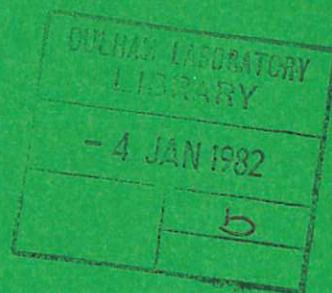




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DETERMINATION OF CURRENT DISTRIBUTION IN A TOKAMAK

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ABSTRACT. It is shown that the equilibrium current distribution in an axisymmetric discharge can, in principle, be determined completely from purely geometric information about the shape of the magnetic surfaces. As these surfaces coincide with those of constant density and temperature it is possible that observations of plasma density and temperature could be sufficient to determine the current distribution.

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1. INTRODUCTION

The current distribution in an axisymmetric tokamak, or equivalently the poloidal field distribution, is an important feature of the discharge. It determines the q -profile, with important consequences for stability and disruptions, as well as the distribution of ohmic heating. Although direct methods for measuring the poloidal magnetic field exist [1], none has yet been developed as a routine diagnostic. However moments of the current defined by

$$u_m = \int \chi_m(R,Z) J_\varphi(R,Z) dRdZ$$

where $\chi_m(R,Z)$ are a set of functions satisfying

$$\nabla \cdot \frac{1}{R^2} \nabla \chi_m(R,Z) = 0$$

(and R, φ, Z are the usual polar coordinates), can readily be obtained from measurements of the poloidal field outside the plasma [2,3].

In this paper we investigate an alternative method for determining the current distribution. We show that the current distribution can be obtained completely from purely geometric information about the shape of the magnetic surfaces. Since both plasma density and temperature are uniform over a magnetic surface it is possible that measurements of the density or temperature surfaces, eg by X-ray emission, cyclotron emission or laser scattering, could alone be sufficient to determine the current distribution.

It may be surprising that information about the shape of the surfaces alone is sufficient to determine current profile in a

tokamak since in a straight and circularly symmetric system (often considered as the large aspect ratio limit of a simple tokamak) it is clearly impossible to infer the current distribution from knowledge of the (circular) shape of the surfaces. However, we shall show that, except in the circular limit, knowledge of the shape of the surfaces is not only sufficient to fully determine the current distribution - it actually leads to a high degree of redundancy in the determination.

The layout of the paper is as follows:- In section 2 we present the basic method for determining the current distribution from the surfaces in a linear system and show that it fails only when the surfaces are circular. We also show that for any non-circular system the information obtained exhibits redundancy. In section 3 we describe the corresponding analysis for a toroidal system. In section 4 we illustrate the basic method by some simple examples. Finally, in section 5, we comment on means of exploiting the redundancy to obtain the best estimates of current distribution from imprecise data on the shape of the surfaces.

2. LINEAR SYSTEMS

In a linear system the magnetic field can be written

$$\underline{B} = \underline{n} \times \nabla \psi + \underline{n} f$$

where \underline{n} is a unit vector in the symmetry direction. A scalar pressure plasma equilibrium then satisfies

$$\nabla^2 \psi = -\mu_0 J(\psi)$$

where $J(\psi)$ is the axial current density in which we are interested.

The function $\psi(r, \theta) = \text{constant}$ defines the magnetic surfaces which, as we have noted, are also pressure, density and temperature surfaces.

Suppose that the shapes of these surfaces are known, but not, of course, the magnetic field associated with them. Then one can specify the surfaces by $V(r, \theta) = \text{constant}$, where V is the volume of each surface. We now show that $\psi(r, \theta)$, and hence all magnetic quantities such as magnetic field and current profiles, can be deduced from the function $V(r, \theta)$.

