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AN INTEGRATOR SUITABLE FOR STATISTICAL MEASUREMENTS OF FLUCTUATIONS IN PULSED DISCHARGES

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A B S T R A C T

We describe an electronic integrator suitable for measuring the mean deviation of a random signal from its mean value and hence obtaining estimates of the r.m.s. level of the signal. We show how this device can be used to measure the correlation between two signals, and describe a modification to introduce an effective time delay of one signal so that delayed correlations and auto-correlations can be measured. We give estimates of the statistical uncertainty of the results in each case and discuss the conditions required for minimising this uncertainty.

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APPENDICES

1. INTRODUCTION

In this report we describe an electronic averaging device suitable for the measurement of the mean value of a fluctuating signal and of its mean deviation. Such a signal might represent, for instance, the random velocity of a turbulent fluid, or the electric field fluctuations in a high current gas discharge.

The device consists basically of two diode rectifiers followed by long time constant integrators; the outputs of the diodes represent the positive and negative-going parts of the signal, with respect to an adjustable zero level. The integrators are simple RC circuits enclosed in feedback loops which simultaneously make the effective time constant very long and provide a means of measuring the voltage on the capacitor, from which we obtain estimates of the time averages of the positive and negative going parts of the signal. Using the theory developed in section 2 we can then estimate the mean level of the signal with respect to the adjusted zero level, and the r.m.s. deviation from the mean. In principle, we need to know the probability distribution of the signal, but the theory shows that the results are not very sensitive to the details of the distribution, which can usually be assumed to be Gaussian without serious error.

The theory is developed in two stages. First we assume that the time averaging is carried out over an infinite time, and that the random signal is stationary in time. The values of the averages are then related only to the probability distribution and the characteristics of the rectifier. Next we estimate the errors introduced by averaging for a finite time, and using the results as estimates of the infinite-time averages.

A common requirement in the study of random signals is to measure the correlation coefficient between two such signals. To obtain this we require the average of the product of the signals. Since the sum and difference of two signals can be obtained electronically more easily than the product, it is convenient to use the 'quarter-square' method: if $x(t)$ and $y(t)$ are the two random signals, then

$$xy = \frac{1}{4}\left[\langle (x+y)^2 \rangle - \langle (x-y)^2 \rangle\right] \qquad \ldots (1.1)$$

where brackets $\langle\,\rangle$ denote infinite time averages. The correlation coefficient φ is then defined by

$$\varphi = \frac{\langle xy \rangle}{\hat{x}\,\hat{y}} \qquad \ldots (1.2)$$

where

$$\hat{x} = (\langle x^2 \rangle)^{\frac{1}{2}}, \quad \hat{y} = (\langle y^2 \rangle)^{\frac{1}{2}} \qquad \ldots (1.3)$$

and we assume that $\langle x \rangle = \langle y \rangle = 0$. In section 3 we discuss the errors in estimates of φ derived from finite time averages, and show that these depend on which, if any, of the four quantities $\langle(x + y)^2\rangle$, $\langle(x - y)^2\rangle$, $\hat{x},\hat{y}$ are measured simultaneously. The device which we have constructed contains two measuring channels, and in particular therefore we discuss the errors when pairs of signals are measured simultaneously, and compare the two possible distinct pairings of signals. We show that it is in general more efficient to measure $\langle(x + y)^2\rangle$ with $\hat{x}$, and $\langle(x - y)^2\rangle$ with $\hat{y}$, than to measure $\langle(x + y)^2\rangle$ with $\langle(x - y)^2\rangle$, and $\hat{x}$ with $\hat{y}$.

In section 4 we describe in detail the instrument which has been constructed for the measurement of fluctuations of electric and magnetic fields in the ZETA discharge (Butt et al, 1958). In this case, casual inspection shows that the amplitude of the fluctuations varies with time after the initiation of the discharge. Accordingly, the theory developed here is not applicable since the signals are not stationary. The appropriate average is then that taken over the ensemble of separate discharges at a fixed time in each. We therefore use an electronic gate to sample the random signal at a fixed time, and integrate over many discharges. This leads to a proper ensemble average if the gate is open for a time so short that we obtain only one effectively independent reading per discharge, in the sense defined by Rusbridge (1962). More commonly, however, we use a longer gate and obtain several effective readings per discharge to conserve experimental time (since we can then average over fewer discharges); this is justifiable as long as the gate is still short compared with the discharge pulse length.

The main requirements on the design of the instrument are (1) the time constant of the integrator must be very long compared with the interval between discharges (about 30 sec), (2) since the gate is open typically for only 10^{-5} of the time, any signal (due, e.g. to thermal noise, ripple at the mains frequency, etc.) entering the integrator when the gate is shut must be much less than 10^{-5} of the typical signal measured, and (3) the leakage current must be sufficiently small to produce a negligible change in the integrator output between pulses. An earlier version of this instrument was referred to by Rusbridge et al (1961); this version, however, did not satisfy these requirements and could not be used on ZETA.

Also in section 4 we describe a modification to this instrument to introduce an effective time delay into one measuring channel, so that delayed correlations or auto-correlations can be measured. The principle of this device is the following. Consider a typical randome signal $x(t)$ (Fig.1(a)), and consider also the signal $x_1(t , t_0 , \Delta t)$ defined by

$$x_1 (t , t_0 , \Delta t) = x (n \Delta t + t_0)$$

where n is such that

$$(n + 1) \Delta t > t - t_0 > n \Delta t ,$$

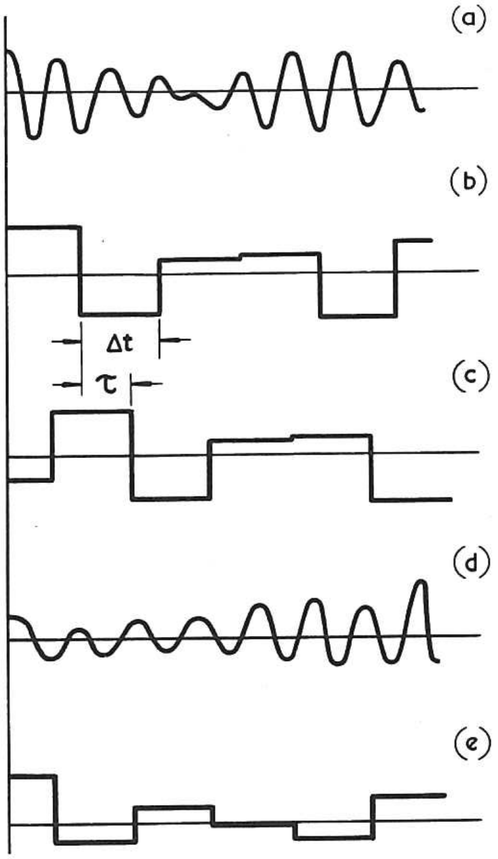

Fig. 1 (CLM-P 68)
The principle of the time-delay system. (a), (d) Input random signals $x(t)$, $y(t)$ respectively (white noise filtered to pass a band of frequencies 14-20 kc/sec). (b) The sampled signal $x_1(t, t_o, \Delta t)$. (c) The delayed version of (b), $x_{11}(t, t_o + \tau, \Delta t)$. (e) The sampled version of $y(t)$.

and t_o is an arbitrary time. This signal is shown in Fig.1(b). Since the change from x to x_1 corresponds merely to sampling x at discrete times, we have clearly

$$\langle x_1^2 \rangle = \langle x^2 \rangle$$

where, as before, brackets $\langle\,\rangle$ denote infinite time averages; and if from a second random signal $y(t)$ we obtain $y_1(t, t_o, \Delta t)$ in the same way we have

$$\langle x_1 y_1 \rangle = \langle xy \rangle$$

i.e. the correlation coefficients are unaffected. Moreover, it is clear from the results of Rusbridge (1962) that the accuracy of a finite time average is unaffected if Δt is small enough compared with the characteristic time of variation of $x(t)$.

Now suppose we repeat the process, sampling the signal $x_1(t, t_o, \Delta t)$ and obtaining the signal $x_{11}(t, t_o + \tau, \Delta t)$ shown in Fig.1(c) and defined by

$$x_{11}(t, t_o + \tau, \Delta t) = x_2(n_1 \Delta t + t_o + \tau, t_o, \Delta t)$$

where n_1 is such that

$$(n_1 + 1)\Delta t > t - t_o - \tau > n_1 \Delta t .$$

We see from Fig.1(c) that this signal is simply x_1 delayed by an amount τ (provided $\tau < \Delta t$). Finally we sample $y(t)$ once, forming $y_1(t, t_o + \tau, \Delta t)$ (Figs.1(d) and 1(e)), and measure the correlation between x_{11} and y_1 in the normal way. It is then easy to see that

$$\langle x_{11}(t, t_o + \tau, \Delta t)\, y_1(t, t_o + \tau\ \Delta t)\rangle = \langle x(t - \tau)\, y(t)\rangle$$

so that we have generated the delayed correlation that we require.

2. THEORY

In this section we shall use the symbol $x(t)$ to denote a random signal, and x to denote any instantaneous value that $x(t)$ may take. We assume $x(t)$ to be stationary in time, and define its probability distribution $f(x)$ by requiring that $f(x)dx$ is the probability that $x(t)$ takes values lying in the range x to $x + dx$.

The signal $x(t)$ is passed through a detector whose output is assumed to depend only on the instantaneous value of $x(t)$ and may therefore be denoted by $d(x)$. Then the average value of this output over all time is given by

$$\Sigma = \int_{-\infty}^{\infty} d(x)\, f(x)\, dx \ .$$

In the case of a square law detector, for example, $d(x) = x^2$ and $\Sigma = \sigma^2$, the variance of $x(t)$.

