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Abstract

It is shown that there is a dynamic friction associated with magnetic field line wandering and that its existence enables one to reconcile the diffusion of field lines with the idea that in some sense the density of field lines is the field strength. It is also suggested how the possible non-occurrence of ergodicity in toroidal geometry might be formally reconciled with a diffusive analysis of field line wandering.

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I INTRODUCTION

This paper is about the statistical geometry of wandering magnetic field lines and shows how the concept of dynamic friction reconciles two apparently contradictory intuitive notions about field lines. These notions are: (a) that the wandering of the field lines will lead them to diffuse, and (b) that, in some sense, the density of the field lines is the field strength, which obviously does not diffuse.

(a) and (b) are stated very imprecisely and it might be thought that any attempt to formulate them more precisely would dissolve the apparent contradiction. But it is argued below that this is not the case: that (a) and (b) can be made mathematically respectable, that the apparent contradiction persists, but that it can be removed by a more careful treatment which leads to the introduction of a dynamic friction.

However, before proceeding with the argument we wish to make it clear that we are not concerned with two other related problems. Firstly we do not consider the effect of magnetic field fluctuations on transport (CALLEN 1977, RECHESTER and ROSENBLUTH 1978, KADOMTSEV and POGUTSE 1979, THYAGARAJA, HAAS and COOK 1980). Secondly, we do not directly discuss the conditions which must be satisfied in order that the field lines behave ergodically (LAVAL and GRESILLON 1980). However, it is possible to formally reconcile the diffusive analysis presented here with the possibility that ergodicity does not occur. This issue is pursued in section 3.

Let the magnetic field be written in the form

$$\underline{B} = \hat{\underline{z}}\, B_z(\underline{r}) + \underline{B}_\perp(\underline{r}) \,, \tag{1}$$

where $\hat{\underline{z}}$ is a unit vector and $B_\perp$ (which will later be chosen to be small) has no z-component:-

$$\hat{\underline{z}} \cdot \underline{B}_\perp = 0 \,. \tag{2}$$

Now let

$$\underline{R}_\perp(z) \equiv \big(X(z),\ Y(z),\ 0\big)\ , \tag{3}$$

be the "trajectory" of a field line. It satisfies

$$\frac{d\,\underline{R}_\perp}{dz} = \frac{\underline{B}_\perp(\underline{R}_\perp(z),z)}{B_z(\underline{R}_\perp(z),z)} \tag{4}$$

If we think of z as a "time", it is useful to define a "fluid velocity" $\underline{U}_\perp$ by

$$\underline{U}_\perp(\underline{r}_\perp,z) \equiv \frac{\underline{B}_\perp(\underline{r}_\perp,z)}{B_z(\underline{r}_\perp,\ z)} \tag{5}$$

Then we can re-write (4) in the suggestive form

$$\frac{d\,\underline{R}_\perp}{dz} = \underline{u}_\perp(\underline{R}_\perp(z),\ z)\ . \tag{6}$$

Let $\rho(\underline{r}_\perp,\ z)$ be the density of an arbitrarily chosen set of field lines. (They might, for example, be labelled (discretely or continously) by their positions as they intersect the $z=0$ plane). From (6) ρ satisfies

$$\frac{\partial\rho}{\partial z} + \underline{\nabla}_\perp\cdot(u_\perp\rho) = 0\ . \tag{7}$$

(See, e.g., KADOMTSEV and POGUTSE 1979).

From this point a conventional development (explicit in Kadomtsev and Pogutse and implicit in other work) would be to split ρ into mean and fluctuating parts

$$\rho \equiv \langle\rho\rangle + \tilde{\rho}\ , \tag{8}$$

and, via a quasi-linear analysis, derive a diffusion equation for $\langle\rho\rangle$:-

$$\frac{\partial\langle\rho\rangle}{\partial z} - D_\perp\nabla_\perp^2\langle\rho\rangle = 0\ , \tag{9}$$

where $D_\perp$ is the line-diffusion coefficient and is an integral over the auto-correlation function of $\underline{u}_\perp$. This seems reasonable and certainly suggests that the lines diffuse.