Since the surfaces $\psi = \text{constant}$ coincide with those of $V = \text{constant}$ there is a functional relation between ψ and V which we must find. Now if $\psi = \psi(V)$ we have

$$\nabla^2 \psi = \frac{d^2 \psi}{dV^2} |\nabla V|^2 + \frac{d\psi}{dV} (\nabla^2 V) = -\mu_0 J(\psi). \quad (1)$$

In this equation the unknowns ψ'' , ψ' and J are uniform over the surfaces $V = \text{constant}$. On the other hand the known quantities $|\nabla V|^2$ and $\nabla^2 V$ (in general) vary over the surfaces $V = \text{constant}$. Defining the average and varying parts of any quantity over a surface by

$$\langle A \rangle = \oint A \frac{d\ell}{|\nabla V|}, \quad \tilde{A} = A - \langle A \rangle$$

we therefore obtain

$$\frac{d^2 \psi}{dV^2} \left(\frac{d\psi}{dV} \right)^{-1} = - \left[\widetilde{\nabla^2 V} \right] \left[\widetilde{|\nabla V|^2} \right]^{-1} \quad (2)$$

If the surfaces specified by $V(r, \theta)$ do indeed represent an equilibrium then the expression on the right side of Eq. (2) will be

a function of V only, and we may write

$$\widetilde{[\nabla^2 V]} [\widetilde{|\nabla V|^2}]^{-1} \equiv \lambda_1(V)$$

Then elementary integration yields the required function $\psi(V)$ in terms of known quantities:

$$\psi(V) = \alpha \int^V \exp \left(- \int^{V'} \lambda_1(V'') dV'' \right) dV' + \beta .$$

The constant β is of no physical significance and α is determined by the total flux. The corresponding current distribution is

$$\mu_0 J = -\alpha [\lambda_1(V) \langle |\nabla V|^2 \rangle + \langle \nabla^2 V \rangle] \exp \left(- \int^V \lambda_1(V') dV' \right) .$$

We see, therefore, that if the shape of each surface is known then the magnetic field and the current distribution can readily be derived. The only situation where this is in principle impossible is when $\widetilde{|\nabla V|^2}$ is zero; this is just the case of circular magnetic surfaces and the usefulness of the method therefore depends on the extent to which the surfaces depart from circular symmetry.

It is not necessary that the given surfaces be specified by their volume; any function would suffice. For if the surfaces are defined by any function $F(r, \theta) = \text{constant}$, then, in analogy to Eq. (1), we have

$$\frac{d^2 \psi}{dF^2} |\nabla F|^2 + \frac{d\psi}{dF} \nabla^2 F = -\mu_0 J(\psi)$$

The functions ψ'' , ψ' and J are uniform over a surface $F = \text{constant}$, so if θ is any convenient angle around such a surface and $h(\theta)$ is any function whose mean value over such a surface is zero, then

$$\frac{d^2\psi}{dF^2} \left(\frac{d\psi}{dF} \right)^{-1} = - \langle h \nabla^2 F \rangle \langle h |\nabla F|^2 \rangle^{-1} \equiv -\lambda_2(F) \quad (3)$$

where in this case we define

$$\langle A \rangle \equiv \oint A d\theta .$$

Eq. (3) determines the required function $\psi(F)$; hence the magnetic field and current profiles are once again determined in terms of known quantities.

There are two important points about the derivations given above. First, only if the given surfaces represent some plasma equilibrium will λ_1 be a function only of V and similarly only then will λ_2 be independent of the "test function" $h(\theta)$. The necessary and sufficient condition for a set of arbitrary surfaces to be magnetic surfaces of a plasma equilibrium is that $\widetilde{(\nabla^2 V)} (\widetilde{|\nabla V|^2})^{-1}$ or $\widetilde{(\nabla^2 F)} (\widetilde{|\nabla F|^2})^{-1}$ be uniform over the surfaces. Second, if the given surfaces do represent some plasma equilibrium, then the basic Eq. (3) can be derived using many alternative test functions $h(\theta)$, and Eq. (2) can be derived by determining $\widetilde{(\nabla^2 V)} (\widetilde{|\nabla V|^2})^{-1}$ at an infinite number of alternative positions on the surfaces. Thus, there is a high degree of redundancy in the information available through F or V . In practice one would know F or V only inexactly and we shall return in section 5 to the question of finding a good estimate of $\lambda_1(V)$ or $\lambda_2(F)$ in this situation.