We shall assume that $f(x)$ is symmetrical about the mean value of x, which for later convenience we shall denote by $-x_o$, and we shall usually require $|x_o| \ll \sigma$. We may write $f(x) = f'(|x + x_o|)$, say, and it is convenient to change the variable to

$$x' = x + x_o \ ,$$

so that the expression for Σ becomes

$$\Sigma = \int_{-\infty}^{\infty} d(x' - x_o)\, f'(|x'|)\, dx' \ . \qquad \ldots (2.1)$$

We shall discuss particularly two forms for $d(x)$:

$$d(x) = |x| \qquad \ldots (2.2)$$

which represents an ideal full-wave rectifier, and

$$d(x) = |x| - \frac{1}{c}\left(1 - e^{-c|x|}\right) \qquad \ldots (2.3)$$

which allows a rather better representation of the behaviour of a real rectifier. We shall assume that $1/c$ is small, or more precisely $1/\sigma c \ll 1$. These functions are shown in Fig.2.

We see from the transformation above that the value $x = x_o$ can also be regarded as the point about which the diodes are biassed to rectify. In this sense, the value of x_o is controllable in our apparatus and we shall treat it as an independent variable.

Consider first the ideal rectifier given by (2.2). Substituting in (2.1) and dropping the dashes we have

$$\Sigma(x_o) = 2\int_{x_o}^{\infty} x f(x)\, dx + 2x_o \int_{0}^{x_o} f(x)\, dx \qquad (2.4)$$

or for the particular case $x_o = 0$,

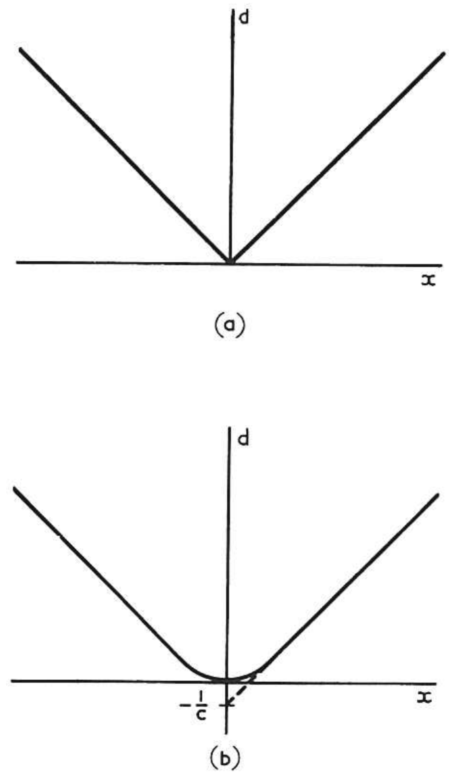

Fig. 2 (CLM-P 68)
(a) The detector function $d(x) = |x|$
(b) The detector function $d(x) = |x| - \frac{1}{c}(1 - e^{-c|x|})$

$$\Sigma(o) = 2\int_0^\infty x\,f(x)\,dx,$$

the mean deviation of x. We shall denote $\Sigma(o)$ simply by Σ. Thus for a Gaussian distribution given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\; e^{-x^2/2\sigma^2} \qquad \ldots (2.5)$$

we obtain the well known result

$$\Sigma = \sigma\sqrt{2/\pi} = 0.796\,\sigma$$

(see, e.g. Kendall and Stuart 1958).

Differentiating (2.4) twice with respect to x_o we obtain

$$\frac{d^2\Sigma(x_o)}{dx_o^2} = 2\,f(x_o) \qquad \ldots (2.6)$$

and using this we expand $\Sigma(x_o)$ for small x_o to obtain

$$\Sigma(x_o) = 2\int_0^\infty x f(x)dx + x_o^2\, f(o) + O(x_o^4) \qquad \ldots (2.7)$$

When $f(x)$ is Gaussian this becomes

$$\Sigma(x_o) \simeq \sigma\sqrt{\frac{2}{\pi}} + \frac{x_o^2}{\sigma\sqrt{2\pi}}$$

$$\simeq 0.796\,\sigma + 0.398\,\frac{x_o^2}{\sigma} \qquad \ldots (2.8)$$

In this case the integrals in 2.4 can be evaluated exactly to give

$$\Sigma(x_o) = \sigma\sqrt{\frac{2}{\pi}}\; e^{-x_o^2/2\sigma^2} + x_o \operatorname{erf}\frac{x_o}{\sigma\sqrt{2}} \qquad \ldots (2.9)$$

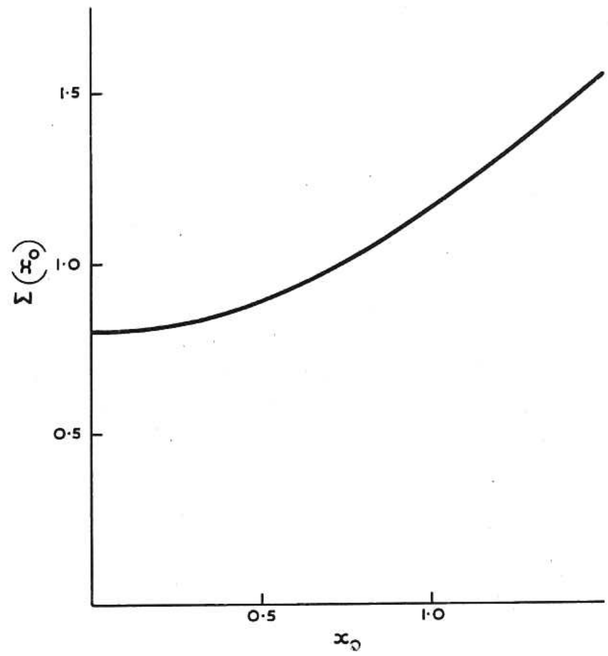

Fig. 3 (CLM-P 68)
Mean value of detector output, Σ, as a function of the bias level x_o for a Gaussian distribution of the input signal with variance $\sigma^2 = 1$.

This function is shown in Fig.3 for $\sigma = 1$. The approximate form (2.8) is accurate to within about 3% for $x_o/\sigma < 1$.

Now, recalling the brief description of our averaging device in section 1, we note that the output is in the form of two quantities I_1 and I_2, which (assuming as usual averaging over all time) are given by

$$I_1 = \int_{x_o}^\infty (x - x_o)\, f(x)\, dx \qquad \ldots (2.10)$$

$$I_2 \int_\infty^{x_o} (x_o - x)\, f(x)\, dx$$

and from these results it can easily be seen that

$$\Sigma(x_o) = I_1 + I_2 \qquad \ldots (2.11)$$
$$x_o = I_2 - I_1$$

We shall show below that to obtain an accurate measurement of σ from (2.8) we require $x_0^2 \ll \Sigma^2$; in this case the second term in (2.8) is small and we can write approximately

$$\Sigma(x_0) \approx 0.796\,\sigma + 0.398\, x_0^2 \frac{0.796}{\Sigma(x_0)} \quad ,$$

thus

$$\sigma \approx \frac{\Sigma(x_0)}{0.796} - \frac{0.398\, x_0^2}{\Sigma(x_0)} \qquad \text{... (2.12)}$$

$$\approx \frac{(I_1 + I_2)}{0.796}\left[1 - \frac{1}{\pi}\frac{(I_2 - I_1)^2}{(I_1 + I_2)^2}\right] \qquad \text{... (2.13)}$$

in terms of the actual measured quantities. Thus quite large differences between I_1 and I_2 have very little effect on the derived value of σ; for $|(I_2 - I_1)|/(I_2 + I_1) \leqslant 1/3$ or $0.5 \leqslant I_1/I_2 \leqslant 2$ the value derived for σ is altered by less than 3% by including the correction term, which may therefore often be neglected since the random errors in the determination of σ will usually be larger than this.

We have investigated the effect of distribution functions other than Gaussian. For each distribution we obtain an equation analogous to (2.8) representing the first two terms of the expansion of $\Sigma(x_0)$

$$\Sigma(x_0) \approx \alpha\sigma + \beta \frac{x_0^2}{\sigma} \quad . \qquad \text{... (2.14)}$$

The coefficients α and β are given in Table 1 for a series of distributions from exponential to square. We write the distribution function in the form

$$f(x) = \frac{1}{\sigma} f_1\left(\frac{x}{\sigma}\right)$$

and tabulate the functions $f_1(u)$.

Table 1

Distribution	$f_1(u)$	α	β
Exponential	$\frac{1}{\sqrt{2}} e^{-\lvert u\rvert/\sqrt{2}}$	0.707	0.707
Gaussian	$\frac{1}{\sqrt{2\pi}} e^{-u^2/2}$	0.796	0.398
Triangular	$\frac{1}{\sqrt{6}} \left(1 - \frac{\lvert u\rvert}{\sqrt{6}}\right), \lvert u\rvert \leqslant \sqrt{6}$	0.818	0.409
Parabolic	$\frac{3}{4\sqrt{5}} \left(1 - \frac{u^2}{5}\right), \lvert u\rvert \leqslant \sqrt{5}$	0.838	0.335
Square	$\frac{1}{2\sqrt{3}}, \lvert u\rvert \leqslant \sqrt{3}$	0.866	0.289

(f_1 is understood to vanish outside the ranges specified for u)

This sequence of distribution functions covers a range much more extreme than any which we might expect to meet in practice, yet the value of α varies by less than 12% from the value for a Gaussian distribution. In addition, we have measured the value of

α for a series of records of electric field fluctuations obtained from the ZETA discharge, obtaining a value of 0.792 ± 0.007. The absolute value of σ is seldom required with great accuracy, and in any series of measurements of the same quantity the distribution function is unlikely to change so much that the variation of α will be significant. The change of β may be larger, but as we have seen the term containing β must always by a small correction in an experiment designed to measure σ. We do not, therefore, believe that the dependence on the distribution function is a serious objection to the use of this method to obtain estimates of σ. In the remainder of this report, and where necessary in experimental work, we use the values appropriate to a Gaussian distribution.

