting double sum over m can be contracted to a single sum of the form

$$\delta W \propto \sum_N \int d\psi \int_{-\infty}^{\infty} d\eta \; e^{2\pi i N n q(\psi)} \; \xi(\eta,\psi)\; \xi(\eta + 2\pi N, \psi) \quad . \tag{4}$$

Ballooning theory, in effect, approximates this infinite sum by its zero-th term, e.g. by invoking the Reimann-Lebesgue lemma [8]. The accuracy of this can be gauged by substituting, a posteriori, the resulting localised $\xi(\psi)$; this shows the error to be exponentially small in n , of order $\exp[-n(\pi\psi q')^2]$.

For "typical" equilibria $(\psi q') \sim 1$ and ballooning theory requires only $n \gg 1$. However in regions of low shear $(\psi q' \ll 1)$ two regimes of high-n modes must be distinguished.

(a) $n \gg (\psi q')^{-2} \gg 1$ when standard ballooning theory is valid.

(b) $(\psi q')^{-2} \gg n \gg 1$ when a new theory is required.

In the following section we outline a theory for this second regime, again by an expansion in powers of $1/n$, but one in which $n\psi q'$ is $O(1)$. The full calculation is similar to that of CHT which should be consulted for mathematical details.

3. HIGH MODE NUMBER INSTABILITIES IN WEAK SHEAR EQUILIBRIA

The general Euler equation for the minimising displacement $X = RB_\chi \xi_\psi$ of a high-n perturbation takes the form

$$(L - \Omega^2 M)X = 0 \tag{5}$$

where the operators L and M are defined by Eq. (15) of CHT. For weak shear equilibria, instead of introducing the ballooning representation, we write the perturbations as

$$X = \hat{X} \exp\left[in \int^{\psi} k(\psi)d\psi - i\frac{m}{q}\int^{\chi} \nu d\chi \right] \tag{6}$$

so that the parallel derivative becomes

$$iJBk_{\parallel} = \frac{\partial}{\partial\chi} + i\frac{\nu}{q}(nq - m) \tag{7}$$

and the transverse derivative

$$\frac{1}{n}\frac{\partial}{\partial\psi} \rightarrow \left[ik - i\,\frac{m}{nq}\left(\int^{\chi} \nu' d\chi - \frac{q'}{q}\int^{\chi} \nu d\chi\right) + \frac{1}{n}\frac{\partial}{\partial\psi}\right] \hat{X} \; . \tag{8}$$

Here k is used instead of the parameter y_o of CHT. [In standard ballooning theory the quantity $|\nu'(y_o)|^{-1} (\partial^2\omega^2/\partial y_o^2)^{\frac{1}{2}}$ in Eq. (3) is then replaced by $(\partial^2\omega^2/\partial k^2)^{\frac{1}{2}}$ which, as we have noted, remains finite as $\nu' \rightarrow 0$.]

Because the global shear is weak, the term $(nq - m)$ in $k_{\parallel}$ is now treated as an $O(1)$ quantity in the $1/n$ expansion (in CHT it was $O(n)$ and was removed by the ballooning transformation). Then Eq. (5) is solved by expansion in powers of $1/n$.

In lowest order one obtains an eigenvalue equation

$$(L_o - \omega_o^2 M_o)\, \hat{X} = 0 \tag{9}$$

where

$$\begin{aligned} L_o = &-\left(\frac{\partial}{\partial\chi} + i\,\frac{\nu}{q}\,\mu\right) \frac{1}{JR^2B_\chi^2}\left[1 + \left(\frac{R^2B_\chi^2}{B}\right)^2 \tilde{k}^2\right]\left(\frac{\partial}{\partial\chi} + i\,\frac{\nu}{q}\,\mu\right) \\ &- \frac{2J}{B^2}\,p'\,\frac{\partial}{\partial\psi}\left(p + \frac{1}{2}B^2\right) + \tilde{k}\,\frac{\partial}{\partial\chi}(Ip'/B^2) \end{aligned} \tag{10}$$

$$M_o = \frac{J}{R^2B_\chi^2}\left[1 + \left(\frac{R^2B_\chi^2}{B}\right)^2 \tilde{k}^2\right] \tag{11}$$

with $\tilde{k} = k - \frac{m}{nq}\left(\int^{\chi} \nu' d\chi - \frac{q'}{q}\int^{\chi} \nu d\chi\right)$, and $\mu \equiv nq - m$.

As in CHT this is an ordinary differential equation in the poloidal coordinate and defines a "local" eigenvalue ω^2 . It differs from the corresponding equation of standard ballooning theory (Eq. (2)) in that:-

(i) Eq. (9) is to be solved in the finite domain with periodic boundary conditions instead of in the infinite domain.