3. TOROIDAL SYSTEMS

It is a straightforward matter to generalise the argument of the previous section to the toroidal case. In a toroidal system R, φ, Z the field is defined by

$$\vec{B} = \frac{\vec{n} \times \nabla \psi}{R} + \frac{\vec{n} f(\psi)}{R}$$

and an equilibrium satisfies

$$\Delta^* \psi = R^2 \nabla \cdot \frac{1}{R^2} \nabla \psi = - \mu_0 R J = - \mu_0 R^2 p'(\psi) - f f'$$

where $p(\psi)$ is the plasma pressure.

If now the surfaces are specified by $F(R, Z) = \text{constant}$ then as in the linear case there is a functional relationship $\psi(F)$ and we have

$$\frac{d^2 \psi}{dF^2} |\nabla F|^2 + \frac{d\psi}{dF} \Delta^* F = - \mu_0 R^2 \frac{dp}{d\psi} - f \frac{df}{d\psi}$$

Once again the unknown functions ψ'' , ψ' , p and f are constant over a surface, whereas the known quantities $\Delta^* F$ and $|\nabla F|^2$ vary over a surface. We now introduce two functions $h_1(\theta)$ and $h_2(\theta)$ (where θ is a convenient poloidal angle) whose mean value over a surface is zero. Then after some manipulation we obtain

$$-\frac{d^2 \psi}{dF^2} \left(\frac{d\psi}{dF} \right)^{-1} = \frac{\langle h_1 \Delta^* F \rangle \langle h_2 R^2 \rangle - \langle h_2 \Delta^* F \rangle \langle h_1 R^2 \rangle}{\langle h_1 |\nabla F|^2 \rangle \langle h_2 R^2 \rangle - \langle h_2 |\nabla F|^2 \rangle \langle h_1 R^2 \rangle} \equiv \lambda(F)$$

which again defines $\psi(F)$ in terms of known quantities:

$$\psi = \alpha \int^F \exp \left(- \int^{F'} \lambda(F'') dF'' \right) dF' + \beta$$

In the toroidal case the contributions to J arising from $p'(\psi)$ and $ff'(\psi)$ can be obtained separately (unlike the linear case where only J itself is relevant) as

$$p'(\psi) = - \frac{\alpha \exp \left(- \int^F \lambda(F') dF' \right)}{\mu_0 \langle h_1 R^2 \rangle} (\lambda \langle h_1 |\nabla F|^2 \rangle + \langle h_1 \Delta^* F \rangle)$$

$$ff'(\psi) = - \frac{\alpha \exp \left(- \int^F \lambda(F') dF' \right)}{\langle h_1 R^{-2} \rangle} (\lambda \langle h_1 R^{-2} |\nabla F|^2 \rangle + \langle h_1 R^{-2} \Delta^* F \rangle)$$

This toroidal calculation appears always to be possible in principle but becomes increasingly difficult and inaccurate as the aspect ratio increases and as the surfaces approach circular form. It is, therefore, most likely to be feasible in systems with small aspect ratio and markedly non-circular cross section - such as JET.