However, we note in passing that an exact expression for σ^2 can be obtained in the form

$$\sigma^2 = 2 \int_0^\infty (\Sigma(x_o) - x_o)dx_o \qquad \text{... (2.16)}$$

This relation can be verified by substituting the value of $\Sigma(x_o)$ from equation (2.4) and changing the order of integration. Higher moments of $f(x)$ are given by similar relations. For large x_o, however, equation (2.16) involves the difference of two large quantities which in the limit are equal, and it is therefore not likely to give an accurate result in practice. An attempt to determine $f(x_o)$ itself by the use of equation (2.6) would encounter similar difficulties.

We shall now quote without detailed derivation the results obtained for the non-ideal rectifier given by equation (2.3). We obtain an expression for $\Sigma(x_o)$ expanded in powers of two small parameters x_o/σ and $1/\sigma c$ in the form

$$\frac{\Sigma(x_o)}{\sigma} \approx 2 \int_0^\infty u f_1(u)du - \frac{1}{\sigma c} + f_1(o) \frac{x_o^2}{\sigma^2} + \frac{2}{\sigma^2 c^2} \qquad \text{... (2.17)}$$

where $f_1(u)$ is defined by (2.15). For the Gaussian distribution

$$\frac{\Sigma(x_o)}{\sigma} = 0.796 - \frac{1}{\sigma c} + 0.398 \left[\frac{x_o^2}{\sigma^2} + \frac{2}{\sigma^2 c^2} \right]$$

and σ is given by an expression analogous to (2.12).

$$\sigma = \frac{\Sigma(x_o) + \frac{1}{c}}{0.796} - \frac{0.398\,(x_o^2 + \frac{2}{c^2})}{\Sigma(x_o)} . \qquad \text{... (2.18)}$$

The two measured quantities I_1 and I_2 are now given by

$$I_1 = \int_{x_o}^\infty \left[(x - x_o) - \frac{1}{c}\left(1 - e^{-c(x - x_o)}\right) \right] f(x)\,dx$$

$$I_2 = \int_{-\infty}^{x_o} \left[(x_o - x) - \frac{1}{c}\left(1 - e^{-c(x_o - x)}\right) \right] f(x)\,dx$$

and while $\Sigma(x_0)$ is still given by (2.11), it is no longer exactly correct to write

$$x_0 = I_2 - I_1 .$$

We obtain instead

$$x_0 \approx (I_2 - I_1)\ (1 + \frac{2\varphi(o)}{\sigma c})$$

$$\approx (I_2 - I_1)\ (1 + \frac{2}{\pi}\frac{1}{\Sigma c}) ,$$

where we have substituted the first approximation for σ. However, since x_0 itself occurs only in a correction term in equation (2.18) its own correction term can be ignored and we finally obtain

$$\sigma = \frac{I_1 + I_2 + \frac{1}{c}}{0.796} - \frac{0.398\ (I_2 - I_1)^2 + \frac{2}{c^2}}{I_1 + I_2} \qquad \text{... (2.19)}$$

We now have to consider the effects of averaging over a finite time. We use the results obtained from such averages as estimates of the averages over all time. The estimates obtained by averaging over separate non-overlapping periods of time are independent and randomly distributed, and provided the time period of each average is great enough the distribution will be normal with mean value approximately equal to the infinite time average. 'Great enough' here means that the effective number N of independent readings in each averaging period, as defined by Rusbridge (1962), must be $\geqslant 100$ (cf. Kendall and Stuart 1958). We shall calculate an estimate of the probable error of a single average considered as an estimate of the infinite time average. We consider first the ideal rectifier for $x_0 = 0$. The variance Ω^2 of the output of the detector is given by

$$\Omega^2 = \int_{-\infty}^{\infty} \left[d(x)\right]^2 f(x)dx - \left[\int_{-\infty}^{\infty} d(x)\ f\ (x)dx\right]^2$$

and since

$$[d(x)]^2 = (|x|)^2 = x^2$$

we see that

$$\Omega^2 = \sigma^2 - \Sigma^2 \qquad \text{... (2.20)}$$

For a Gaussian distribution this gives

$$\Omega = 0.605\,\sigma \qquad \text{... (2.21)}$$

The p.e. of the mean of N effectively independent readings is then given by $0.6745\ \Omega/\sqrt{N}$; The ratio of the p.e. to the mean is $0.6745\ \varepsilon/\sqrt{N}$ where

$$\varepsilon = \frac{\Omega}{\Sigma} = 0.757 \qquad \text{... (2.22)}$$

so that for $N = 100$ the p.e. is about 5%.

The corresponding result for small x_0 is derived in Appendix I. It can be written in the form

$$\varepsilon \approx \varepsilon_0\ (1 + \Upsilon\ \frac{x_0^2}{\Sigma^2}) \qquad \text{... (2.23)}$$

where for a Gaussian distribution $\Upsilon = 0.635$, and $\varepsilon_0 = 0.757$ is the value of ε given by (2.22) for $x_0 = 0$. Recalling that $x_0 = I_2 - I_1$, $\Sigma = I_1 + I_2$, we find that $(\varepsilon - \varepsilon_0)$ is less than 10% of ε_0 as long as

$$0.43 < {}^{I_1}/I_2 < 2.3 \qquad \dots (2.24)$$

This condition is easy to satisfy.

For the detector defined by equation (2.3) we shall consider only the case $x_0 = 0$. The result, also derived in Appendix I, is

$$\varepsilon = \varepsilon_0 \left(1 + \frac{2\Upsilon}{\Sigma^2 c^2}\right) \qquad \dots (2.25)$$

where Υ is the same quantity that appears in (2.23). Thus $(\varepsilon - \varepsilon_0)$ is less than 10% to ε_0 as long as

$$\Sigma c > \sqrt{20\Upsilon} = 3.6$$

or if $I_1 \sim I_2$, as long as

$$I_1 > 1.8 \frac{1}{c} . \qquad \dots (2.26)$$

Again this should be very easy to satisfy.

Finally, we summarize the limitations on the values of I_1 and I_2 which have been derived above:

(1) The correction term in equation (2.13) is less than 3% as long as

$$0.5 \leqslant I_1/I_2 \leqslant 2 .$$

(2) For $I_1 = I_2$ the corresponding correction term in equation (2.19) is less than 3% as long as

$$I_1 > \frac{3}{\sqrt{2}} \frac{1}{c} = \frac{2.1}{c} .$$

(3) The error estimate given by (2.23) increases by less than 10% as long as

$$0.43 \leqslant {}^{I_1}/I_2 \leqslant 2.3 .$$

(4) For $I_1 = I_2$ the error estimate increases by less than 10% as long as

$$I_1 > \frac{1.8}{c} .$$

3. THE ACCURACY OF MEASUREMENTS OF THE CORRELATION COEFFICIENT

Suppose the random signals to be correlated are denoted by $x(t)$ and $y(t)$, and write

$$\begin{aligned} \pi(t) &= x(t) + y(t) \\ \mu(t) &= x(t) - y(t) . \end{aligned} \qquad \dots (3.1)$$

We shall assume that $x(t)$ and $y(t)$ have zero mean, and then so also will $\pi(t)$ and $\mu(t)$. As in section 1 we define

$$\hat{x} = [\langle x^2(t) \rangle]^{\frac{1}{2}} \qquad \dots (3.2)$$

with similar definitions for $\hat{y}, \hat{\pi}, \hat{\mu}$. Then from equations (1.1) and (1.2) the correlation coefficient φ between $x(t)$ and $y(t)$ is given by

$$\varphi = \frac{\hat{\pi}^2 - \hat{\mu}^2}{4\,\hat{x}\,\hat{y}} \,. \qquad \ldots (3.3)$$

In practice we make estimates of the quantities x, y, π, μ by averaging over a finite time τ, and Rusbridge (1962) has shown that this is equivalent to making a certain number n of independent measurements and averaging over these, where n is a function of τ. Let us suppose therefore that we have made n independent measurements of $x(t)$ and denote these by x_i, $i = 1 \ldots n$. The operation of the integrator gives an estimate $\hat{x}_n$ of $\hat{x}$ which is approximately equivalent to

$$\hat{x}_n = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{n} \sum_i |x_i| \,.$$

It is however, mathematically simpler to discuss first the case of a true square-law detector for which

$$\hat{x}_n = \left[\frac{1}{n} \sum_i x_i^2 \right]^{1/2}$$

and afterwards to give the modifications required for the case of the rectifier which are numerically small in all cases. Now $\hat{x}_n$ is not quite an unbiassed estimate of $\hat{x}$; it can be shown that the mean of $\hat{x}_n$, taken over all independent samples of size n, is given approximately for large n by

$$\hat{x}_n = \hat{x}(1 - \frac{1}{4n})$$

but since we deal in all cases with values of $n > 100$ the correction is negligible, and if we write

$$\delta\hat{x}_n = \hat{x}_n - \hat{x}$$

we may assume $\delta\hat{x}_n = 0$, while the variance is given by the well-known formula (Kendall and Stuart 1958)

$$\delta\hat{x}_n^2 = \frac{\hat{x}^2}{2n} \,. \qquad \ldots (3.4)$$

From our estimates of $\hat{x}$ etc, we obtain an estimate of F_n of φ by

$$F_n = \frac{\hat{\pi}_n^2 - \hat{\mu}_n^2}{4\hat{x}_n\;\hat{y}_n}$$

and we write $\delta F_n = F_n - \varphi$. We suppose, as is always the case in practice, that n is sufficiently large that $\delta\hat{x}_n \ll \hat{x}_n$ so that in expressions for δF_n and δF_n^2 we may replace $\hat{x}_n$ by $\hat{x}$, F_n by φ, etc. Then

$$\delta F_n \simeq \frac{1}{4\,\hat{x}\hat{y}} \left[2\,(\hat{\pi}\;\delta\hat{\pi}_n - \hat{\mu}\,\delta\,\hat{\mu}_n) - \frac{\hat{\pi}^2 - \hat{\mu}^2}{\hat{x}\hat{y}}\,(\hat{y}\;\delta\hat{x}_n + \hat{x}\delta\hat{y}_n) \right]$$