On the other hand, we know that div $\underline{B} = 0$ and this can be written, in the present case, as

$$\frac{\partial B_z}{\partial z} + \nabla_\perp . B_\perp = 0 . \tag{10}$$

But, using the definition (5), (10) can be re-written as

$$\frac{\partial B_z}{\partial z} + \underline{\nabla}_\perp . (\underline{u}_\perp B_z) = 0 . \tag{11}$$

So we see that B_z satisfies (7). Thus there is a <u>particular</u> density of field lines ($\rho \propto B_z$) which (in a limited way) makes precise the notion that the density of lines is the field strength.

We now observe that if the reasoning leading from (7) to (9) were valid, we would expect (because of (11)) that $\langle B_z \rangle$ would satisfy a diffusion equation. But we know that this is not so. So we can conclude that the apparent conflict between intuitive notions (a) and (b) persists even when they are considerably sharpened up. In the next section we dissolve this apparent contradiction: the essence of the matter is the recognition that in general the "flow" is compressible:

$$\underline{\nabla}_\perp . \underline{u}_\perp \neq 0 . \tag{12}$$

II DIFFUSION AND FRICTION

We shall treat the problem in the simplest geometry that suffices to illustrate the essential point. So let

$$\langle B_z \rangle = B_o(x) , \tag{13}$$

$$\tilde{B}_z = 0 , \tag{14}$$

$$< \underline{B}_\perp > = 0 \tag{15}$$

and

$$|\tilde{\underline{B}}_\perp| \ll B_o , \tag{16}$$

where angular brackets denote ensemble averages and tildes denote zero-mean random fluctuations. The inequality (16) is to be understood metaphorically: the true criterion for the validity of a quasi-linear analysis is that the field lines diffuse by less than a perpendicular correlation length in one parallel correlation length.

Eq. (7) now becomes

$$\frac{\partial \rho}{\partial z} + \nabla_\perp \cdot (\tilde{\underline{u}}\, \rho) = 0 , \tag{17}$$

where

$$\tilde{\underline{u}} \equiv \frac{\tilde{B}_\perp}{B_o} . \tag{18}$$

Averaging (17),

$$\frac{\partial <\rho>}{\partial z} + \underline{\nabla}_\perp . <\tilde{\underline{u}}\, \tilde{\rho}> = 0 . \tag{19}$$

Subtracting (19) from (17) and neglecting the terms quadratic in the fluctuating quantities,

$$\frac{\partial \tilde{\rho}}{\partial z} + \nabla_\perp . \left\{ \tilde{\underline{u}} <\rho> \right\} = 0 \tag{20}$$

(20) may more conveniently be written as

$$\frac{\partial \tilde{\rho}}{\partial z} + \tilde{\underline{u}} . \underline{\nabla}_\perp <\rho> + (\underline{\nabla}_\perp . \tilde{\underline{u}}) <\rho> = 0 \tag{21}$$

Operating on (18) with $\underline{\nabla}_\perp$ and using $\underline{\nabla}_\perp . \tilde{\underline{B}}_\perp = 0$, we can easily show that

$$\underline{\nabla_\perp} . \underline{\tilde{u}} = - \underline{\tilde{u}} . \hat{x} \left(\frac{d}{dx} \ell n\, B_o \right) \qquad (22)$$

So, substituting (22) into (21) we obtain

$$\frac{\partial \tilde{\rho}}{\partial z} = \underline{\tilde{u}} . \left\{ \left(\hat{x} \frac{d}{dx} \ell n\, B_o - \underline{\nabla_\perp} \right) <\rho> \right\} \qquad (23)$$

We now integrate (23) from z_o to z, multiply by $\tilde{u}^{\alpha}(\underline{r_\perp}, z)$ and average, to obtain

$$<\tilde{u}^{\alpha} \tilde{\rho}> = - \int_{z_o}^{z} ds <\tilde{u}^{\alpha}(\underline{r_\perp}, z) \tilde{u}^{\beta}(\underline{r_\perp}, s)> \left\{ \delta^{\beta 1} \frac{d}{dx} \ell n\, B_o - \nabla_\perp^{\beta} \right\} <\rho(\underline{r_\perp}, s)>$$

$$+ <\tilde{u}^{\alpha}(\underline{r_\perp}, z) \tilde{\rho}(\underline{r_\perp}, z_o)> \qquad (24)$$