(ii) In the operators L_o and M_o , since the global (or average) shear is small, only the local shear remains; on the other hand the rotational transform itself now appears through $(nq - m) \equiv \mu$.

(iii) The mode numbers n and m also enter directly through μ so that even the lowest order eigenvalue is a function of toroidal mode number, $\omega^2 = \omega^2(\psi,k,n)$. [In standard ballooning theory there is no unique poloidal mode number m and the lowest order eigenvalue is independent of n , corresponding to the limit $n \to \infty$; the present result corresponds to the limit $n \to \infty$ but with $(nq - m)$ remaining finite.]

The variation of $\omega^2(\psi)$, at fixed k and n , has two distinct contributions. The first, due to shear, arises because $(nq - m) \equiv \mu$ depends on ψ . It produces a periodic variation in the lowest eigenvalue (since when $nq(\psi)$ changes by ± 1 a corresponding change in m restores μ to its previous value). This periodic contribution has no counterpart in standard ballooning theory. The second contribution arises because other equilibrium quantities, such as pressure gradient or field curvature, depend on ψ . It produces a variation in $\omega^2(\psi)$ similar to that arising in standard ballooning theory.

Of these two contributions the first depends on toroidal mode number but the second does not. Roughly speaking the total variation of $\omega^2(\psi)$ is of the form $\omega^2(\psi) \sim A + B(\psi - \psi_o)^2 + C(nq - m)^2$ and resembles Fig. 1. The two contributions can be distinguished by regarding ω^2 as a function of the variables (ψ,k,μ) rather than (ψ,k,n). Then

$$\left.\frac{\partial\omega^2}{\partial\psi}\right|_n = \left.\frac{\partial\omega^2}{\partial\psi}\right|_\mu + nq' \frac{\partial\omega^2}{\partial\mu} \tag{12}$$

where the derivatives on the right hand side are independent of mode number.

Higher Order Theory

As in standard ballooning theory, the solution of the lowest order Eq. (9) contains an unknown amplitude $A(x)$ and does not determine the value of k or ψ. These are determined by the higher order theory in the expansion in $1/n$.

In higher orders one of the consequences of the integrability conditions is that ψ and k must be at a minimum of $\omega^2(\psi,k,n) \equiv \omega_o^2(n)$. In many cases (though not all), the minimum with respect to k will be at $k = 0$ if the equilibrium has symmetry about the equator $\chi = 0$. The variation of ω^2 with ψ has been discussed above (see Fig. 1). It is clear from that discussion that the value of ω_o^2 depends on the relative location of the minima of $\omega^2(\psi,k,\mu)$ with respect to ψ at fixed μ and the minima with respect to μ at fixed ψ; that is on the location of the rational surfaces $(nq(\psi) - m) = 0$. Thus $\omega_o^2(n)$ will oscillate as a function of n, regarded as a continuous variable, as each successive rational surface passes through ψ_o (where ψ_o is that flux surface on which $\omega^2(\psi,k,\mu = 0)$ has its minimum). The amplitude of these oscillations will decrease as n increases because the rational surfaces become more closely spaced in ψ, but the period remains fixed, $\Delta n \sim 1/q$.

Usually integer values of n will not lie at the peaks or troughs of the oscillations in ω_o^2 but during the evolution of a discharge the rotational transform will change slowly and the rational surfaces will pass through ψ_o, giving an effect similar to that of a continuous n . Hence we shall continue to regard n as a continuous parameter.

A further consequence of the higher order integrability conditions, (and one which completes the theory) is that the amplitude $A(x)$ is given by a Weber equation. When a rational surface coincides with the surface ψ_o - which is an important case because it usually corresponds to an unstable peak in the oscillations of $\omega^2(n)$ - the equation for $A(x)$ takes the same form as in standard ballooning theory;

$$\left(\frac{\partial^2\omega^2}{\partial k^2}\right)_n \frac{d^2A}{dx^2} + \left[\ 2n(\Omega^2 - \omega_o^2) - \left(\frac{\partial^2\omega^2}{\partial\psi^2}\right)_n x^2\ \right] A = 0 \tag{13}$$

Thus in this case Ω^2 is again determined entirely by the zero-order eigenvalue ω^2 and its derivatives and

$$\Omega^2 = \omega_o^2 + \frac{1}{2n}\left\{\left.\frac{\partial^2\omega^2}{\partial k^2}\right|_n \left.\frac{\partial^2\omega^2}{\partial\psi^2}\right|_n\right\}^{\frac{1}{2}} \quad . \tag{14}$$

In the general case $A(x)$ is still given by a Weber equation but this contains additional terms arising from the operator $\tilde{L}_2$ of CHT. [These terms have no effect in standard ballooning theory and do not arise in the WKB version of Ref. [3], but in the present theory they vanish only when $(nq(\psi_o) - m) = 0$].