$$\simeq \frac{1}{\hat{x}\;\hat{y}} \left[\frac{\hat{\pi}\;\delta\hat{\pi}_n - \hat{\mu}\;\delta\hat{\mu}_n}{2} - \varphi\;(\hat{y}\;\delta\hat{x}_n + \hat{x}\;\delta\hat{y}_n) \right]$$

and

$$\langle \delta F_n^2 \rangle = \frac{1}{\hat{x}^2 \hat{y}^2} \left\{ \frac{1}{4} \left[\hat{\pi}^2 \langle \delta\hat{\pi}_n^2 \rangle - 2\hat{\pi}\hat{\mu} \langle \delta\hat{\pi}_n \, \delta\hat{\mu}_n \rangle + \hat{\mu}^2 \langle \delta\hat{\mu}_n^2 \rangle \right] \right.$$

$$+ \varphi^2 \left[\hat{y}^2 \langle \delta\hat{x}_n^2 \rangle + 2\,\hat{x}\hat{y} \langle \delta\hat{x}_n \, \delta\hat{y}_n \rangle + \hat{x}^2 \langle \delta\hat{y}_n^2 \rangle \right]$$

$$- \varphi \left[\hat{y}\hat{\pi} \langle \delta\hat{x}_n \, \delta\hat{\pi}_n \rangle - \hat{y}\hat{\mu} \langle \delta\hat{x}_n \, \delta\hat{\mu}_n \rangle + \hat{x}\hat{\pi} \langle \delta\hat{y}_n \, \delta\hat{\pi}_n \rangle \right.$$

$$\left. \left. - \hat{x}\,\hat{\mu} \langle \delta\hat{\mu}_n \, \delta\hat{y}_n \rangle \right] \right\} . \quad \ldots (3.6)$$

Within our approximation $\langle \delta F_n \rangle = 0$, i.e. F_n is an unbiassed estimate of φ.

The details of the evaluation of equation (3.6) are given in Appendix II. Here we simply quote the results. We distinguish four cases differing practically in the method of sampling and mathematically in the selection of 'cross-terms' to be retained and evaluated in equation (3.5).

<u>Case 1:</u> The quantities $\hat{x}_n$, $\hat{y}_n$, etc. are obtained from independent samples of n measurements. Then all the cross-terms disappear and we obtain

$$\langle \delta F_n^2 \rangle = \frac{1}{2n} \left[\frac{1}{2} \left(\frac{\hat{x}^2}{\hat{y}^2} + \frac{\hat{y}^2}{\hat{x}^2} \right) + 1 + 4\,\varphi^2 \right] . \quad \ldots (3.7)$$

The minimum value is given by $\hat{x} = \hat{y}$ and is

$$\langle \delta F_n^2 \rangle = \frac{1}{n} (1 + 2\varphi^2) . \quad \ldots (3.8)$$

<u>Case 2(a):</u> The quantities $\hat{x}_n$ and $\hat{y}_n$ are obtained from averages over the same period of time and are therefore in general correlated. $\hat{\pi}_n$ and $\hat{\mu}_n$ are similarly paired. Then

$$\langle \delta F_n^2 \rangle = \frac{1}{n} (1 + \varphi^2)^2 . \quad \ldots (3.9)$$

<u>Case 2(b):</u> Now we measure $\hat{x}_n$ and $\hat{\pi}_n$ simultaneously, followed by the other pair $\hat{y}_n$ and $\hat{\mu}_n$. Then

$$\langle \delta F_n^2 \rangle = \frac{1}{2n} \left[\frac{1}{2} \left(\frac{\hat{x}^2}{\hat{y}^2} + \frac{\hat{y}^2}{\hat{x}^2} \right) + 1 - \left(\frac{\hat{x}}{\hat{y}} - \frac{\hat{y}}{\hat{x}} \right) \varphi (1 - \varphi^2) \right] . \quad \ldots (3.10)$$

When $\hat{y} = \hat{x}$ this reduces to

$$\langle \delta F_n^2 \rangle = \frac{1}{n} . \quad \ldots (3.11)$$

By minimising $\langle \delta F_n^2 \rangle$ with respect to $\hat{x}/\hat{y}$ for each value of φ we can in principle improve on (3.11), but the absolute minimum value of $\langle \delta F_n^2 \rangle$, attained for

$$\varphi = \frac{1}{\sqrt{3}} , \frac{\hat{x}}{\hat{y}} = 1.21 \quad \left(\text{or} \quad \varphi = -\frac{1}{\sqrt{3}} , \frac{\hat{x}}{\hat{y}} = 0.83\right) \quad \text{is}$$

$$\langle \delta F_n^2 \rangle = \frac{26}{27} \cdot \frac{1}{n} \quad \ldots (3.12)$$

so the improvement is negligible.

We can also evaluate the limits within which $\hat{x}/\hat{y}$ must lie if $\langle \delta F_n^2 \rangle$ is not to exceed the value given by equation (3.11) by more than 10%; for $\varphi = 0$ or ± 1 they are

$$0.74 < \hat{x}/\hat{y} < 1.36 \qquad \ldots (3.13)$$

Case 3: All four quantities $\hat{x}_n, \hat{y}_n, \hat{\pi}_n, \hat{\mu}_n$, are measured from averages over the same period of time and are correlated, so that all the cross terms must be retained in equation (3.6). This gives the result usually quoted (Kendall and Stuart 1958)

$$\langle \delta F_n^2 \rangle = \frac{1}{n} (1 - \varphi^2)^2 \quad . \qquad \ldots (3.14)$$

With only two measurement channels on our device we are restricted to the first three cases, and it is clear that in general Case 2(b) represents the most efficient arrangement provided that the ratio $\hat{x}/\hat{y}$ is sufficiently under our control to be kept within certain limits which depend on φ. For example, with our arrangement $\hat{x}/\hat{y}$ can be kept within limits $0.89 < \hat{x}/\hat{y} < 1.12$ by suitably choosing attenuator settings, and if we choose, for example, $\hat{x}/\hat{y} = 1.12$ then Case 2(b) is more efficient for all φ outside the range $0.05 > \varphi > -0.1$. Even within this range, although case 2(a) is theoretically more efficient, the difference is negligible and it is preferable to use the arrangement of case 2(b), because the signals $\pi(t)$ and $\mu(t)$ can be measured on the same channel, so that the effect of any systematic differences between the channels is minimised.

The case of the full-wave rectifier is also considered in Appendix II. In general the results cannot be given analytically, but approximate forms valid for $\varphi < 0.6$ can be given for cases 1 and 2(a):

Case 1:

$$\langle \delta F_n^2 \rangle \approx (\pi - 2) \frac{1}{n} (1 + 2\varphi^2) \qquad \ldots (3.15)$$

Case 2(a):

$$\langle \delta F_n^2 \rangle \approx \frac{1}{2n} \left[\frac{\pi - 3}{2} \left(\frac{\hat{x}^2}{\hat{y}^2} + \frac{\hat{y}^2}{\hat{x}^2} \right) + (\pi - 1) + 4 (\pi - 2) \varphi^2 + 2 \varphi^4 \right] \qquad \ldots (3.16)$$

Note that replacing π by 3 reduces these to (3.8) and (3.9) respectively, and provided $\hat{x}/\hat{y} \approx 1$ the effect is to increase $\langle \delta F_n^2 \rangle$ by about 14%.

We have evaluated $\langle \delta F_n^2 \rangle$ for Case 2(b) numerically and find essentially the same result; over the whole range of φ, $\langle \delta F_n^2 \rangle$ differs from $(\pi - 2)/n$ by less than 3%, for $\hat{x}/\hat{y} = 1$. The result that the arrangement of case 2(b) is in general more efficient therefore still holds.

4. CONSTRUCTION AND OPERATION OF THE APPARATUS

There are two basic systems in the apparatus, the signal system and the control system. Block diagrams of these systems are shown in Figs.8 and 9 respectively. Most of the blocks represent conventional or straightforward circuits which do not need individual description, but the long-period integrator (l.p.i.) itself and the time delay unit must be described in some detail before we discuss the functioning of the two systems.

(a) The Long-period Integrator

The circuit diagram is shown in Fig.4. The signal to be measured charges the capacitor C1 through the diode V4, so that only the negative going part of the signal is effective.