We now assume that $\tilde{u}(z)$ is a stationary random function of z, with finite parallel correlation length $L_{\|}$ and that $<\rho>$ varies very slowly on this length scale. Then choosing $z - z_o > L_{\|}$, the last term of (24) can be dropped and (24) re-written (after a change in variable of integration):-

$$<u^{\alpha} \tilde{\rho}> = - \int_{o}^{\infty} d\ell <\tilde{u}^{\alpha}(\underline{r_\perp}, \ell) \tilde{u}^{\beta}(\underline{r_\perp}, o)> \left\{ \delta^{\beta 1} \frac{d}{dx} \ell n\, B_o - \nabla_\perp^{\beta} \right\} <\rho(r_\perp, z)>$$

(25)

We now assume (just for simplicty - it is not necessary) that $<\tilde{u}^{\alpha} \tilde{u}^{\beta}>$ is diagonal, and substitute (25) into (19). The result is the desired equation for $<\rho>$:-

$$\frac{\partial}{\partial z} <\rho> = \frac{\partial}{\partial x} \left\{ D_x \left[\frac{\partial}{\partial x} <\rho> - \left(\frac{1}{B_o} \frac{dB_o}{dx} \right) <\rho> \right] \right\} + \frac{\partial}{\partial y} \left\{ D_y \frac{\partial}{\partial y} <\rho> \right\}, \qquad (26)$$

where

$$D_x \equiv \int_0^\infty d\ell \, < \tilde{u}_x \, (\underline{r_\perp},\ell) \, \tilde{u}_x \, (\underline{r_\perp},o) > \quad , \tag{27}$$

and

$$D_y \equiv \int_0^\infty d\ell \, < \tilde{u}_y \, (\underline{r_\perp},\ell) \, \tilde{u}_y \, (\underline{r_\perp},o) > \quad . \tag{28}$$

It is apparent that a solution of (26) is

$$< \rho \, (x,y,z) > \; = \lambda \, B_o(x) \quad , \tag{29}$$

where λ is any constant. So the presence of the dynamic friction term in (26) (which arose naturally and inevitably from the analysis) enables us to preserve the idea that there is a mean density of field lines which is also the mean field strength, without having to abandon the idea that the field lines diffuse.

The price paid is that the diffusion of field lines is not unhindered. The lines have a systematic tendency to move up the mean field gradient and in addition perform a random walk about this systematic motion.

III THE POSSIBLE NON-OCCURRENCE OF DIFFUSION

It is well-known that in toroidal geometry the existence of fluctuating magnetic fields does not necessarily lead to field line diffusion. This phenomenon is not yet understood but it can be formally accommodated within the foregoing analysis by supposing that the analogues of the integrals (27) and (28) vanish. This will occur if the auto-correlation functions in question have a negative tail equal in area to the central positive hump. We do not know why this should sometimes be the case, but it is reassuring to know that a similar phenomenon occurs in a much more familiar system -

the motion of a particle in a 1-D random potential. This is demonstrated below, where we show that a particle moving in one dimension in a random potential does not diffuse in velocity space, because the structure of the force auto-correlation is as discussed above.

Let the equation of motion be

$$\ddot{X}(t) = \tilde{F}(X(t)) \qquad , \qquad (30)$$

where the force $\tilde{F}$ is a stationary random function and is derivable from a potential:-

$$\tilde{F}(x) = \frac{\partial \tilde{\phi}(x)}{\partial x} \qquad . \qquad (31)$$

To ensure the validity of a quasi-linear analysis, we stipulate that $|\tilde{\phi}|$ is bounded and "small" (ie. that the potential energy is much smaller than the kinetic energy).