Eqs. (13) and (14) involve the derivatives of $\omega^2(\psi,k,n)$ at constant n and so contain the implicit dependence on mode number described above Eq. (12). It is therefore convenient to express (13) in terms of $\omega^2(\psi,k,\mu)$ as

$$\Omega^2 = \omega_o^2 + \frac{1}{2n}\left\{\omega_{kk}^2\,[\omega_{\psi\psi}^2 + (nq')^2\,\omega_{\mu\mu}^2\,]\right\}^{\frac{1}{2}} \tag{15}$$

where the derivatives are now independent of n. Recalling that this result is valid for $n\psi q' \sim 1$ we see that when $n\psi q'$ becomes small, Ω^2 is a linear function of $1/n$,

$$\Omega^2 = \omega_o^2 + \frac{1}{2n}(\omega_{kk}^2 \omega_{\psi\psi}^2)^{\frac{1}{2}} \tag{16}$$

as in the standard ballooning theory, whereas when $n\psi q'$ becomes large

$$\Omega^2 = \omega_o^2 + \frac{|q'|}{2}(\omega_{kk}^2 \omega_{\mu\mu}^2)^{\frac{1}{2}} \tag{17}$$

and Ω^2 is then independent of n.

In comparing these results with those of standard ballooning theory one must remember that ω^2 is defined by a different equation with different boundary conditions in the two cases - though it can be shown that the two theories differ only by terms of $O(q'^2)$ as $n \to \infty$.

4. VARIATION WITH n OF HIGH-n INSTABILITIES

Using the results of the previous sections we are now in a position to give a qualitative description of the variation with n of the growth rate of high-n instabilities.

When all equilibrium scale lengths, including shear, are $O(1)$ the variation is given by standard ballooning theory, Eq. (3), and is linear with $1/n$.

When the shear is very weak a more complex picture emerges. At the largest values of n, $(n \gg (\psi q')^{-2} \gg 1)$ the standard theory is again valid and $\Omega^2(n)$ is again linear in $1/n$. At intermediate values of n $((\psi q')^{-2} \gg n \gg 1)$ the new theory outlined in section 3 is applicable. This shows that $\Omega^2(n)$ is an oscillatory function of n (regarded as a continuous variable). These oscillations, which reflect the passage of rational surfaces across the equilibrium profile as n varies, decrease in amplitude like $1/n^2$ and are negligible in the regime of validity of standard ballooning theory. The period of the oscillations is constant in n so that in terms of $1/n$ the period also decreases like $1/n^2$. The envelope of the oscillations, on the more unstable side, is given by Eq. (15). This envelope is also linear in $1/n$ when $n(\psi q')$ is small but is independent of n when $n(\psi q')$ is large.

These features of the variation of Ω^2 with mode number n are shown schematically in Fig. 2. They are very similar to the features observed recently in numerical computations using the new, more powerful, version of the PEST computer code [6].

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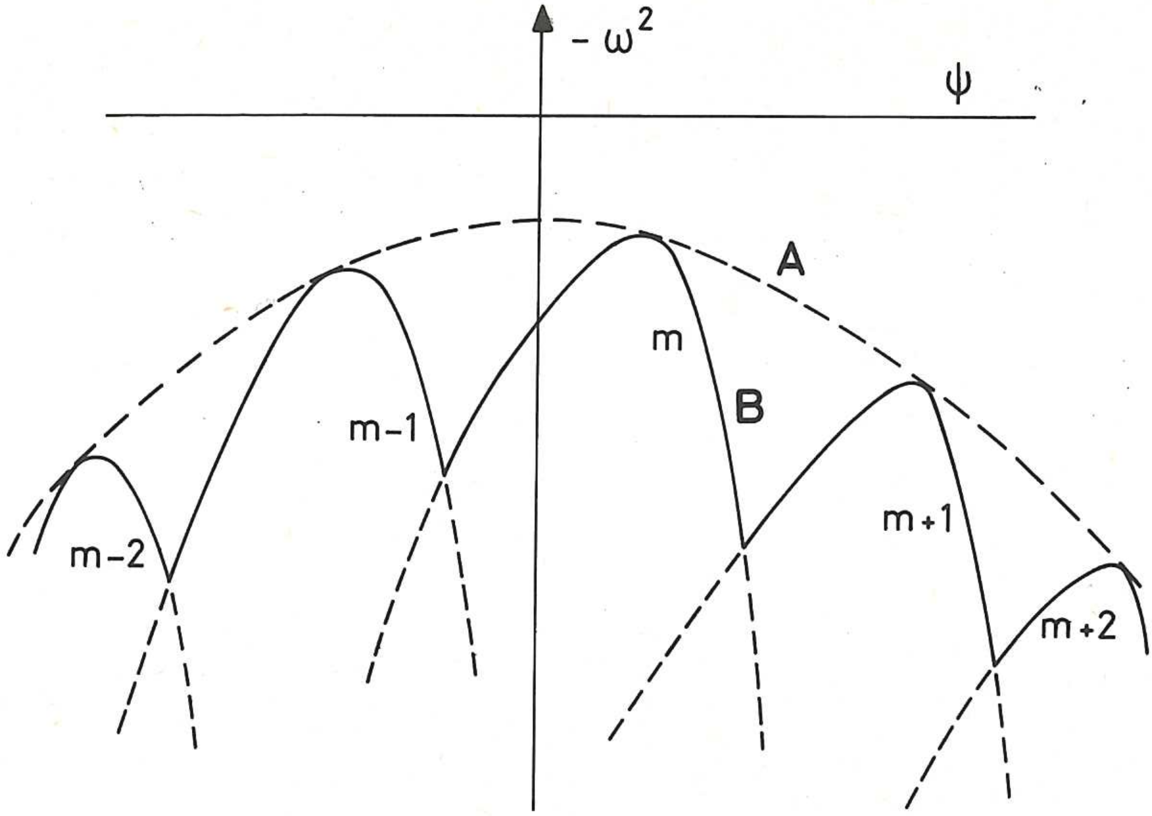