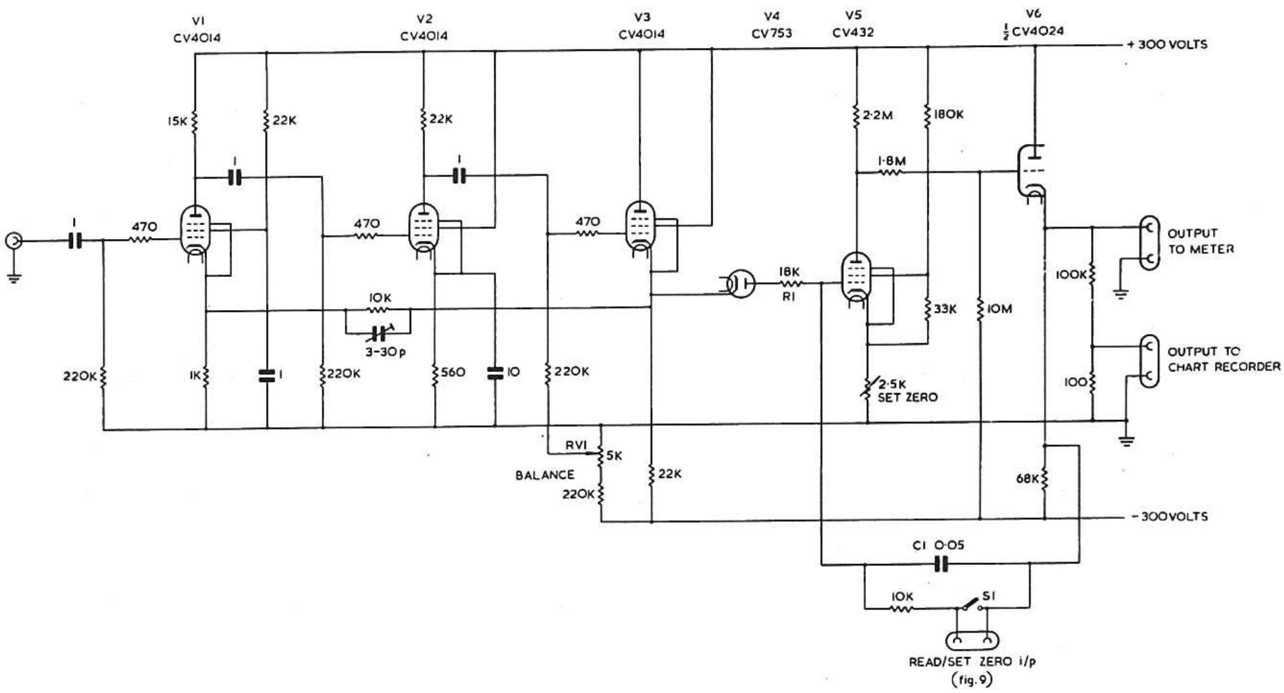

Fig. 4 (CLM-P 68)
Circuit diagram of the long-period integrator.

The integrating capacitor C1 is enclosed in a feedback loop formed by V5 and V6; the grid of the electrometer valve V5 is a virtual earth and the voltage across the diode remains constant as the charge on the capacitor increases. This loop also provides means of measuring the capacitor voltage.

There are two important design criteria. The device is required to integrate signals gated for short periods occurring as rarely as once in half a minute over a period of up to 30 minutes, so that the leakage time constant of the capacitor must be several hours. The gate periods may be as short as 50 μsec, but are more usually set at 500 μsec, and for our applications we required that the capacitor should be fully charged by an input sinusoidal signal of 10 volts peak-to-peak applied over a total time of 5 millisec, i.e. 10 pulses of 500 μsec, or 100 of 50 μsec.

It is desirable for convenience that the diode should have a linear characteristic, although as we showed for a specific case in section 2 small departures from linearity can be tolerated and corrected for if necessary by approximating the characteristic by some simple analytic form. It is more important that the diode have a high reverse resistance and low leakage.

The electrometer valve V5 is a CV432 for which the reverse grid current is about 10^{-12} amps, and C1 is a polystyrene capacitor of high leakage resistance. With these components the principal source of leakage was in fact the diode reverse current, but this could be balanced out by adjusting the cathode potential of V5 by RV1. When the balance is correct the charge on C1 remains steady within a few percent over several hours. With this correct setting, however, the diode characteristic would be similar to that shown in Fig.2(b), with the linear part effectively biassed about 1 volt positive. We therefore add a pedestal to the signal during the gate period to cancel out this bias. Besides reducing the leakage current this arrangement has a further advantage, in that in the long intervals between pulses small signals due to amplifier noise, ripple at the mains frequency, etc, are completely excluded from the integrator, i.e. the system of biassed diode and pedestal forms an additional gate. This is necessary because the ratio of the period for which the gate is open to that for which it is shut may be as small as 1/50,000, so that a continuous signal entering the integrator of even a few millivolts would produce an effect comparable to that of the signal to be measured.

The equipment preceding the integrator is capable of handling signals up to a maximum size of the order of 10 volts, and if we assume that the input signal is Gaussian, the r.m.s. level must be less than 4 volts if this maximum size is to be exceeded less than 1% of the time. Since the cut-off of the diodes occurs over a range of about 1 volt we could not neglect the curvature of the diode characteristic if we fed the signal straight into the diode. From the discussion in section 2 we know the form of the corrections, but it would clearly be more convenient to be able to assume a linear characteristic at least as a first approximation. We have found it possible to do this by inserting a feed-back amplifier with a gain of 10, so that the signal at the diode now has an r.m.s. level of up to 40 volts. The values V1 - V3 form this amplifier; V3 is a cathode follower which also drives the integrator.

The frequency response of the amplifier was deliberately limited to about 1 Mc/sec. At higher frequencies we encounter two limitations on the rate at which the integrating capacitor can be charged:

(1) The charging rate is limited by the rate at which the cathode potential of V3 can change, which in turn depends on the stray capacity and the value of R1 (cf. the situation in counting rate meters, Millman and Taub, 1950).

(2) It is also limited by the stray capacity of the anode of the diode which takes a finite time to discharge into C1 through R1. As a consequence the diode may cut-off prematurely if the cathode potential rises too fast. Both these effects can be reduced and the frequency response increased by reducing the charging resistor R1, but when this is done the overall characteristic tends to become less linear. Since we intended to use the instrument chiefly for the investigation of phenomena in the frequency range 10 - 100 kc/sec the limit of 1 Mc/sec is quite adequate. The overall frequency response curve is shown in Fig.5. We shall show below how this frequency response can be improved.

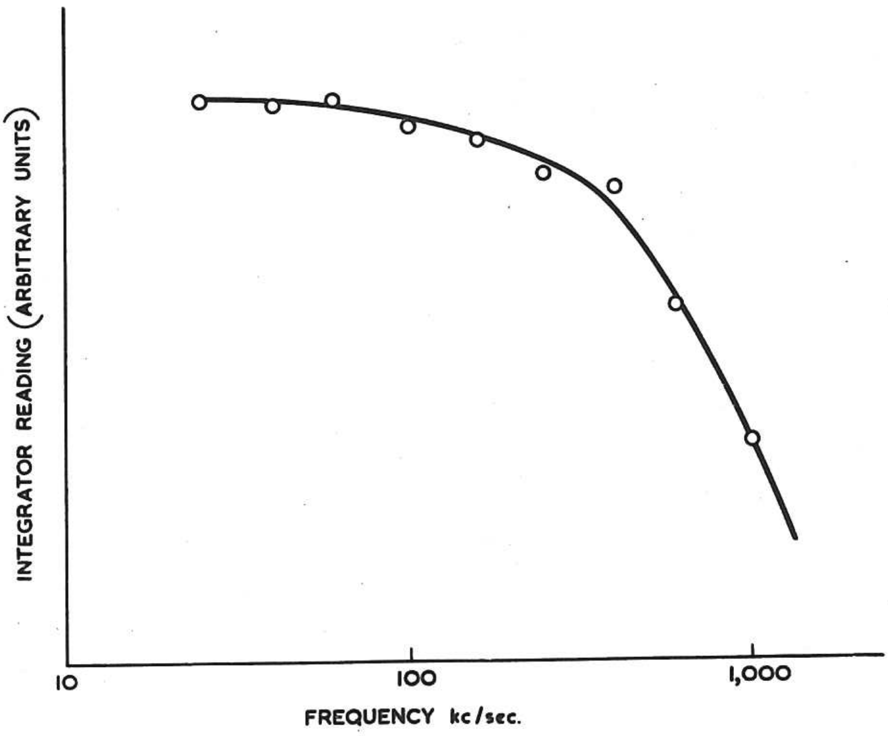

Fig. 5 Overall frequency response. (CLM-P 68)

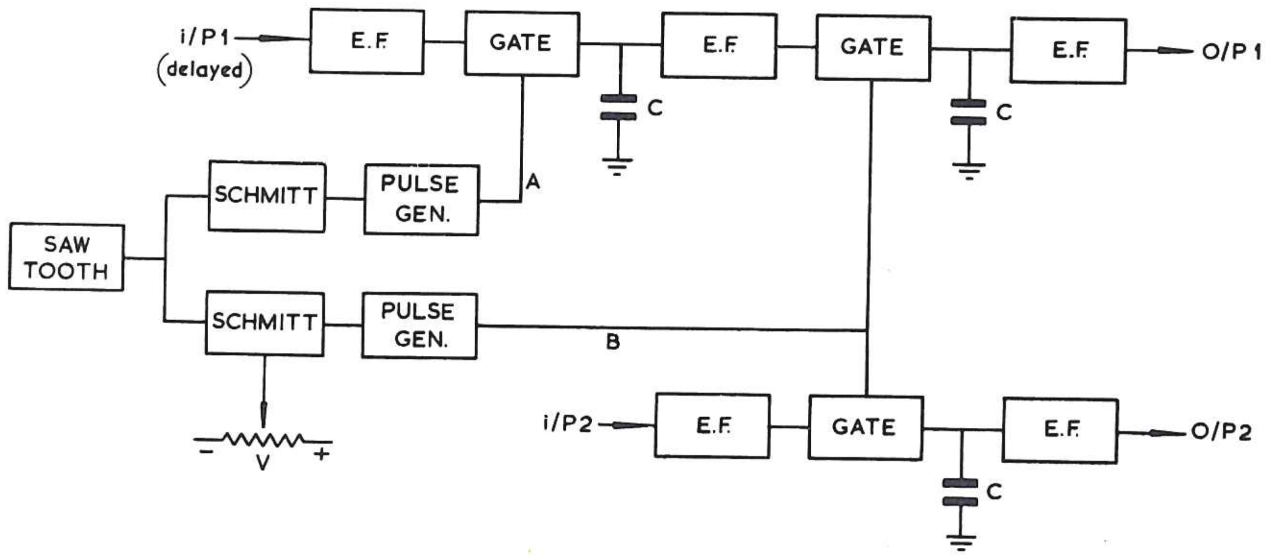

Fig. 6 Block diagram of time-delay unit. (CLM-P68)