The first integral of (30) is

$$\frac{1}{2} V^2(t) + \tilde{\phi}(X(t)) = C \qquad (32)$$

where $V \equiv \dot{X}$ and C is a constant. As $|\tilde{\phi}|$ is bounded (and "small", so the particle suffers no reflections), we can infer from (32) that $V(t)$ never departs much from its initial value, ie. that diffusion in velocity space does not occur though orbit fluctuations do. We show below that this is quite reconcilable with the quasi-linear analysis.

Integrating (30) from t_o to t :-

$$V(t) = V_o + \int_{t_o}^{t} ds\, \tilde{F}(X(s)) \qquad . \qquad (33)$$

Multiplying (33) by (30) and averaging:-

$$< \dot{V}(V - V_o) > = \int_{t_o}^{t} ds < \tilde{F}(X(t)) \tilde{F}(X(s)) > \qquad (34)$$

We now define $\Delta V \equiv V - V_o$ and make the "quasi-linear" approximation of replacing $X(s)$ by $X(t) - V_o(t-s)$. Then (34) can be re-written as

$$\frac{1}{2} \frac{d}{dt} < (\Delta V)^2 > = \int_{t_o}^{t} ds < \tilde{F}(V_o(t-s)) \tilde{F}(o) > \quad , \tag{35}$$

where we have used the stationary property of $\tilde{F}$. Changing the variable of integration to $\tau \equiv t - s$ and letting $V_o(t - t_o)$ be much greater than the correlation length of $\tilde{F}(x)$, <u>if it exists</u>, we obtain:

$$\frac{1}{2} \frac{d}{dt} < (\Delta V)^2 > = \int_{o}^{\infty} d\tau < \tilde{F}(V_o\tau) \tilde{F}(o) > \quad . \tag{36}$$

This is the conventional result (STURROCK 1966, COOK 1978) which is often written in the alternative form

$$\frac{1}{2} \frac{d}{dt} < (\Delta V)^2 > = \int_{-\infty}^{\infty} dk \; Q_k \; \delta(kV_o) \quad , \tag{37}$$

where Q_k is the Fourier transform of $< \tilde{F}(x) \tilde{F}(o) >$.

(36) is analogous to (27) and (28) above and <u>appears</u> to imply that the particle diffuses in velocity space, which we have seen to be untrue. However, let us introduce the auto-correlation function of the potential, by the definition

$$C(x) \equiv < \tilde{\phi}(x+x_o) \tilde{\phi}(x_o) > \quad . \tag{38}$$

From stationarity one can readily show that

$$\left. \frac{dC}{dx} \right|_{x=o} = 0 \quad , \tag{39}$$

and

$$- \frac{d^2C}{dx^2} = < \tilde{F}(x) \tilde{F}(o) > \tag{40}$$

So (36) may be written

$$\frac{1}{2}\frac{d}{dt} < (\Delta V)^2 > = \frac{1}{V_o}\int_o^\infty dx\left(-\frac{d^2C(x)}{dx^2}\right) \quad . \tag{41}$$

$$= \frac{1}{V_o}\left\{C'(o) - C'(\infty)\right\} \quad . \tag{42}$$

We have already seen (39) that $C'(o)$ vanishes. $C'(\infty)$ vanishes by hypothesis.

So, in spite of appearances, a quasi-linear analysis correctly predicts that there is in fact no diffusion in the present case. The reader may confirm, by constructing examples based on (40), that this is because the contribution to the integral in (36) from the tail of the force correlation function exactly cancels the contribution from the central hump.

IV CONCLUSIONS

The possibility that magnetic field line wandering may account for the presence of anomalous transport in Tokamaks has received extensive discussion (CALLEN 1977, KADOMTSEV and POGUTSE 1979, LAVAL and GRESILLON 1980, RECHESTER and ROSENBLUTH 1978). It is a complex problem in plasma physics, in which the dynamics of particles and fields must receive a self-consistent treatment. Yet we believe that the arguments in section II of this paper show that the concepts of statistical geometry (or kinematics), which underpin the whole subject, have not been treated with sufficient care. An extension of our ideas to embrace particle motion in more realistic field geometry is required.

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