4. EXAMPLES

(i) The simplest example is that of elliptical surfaces in a straight system. Suppose the surfaces are specified by

$$F(r, \theta) = r^2(1 - \mu \cos 2\theta)$$

Then

$$|\nabla F|^2 = \frac{4F}{1 - \mu \cos 2\theta} (1 - 2\mu \cos 2\theta + \mu^2)$$

and

$$\nabla^2 F = 4.$$

Hence, provided $\mu \neq 0$ (ie that the surfaces are not in fact circular) we have, using any suitable test function $h(\theta)$ such as $h(\theta) = \cos 2p\theta$;

$$\langle h \nabla^2 F \rangle = 0 \quad \langle h |\nabla F|^2 \rangle \neq 0$$

so that Eq. (3) gives

$$\psi(r, \theta) = \alpha F(r, \theta) + \beta$$

and

$$J = 4\alpha$$

(ii) A second simple example, with more complex surfaces, is given by

$$F(r, \theta) = [J_0(r) + \sum_m a_m J_m(r) \cos m\theta]^2.$$

In this case we have

$$|\nabla F|^2 = 4F \left\{ \left(J_0'(r) + \sum_m a_m J_m'(r) \cos m\theta \right)^2 + \left(\sum_m \frac{m}{r} a_m J_m(r) \sin m\theta \right)^2 \right\}$$

$$\nabla^2 F = \frac{1}{2F} |\nabla F|^2 + 2F$$

Hence

$$\langle h(\theta) \nabla^2 F \rangle = \frac{1}{2F} \langle h(\theta) |\nabla F|^2 \rangle$$

and Eq. (3) becomes

$$\frac{d^2 \psi}{dF^2} = - \frac{1}{2F} \frac{d\psi}{dF}$$

so that

$$\psi(r, \theta) = \alpha (J_0(r) + \sum_m a_m J_m(r) \cos m\theta) + \beta$$

and

$$\mu_0 J(r, \theta) = \psi(r, \theta).$$

5. USE OF INEXACT DATA

We have pointed out that if the surface shapes are defined precisely then this leads to much redundant information. In the linear system, for example, it would be enough to know only the moments $\langle h \nabla^2 F \rangle$ and $\langle h |\nabla F|^2 \rangle$ on each surface, or alternatively to know the mean values of $\nabla^2 V$ and $|\nabla V|^2$ together with their actual values at only one point on each surface. In practice the surfaces and the derivatives will not, of course, be known accurately, and the problem becomes one which is overdetermined but inexact. One can then exploit the redundancy to reduce the influence of errors in the data. Thus in the linear system a "least squares" estimate for $\lambda_1(V)$, given by

$$\lambda_1(V) = \left[\oint \left(\frac{\nabla^2 V}{|\nabla V|^2} \right)^2 \frac{d\ell}{\nabla V} \right]^{\frac{1}{2}}$$

would seem to be appropriate. In the toroidal system the redundancy could be exploited by using a sequence of test functions h_n , h_m so obtaining a sequence of estimates $\lambda_{nm}(F) = \lambda_{mn}(F)$, $m \neq n$. Then the least squares estimate would be

$$\lambda_2(F) = \left[\frac{2}{N(N-1)} \sum_{n=2}^N \sum_{m=n+1}^N \lambda_{mn}^2 \right]^{\frac{1}{2}}$$

The errors in these estimates will depend on many factors including the way in which the data is processed to produce λ_{mn} , the instrumental errors themselves and the nature of the configuration itself, particularly its aspect ratio and departure from cylindrical symmetry.

6. CONCLUSIONS

We have shown how the current distribution in an axisymmetric tokamak discharge can, in principle, be determined completely from purely geometric information about the shape of the magnetic surfaces. This information might itself be obtained from experimental observations of the temperature or density profiles. If the surfaces are specified by $F(R,Z) = \text{constant}$ then the poloidal flux, and hence the magnetic field and current distribution, is given in terms of a single function $\lambda(F)$ which can be computed on each surface from the Laplacian and gradient of F . Since $\lambda(F)$ depends on the variation of ∇F around a magnetic surface the method appears most suitable for discharges with small aspect ratio and markedly non-circular cross section. It fails for a straight system with circular surfaces.

We have illustrated the method by algebraic examples, but in practice it would have to be implemented numerically and would be subject to errors - both numerical and instrumental. However the information contained in $F(R,Z)$ leads to redundancy which could be used to minimise the effect of these errors. The practical aspects of the problem will be discussed elsewhere.

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