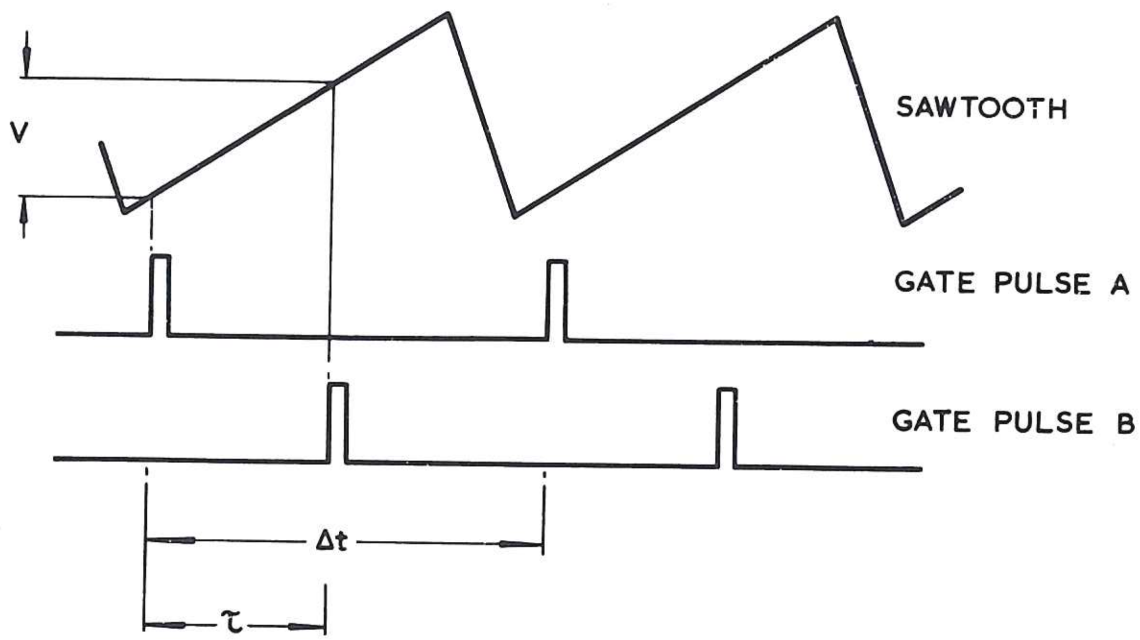

Fig. 7 Time relations of sawtooth and gate pulses (CLM-P68)

(b) <u>The Time-delay Unit</u>

This unit realises the method of obtaining a time-delay described in section 1 by generating the signals shown in Fig.1. At time $t = 0$ the first input, representing the signal $x(t)$, is sampled and its amplitude stored as a voltage on a capacitor. After a variable time delay τ, the second input and the stored version of the first input are sampled. The process is repetitive with frequency $1/\Delta t$ where we require $\Delta t > \tau$. The outputs from the second sampling are the signals to be correlated.

Fig.6 shows the arrangement of the system and Fig.7 illustrates the timing and gating waveforms. The time delay between the gating pulses A and B can be adjusted from zero to about 60% of Δt by adjustment of the Schmitt trigger voltage V. When each gate is open the signals charge the capacitors C to the signal voltage through an emitter follower with output impedance less than 100 Ω. The capacitor voltage is sensed by a second emitter follower with input impedance greater than 1 MΩ. The time constant of the capacitor with the gate shut is about 50 millisec, and this determines the maximum storage time. The gate pulse is 1 μsec long. The repetition time Δt is variable from about 5 μsec to about 250 μsec., which covers the range of maximum delays required for our experiments.

The sawtooth sweep is just over 2 volts peak-to-peak amplitude, and the Schmitt discriminators are adjustable from −1 V to +1 V. The emitter followers, gates and amplifiers are designed to handle signals from −1 to +1 V, and the entire unit runs off a +5 V supply.

The time required to charge the storage capacitors, which are 0.01 μF, when the gate is opened, limits the frequency response of the original unit to about 500 kc/sec. This could be improved by reducing the size of the storage capacitor. This would of course reduce the maximum storage time. It would, however, provide a means for increasing the effective overall frequency response of the integrator system, since the signal after sampling and storing, though now a low frequency signal which can be measured by the integrator, contains information describing the original high frequency signal. The obvious cost is that not all of the information can be transformed in this way, so that there is a loss of statistical accuracy.

(c) The Signal System

The signal system (Fig.8) consists of two channels, identical apart from the time-delay unit. In each, the entering signal, which is assumed to be balanced, is filtered to remove any quasi-d.c. level (due, for example, to pick-up) and, if required, to isolate a particular frequency band. It is then amplified and transferred by a cathode follower to a low impedance logarithmic stepped attenuator which covers a range of 10:1 in steps of 1.25 : 1; by providing such narrow steps we can easily satisfy conditions such as (3.13). At this stage each signal should have an r.m.s. level of about 0.2 volt. If the time-delay unit is used it is inserted here; otherwise the next stage is a mixing box and two standard adding amplifiers (Butt and Gillespie, 1960, Gillespie, 1961). If the input signals in the two channels are denoted by x(t) and y(t), the following pairs of output signals can be selected by a switch at the mixing box:

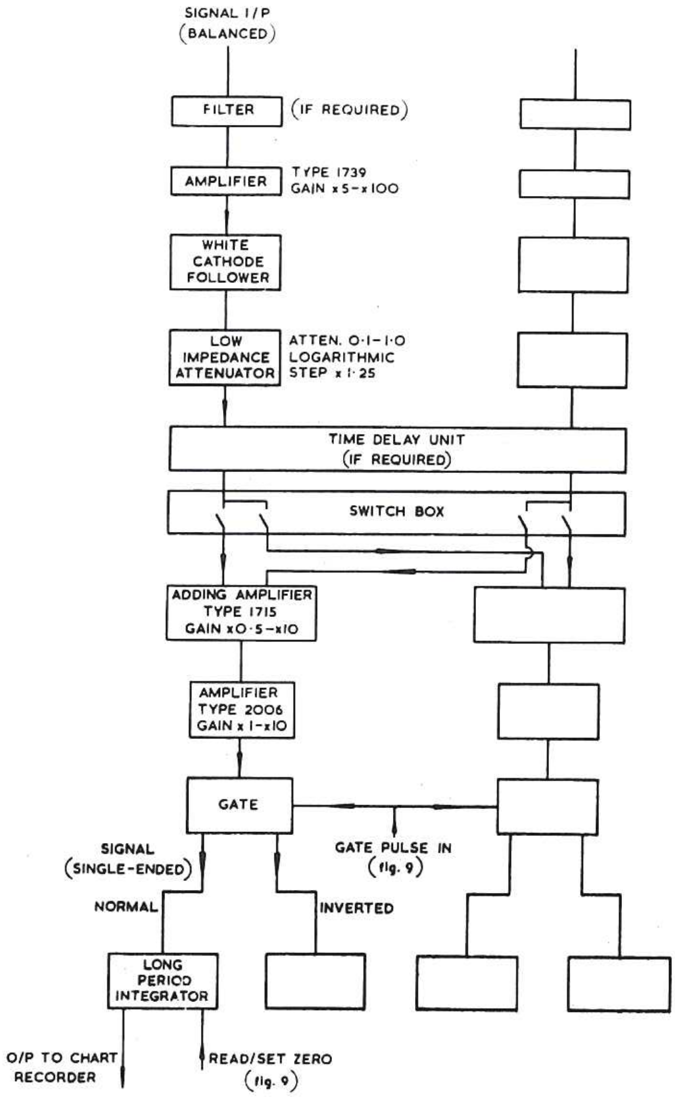

Fig. 8 Block diagram of signal system. (CLM-P 68)

Position	Output 1	Output 2
1	x(t)	y(t)
2	x(t) + y(t)	x(t) - y(t)
3	x(t) + y(t)	x(t)
4	x(t) - y(t)	y(t)

By selecting successively positions 1 and 2 we realise Case 2(a) of section 3, while Case 2(b) corresponds to positions 3 and 4.

The outputs are further amplified and passed through six-diode gates (Millman and Taub, 1956, p.445). At this stage the signals are still balanced, but during the open period of the gates an adjustable pedestal of height about -0.1 V is added to each side of the signal. When this pedestal is amplified in the l.p.i. it suffices to cancel the bias on the diode mentioned above. After the gate each side of each balanced signal is taken to a l.p.i., so that a pair of l.p.i's measures the positive and negative going halves of each signal. The outputs of the l.p.i's are recorded on a chart recorder at the end of a run.

The overall gain in the signal system can be up to 10^4 (excluding the amplifiers in the l.p.i's); the sensitivity is such that signals as small as 0.1 millivolt r.m.s. can easily be detected and clearly separated from amplifier noise. The limiting factor in the present apparatus is in fact 50 c/s ripple rather than noise, and special precautions have to be taken to reduce this ripple by inserting suitable filters at several points in the circuit.

(d) The Control System

The basic function of the control system (Fig.9) is to provide the gate pulse to the six diode gates, at the time and for the duration required. This basic function is provided by the delay and gate generator units; the gate generator is a standard phantastron circuit producing a negative going square pulse of height 20V, whose length can be varied from 35 μsec to 2.8 msec.