Fig.1 Variation of $\omega^2(\psi)$. A. Variation due to changes in parameters such as pressure gradient etc. B. Variation due to change in rotational-transform.

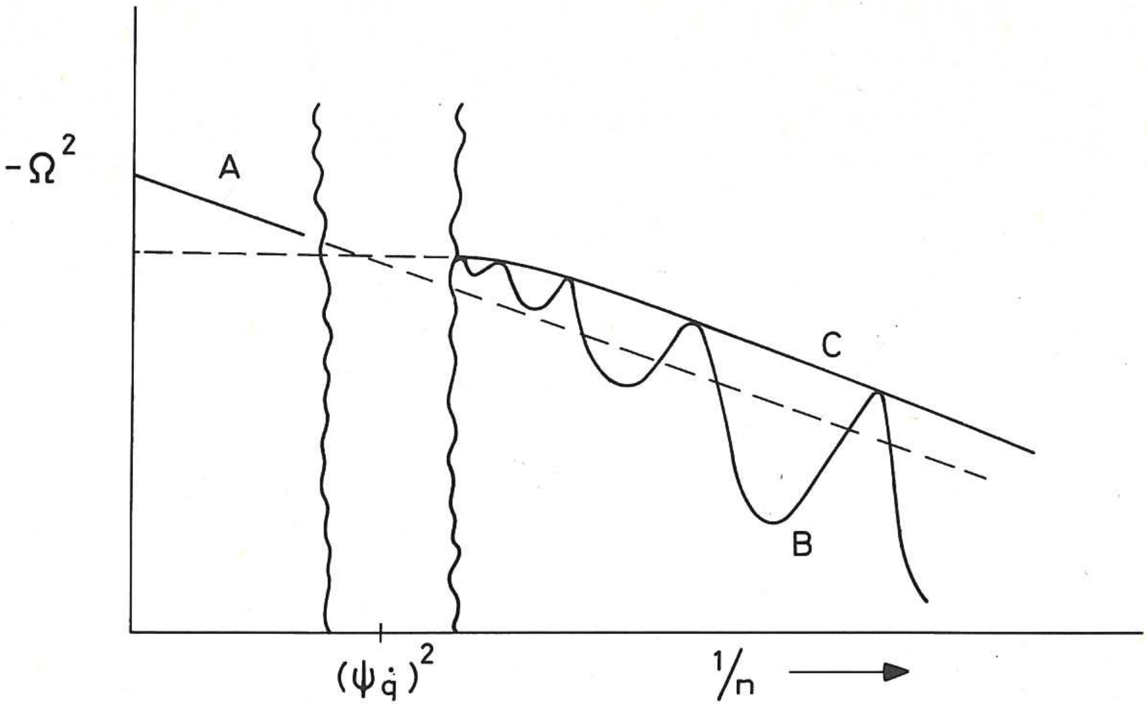

Fig.2 Schematic diagram for variation of Ω^2 with inverse mode number 1/n. A. Standard ballooning theory (Eq.(3), valid for $1/n < (\psi q)^2$). B. Oscillations in $\Omega^2(n)$, amplitude $\sim 1/n^2$. C. Envelope of oscillations (Eq.(15) valid for $1/n > (\psi q)^2$).