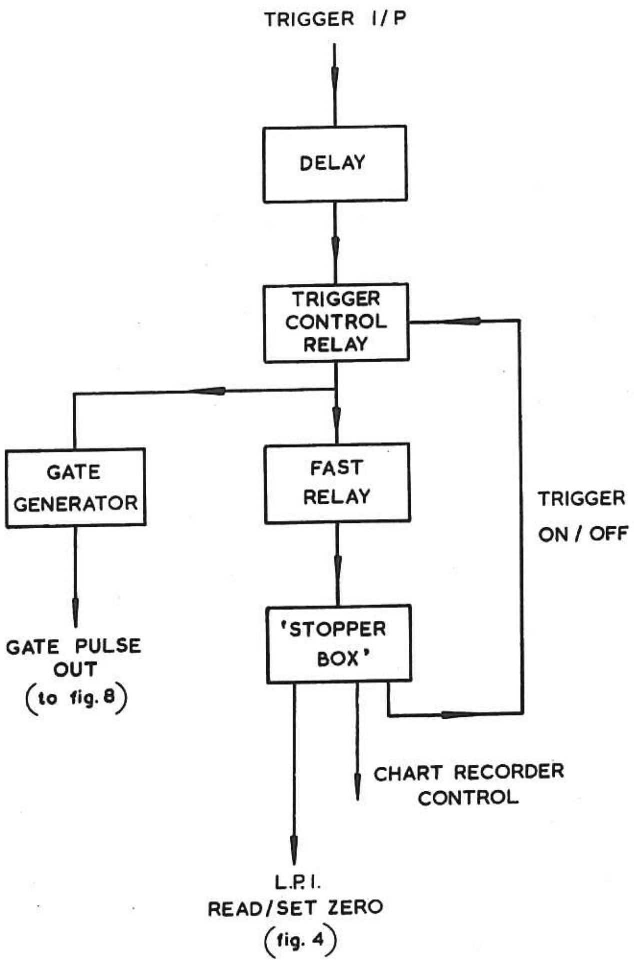

Fig. 9 Block diagram of control system. (CLM-P 68)

The delay unit also operates a fast relay. The output pulse from this relay operates a uniselector in the 'stopper box', which terminates the measurement after a preselected number of pulses. When this number is reached further trigger pulses are interrupted, the uniselector resets itself to zero, and a pulse from the 'stopper box' starts the chart recorder. This moves on one inch of chart and is successively connected to the outputs of the four l.p.i's. When this is complete, the final operation of the stopper box is to reset all the l.p.i's to zero by closing a relay in parallel with the switch S1 (Fig.4). A measurement is started by a push-button which opens this relay and closes the trigger control relay. Further push-button controls are provided to stop a measurement before the preselected number of pulses, either with or without operation of the chart recorder.

(e) Calibration

Calibration is conveniently carried out using a sinusoidal signal of known amplitude, and measuring the output of the integrators as a function of amplitude for a fixed number of pulses at fixed gate width. A pulse generator provides a source of trigger pulses at a rate of about 1 per sec. A typical calibration curve is shown in Fig.10, for 20 pulses and a gate width of 500 μsec.

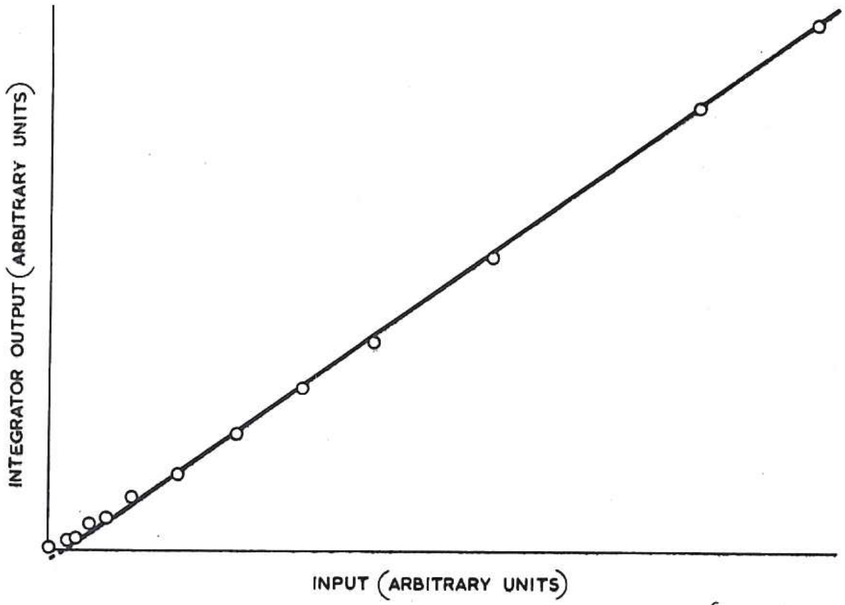

Fig. 10 (CLM-P 68)
Overall calibration curve for long period integrator. Sinusoidal input

For absolute calibrations we have to take account of the distribution of the signal as shown in section 2. For a sinusoidal signal $x(t)$ with amplitude x_0 this is of the form

$$f(x) = \frac{1}{\pi\sqrt{x_0^2 - x^2}}$$

and is very different from a Gaussian. For this distribution

$$\Sigma = 2\int_0^{x_0} \frac{x}{\pi\sqrt{x_0^2 - x^2}}\, dx = \frac{2x_0}{\pi}$$

Thus for a Gaussian distribution, the value of σ giving the same value of Σ as the sinusoidal signal is

$$\sigma = \sqrt{\frac{2}{\pi}}\, x_0 \quad .$$

The calibration curve is approximately linear over most of the range; for small signals however it can be represented by a form such as 2.3, but the constant 1/c can be made small by adjusting the pedestal height, which has the effect of moving the whole curve parallel to the absicissa.

5. CONCLUSION

The apparatus described here has been operated successfully for many experiments on fluctuations of magnetic and electric fields in the ZETA discharge (Butt et al, 1958). We have compared it with the only alternative method available to us, that of photographic recording of the fluctuating signals. In principle this is preferable because it retains more information per discharge - in general less than half the number of discharges is required to give accuracy comparable to that obtained with the integrator. In practice however we have found this advantage to be overwhelmed by the convenience of having the data presented immediately, in a form which enables at least a preliminary estimate of the experimental results to be obtained as the experiment proceeds. This allows a degree of control over the course of the experiment which cannot possibly be obtained with the photographic method when the results are not known until after the experiment because of the time required for analysis. We have usually found in this case that either the experiment is incomplete and must be repeated, or the results contain a large amount of redundant material.

Since the apparatus was constructed squaring and multiplying circuits have become available and could in principle be used in conjunction with our integrator to measure either the r.m.s. level of a signal or the mean product of two signals directly. This would give a slight improvement in statistical accuracy and remove the possibility of systematic error which arises if the distributions are not Gaussian. At the same time it would increase the complexity of the device (note that the diode cannot be eliminated from the integrator even if the functional non-linear device is elsewhere in the circuit) and remove a valuable feature of the present apparatus, that it contains a check of the quasi-d.c. level of a signal which might arise from electrical pick-up even after filtering. If such a d.c. level should exist, the outputs of the two integrators measuring a given signal will differ systematically, as shown in section 2. Accordingly we have not attempted to modify our original scheme.

6. ACKNOWLEDGEMENTS

We wish to thank Mr L.R. Jenkin who designed and constructed the time delay unit, and Mr G.L. Godfrey for the loan of the gate units developed by him.

7. REFERENCES

Butt, E.P. et al. Proc. 2nd International Conference on Peaceful uses of Atomic Energy, Geneva, vol.32, 42, 1958.

Butt, E.P. and Gillespie, A.B. Proc. Fifth International Instruments and Measurements Conference, Stockholm, 1960, p.938.

Gillespie, A.B. Joint Nuclear Instrumentation Symposium, Rayleigh, N.C., 1961, paper 3-3.

Kendall, M.G. and Stuart, A. The advanced theory of statistics, vol.1, London, Charles Griffin and Co. Ltd., 1958.

Laning, J.H. and Battin, R.H. Random processes in automatic control, New York, McGraw-Hill, 1956.

Millman, J. and Taub, H. Pulse and Digital Circuits. New York, McGraw-Hill, 1956.

Rice, S.O. Bell Syst. Techn. J., 24, 46, 1945.

Rusbridge, M.G., Lees, D.J. and Saunders, P.A.H. I.A.E.A. Conference on Plasma Physics and Controlled Nuclear Fusion, Salzburg, paper CN-10/58, 1961.

Rusbridge, M.G. The statistical accuracy of measurements of fluctuating quantities in pulsed devices. London, H.M.S.O. 1962. CLM -R 22.

APPENDIX I

Expression (2.22) gives the ratio of the p.e. of an estimate of Σ to the mean value, Σ, in the form

$$\frac{0.6745\ \varepsilon}{\sqrt{N}} \qquad \ldots \text{(AI.1)}$$

where

$$\varepsilon = \frac{\Omega}{\Sigma} \ .$$

We shall call (AI.1) the fractional error in Σ. When $\sigma \propto \Sigma$, as in the case of the ideal rectifier with $x_o = o$, the same expression gives the fractional error of the estimate of σ derived from the initial value of Σ. In the general case, the fractional error of σ is given by (AI.1) with

$$\varepsilon = \frac{\Omega}{\sigma \frac{d\Sigma}{d\sigma}} \qquad \ldots \text{(AI.2)}$$

(we assume of course that N is large and the actual error small).

When $x_o \neq o$, equation (2.20) becomes

$$\Omega^2 = \sigma^2 + x_o^2 - [\Sigma(x_o)]^2 \ . \qquad \ldots \text{(AI.3)}$$

We now substitute the general expression for $\Sigma(x_o)$ given by equation (2.14) for small x_o, in (AI.3), and retaining terms of order x_o^2 obtain

$$[\Sigma(x_o)]^2 = \alpha^2\sigma^2 + 2\,\alpha\beta x_o^2$$

$$\Omega^2 = \sigma^2(1 - \alpha^2) + x_o^2(1 - 2\,\alpha\beta)$$

whence

$$\Omega = \alpha\sqrt{1 - \alpha^2}\left[1 + \frac{1}{2}\,\frac{x_o^2}{\sigma^2}\,\frac{1 - 2\alpha\beta}{1 - \alpha^2}\right] .$$

Also

$$\sigma\,\frac{d\Sigma}{d\sigma} = \alpha\sigma - \beta\,\frac{x_o^2}{\sigma}$$

$$\frac{1}{\sigma\frac{d\Sigma}{d\sigma}} \approx \frac{1}{\alpha\sigma}\ \left(1 + \frac{\beta}{\alpha}\,\frac{x_o^2}{\sigma^2}\right) .$$

Thus

$$\varepsilon = \frac{\Omega}{\sigma\frac{d\Sigma}{d\sigma}} = \frac{\sqrt{1-\alpha^2}}{\alpha}\left[1 + \left\{\frac{1}{2}\cdot\frac{1 - 2\alpha\beta}{1 - \alpha^2} + \frac{\beta}{\alpha}\right\}\frac{x_o^2}{\sigma^2}\right]$$

or, substituting for σ in terms of Σ

$$\varepsilon = \frac{\sqrt{1-\alpha^2}}{\alpha}\left[1 + \left\{\frac{\alpha^2}{2}\,\frac{(1 - 2\alpha\beta)}{1 - \alpha^2} + \alpha\beta\right\}\frac{x_o^2}{\Sigma^2}\right] \qquad \ldots \text{(AI.4)}$$

which has the form of (2.23) with

$$\varepsilon_o = \frac{\sqrt{1 - \alpha^2}}{\alpha} \qquad \ldots \text{(AI.5)}$$

and

$$\Upsilon = \frac{\alpha^2 - 4\alpha^3\beta + 2\,\alpha\beta}{2(1 - \alpha^2)} \ . \qquad \ldots \text{(AI.6)}$$

Inserting the values of α and β for a Gaussian distribution from Table 1 we obtain the values $\Upsilon = 0.635$, $\varepsilon_o = 0.757$ quoted in section 2. For the detector defined by equation (2.3) we have, assuming $x_o = o$,

$$\Omega^2 = 2 \int_o^\infty (x - \frac{1}{c} + \frac{1}{c} e^{-cx})^2 f(x) \, dx - \Sigma^2 .$$

The exponential in the first term makes no contribution to order $1/c^2$, and to this order we have

$$\Omega^2 = \sigma^2 - \frac{2\Sigma}{c} + \frac{1}{c^2} - \Sigma^2 \qquad \text{... (AI.7)}$$

The appropriate expression for Σ is obtained from equation (2.17) with $x_o = o$:

$$\Sigma = \alpha\sigma - \frac{1}{c} + \frac{2\beta}{\sigma c^2} . \qquad \text{... (AI.8)}$$

Thus

$$\Sigma^2 = \alpha^2\sigma^2 - \frac{2\alpha\sigma}{\theta} + \frac{1}{\theta^2} + \frac{4\alpha\beta}{c^2} \qquad \text{... (AI.9)}$$

and

$$\sigma \frac{d\Sigma}{d\sigma} = \alpha\sigma - \frac{2\beta}{\sigma c^2} . \qquad \text{... (AI.10)}$$

Substituting equations (AI.8) and (AI.9) in (AI.7) we find that the term in $\frac{1}{c}$ vanishes and we obtain

$$\Omega = \sigma \sqrt{(1 - \alpha^2)} \left[1 + \frac{1}{\sigma^2 c^2} \cdot \frac{1 - 2\beta\alpha}{1 - \alpha^2} \right] .$$

From this and AII.10 we obtain

$$\varepsilon = \frac{\Omega}{\sigma \frac{d\Sigma}{d\sigma}} = \frac{\sqrt{1 - \alpha^2}}{\alpha} \left[1 + \frac{1}{\Sigma^2 c^2} \left\{ \frac{\alpha^2 - 2\beta\alpha^3}{1 - \alpha^2} + 2\beta\alpha \right\} \right]$$

which reduces to

$$\varepsilon = \varepsilon_o \left(1 + \frac{2\Upsilon}{\Sigma^2 c^2} \right)$$

where ε_o and Υ are defined by AI.5 and AI.6.

Note that these results are largely independent of the detailed form of (2.3) near $x = o$; this detailed form would influence only higher order correction terms. The important quantity is $1/c$, the intercept on the α-axis when the linear part of (2.3) is extrapolated back to $x = o$, and any form for $d(x)$ which had the same slope for the linear portion and the same value for $1/c$ would be expected to give the same results to the order that we have calculated.

APPENDIX II

We have to evaluate averages of the form $\langle \delta\hat{x}_n \, \delta\hat{y}_n \rangle$ where

$$\delta\hat{x}_n = \hat{x}_n - \hat{x}$$

and

$$\hat{x}_n = \left[\frac{1}{n} \sum_{i=1}^{n} x_i^2 \right]^{\frac{1}{2}} \qquad \ldots \text{(AII.1)}$$

$$\hat{x}^2 = \langle x^2 \rangle .$$

Define

$$\delta(\hat{x}_n^2) = \hat{x}_n^2 - \hat{x}^2 \qquad \ldots \text{(AII.2)}$$

$$= 2\hat{x} \, \delta\hat{x}_n$$

for small $\delta\hat{x}_n$. Thus

$$\delta\hat{x}_n = \frac{\delta(\hat{x}_n^2)}{2\hat{x}}$$

and

$$\langle \delta\hat{x}_n \, \delta\hat{y}_n \rangle = \frac{1}{4 \, \hat{x} \, \hat{y}} \langle \delta(\hat{x}_n^2) \, \delta(\hat{y}_n^2) \rangle . \qquad \ldots \text{(AII.3)}$$

Substituting from equations (AII.2) and (AII.1) and remembering that the individual measurements x_i and y_i are correlated in general, but x_i and y_j are independent for $i \neq j$, we can reduce this to the form

$$\langle \delta\hat{x}_n \, \delta\hat{y}_n \rangle = \frac{1}{4n \, \hat{x} \, \hat{y}} \left(\langle \hat{x}^2 y^2 \rangle - \hat{x}^2 \, \hat{y}^2 \right) . \qquad \ldots \text{(AII.4)}$$

$\langle x^2 y^2 \rangle$ can be evaluated from results given by Rice (1945), or directly by integrating over a bivariate normal distribution for x and y. The result is

$$\langle \delta\hat{x}_n \, \delta\hat{y}_n \rangle = \frac{1}{2n} \varphi^2 \, \hat{x} \, \hat{y} \qquad \ldots \text{(AII.5)}$$

It is now a matter of simple algebra to obtain expressions for the other cross-terms appearing in equation (3.6). Thus, for example, defining $\varphi_{\pi\mu}$ as the correlation coefficient between the sum and difference signals π and μ, we have

$$\langle \delta\hat{\pi}_n \, \delta\hat{\mu}_n \rangle = \frac{1}{2n} \varphi_{\pi\mu}^2 \, \hat{\pi} \, \hat{\mu}$$

where

$$\hat{\pi}^2 = \langle (x + y)^2 \rangle$$

$$= \hat{x}^2 + 2\varphi \, \hat{x} \, \hat{y} + \hat{y}^2$$

and similarly

$$\hat{\mu}^2 = \hat{x}^2 - 2\varphi \, \hat{x} \, \hat{y} + \hat{y}^2$$

and

$$\varphi_{\pi\mu} = \frac{\langle \pi\mu \rangle}{\hat{\pi} \, \hat{\mu}} = \frac{\langle (x + y)(x - y) \rangle}{\hat{\pi} \, \hat{\mu}}$$

$$= \frac{\hat{x}^2 - \hat{y}^2}{\hat{\pi} \, \hat{\mu}} .$$

In just the same way we obtain

$$\varphi_{\pi x} = \frac{\hat{x}^2 + \varphi\,\hat{x}\,\hat{y}}{\hat{x}\hat{\pi}} = \frac{\hat{x} + \varphi\,\hat{y}}{\hat{\pi}}$$

$$\varphi_{\mu x} = \frac{\hat{x} - \varphi\hat{y}}{\hat{\mu}}$$

$$\varphi_{\pi y} = \frac{\hat{y} + \varphi\,\hat{x}}{\hat{\pi}}$$

$$\varphi_{\mu y} = \frac{\varphi\hat{x} - \hat{y}}{\hat{\mu}}$$

and thus, to take an example,

$$\langle \delta\hat{\pi}_n\ \delta\hat{x}_n \rangle = \frac{1}{2n}\ \varphi^2_{\pi x}\ \hat{\pi}\ \hat{x}$$

$$= \frac{1}{2n}\ (\hat{x} + \varphi\ \hat{y})^2\ \frac{\hat{x}}{\hat{\pi}}\ .$$

Now substituting these and similar expressions into equation (3.6) we obtain the results quoted in section 3 by straightforward algebra.

In the case of the full-wave rectifier we have

$$\hat{x}_n = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{n} \sum_i |x_i|\ .$$

The analysis runs parallel to that for the square-law detector in place of equation (AII.4) we obtain

$$\langle \delta\hat{x}_n\ \delta\hat{y}_n \rangle = \frac{1}{n}\left(\sqrt{\frac{\pi}{2}}\ \langle |xy| \rangle\ -\ \hat{x}\ \hat{y}\right)$$

$\langle |xy| \rangle$ is evaluated for a bivariate Gaussian distribution by Laning and Battin (1956). From their result we obtain

$$\langle \delta\hat{x}_n\ \delta\hat{y}_n \rangle = \frac{1}{n}\ \varphi^*\ \hat{x}\ \hat{y}$$

where

$$\varphi^* = \sqrt{1 - \varphi^2} + \varphi\ \sin^{-1}\ \varphi - 1$$

and the difference from the square law detector is contained entirely by the replacement of $\frac{1}{2}\ \varphi^2$ by φ^* in equation (AII.5) and other similar expressions. φ^* is in fact very close to $\varphi^2/2$ for small φ, the difference being less that 3% for $\varphi < 0.6$, and reaches a value of $\frac{1}{2}\ (\pi - 2)$ for $\varphi = 1$, so that, for example,

$$\langle \delta\hat{x}^2_n \rangle = (\pi - 2)\ \frac{\hat{x}^2}{2n}\ .$$

As we mentioned in section 3, the net effect is to increase the calculated value of $\langle \delta F^2_n \rangle$ by a factor of about $(\pi - 2)$.