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MAXIMUM ENTROPY SPECTRAL ANALYSIS IN MICHELSON INTERFEROMETRY

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Abstract

A Rapid-Scan Michelson Interferometer is used to measure the spectrum of Electron Cyclotron Emission from the plasma in the DITE Tokamak at the Culham Laboratory. Instruments of this type can only be used to measure the autocorrelation function up to a certain maximum value of the time delay in one of the wave trains; this is determined by the maximum path difference which can be introduced. If conventional Fourier methods are then used to invert the data, false sidelobe structures often appear on the transformed spectra. We describe the application of an alternative method of analysis, the Maximum Entropy Method. The method gives a positive-valued spectrum having the maximum spectral resolution consistent with the given autocorrelation data set but has the minimum of sidelobes and false structures. Because the numerical algorithm is non-linear, we are able to 'bootstrap' the data inversion scheme in order to self-calibrate it with respect to the 'zero-path-difference' correction. We present Electron Cyclotron Emission spectra from the DITE Tokamak plasma and compare results with those obtained from the conventional method.

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1. Introduction

The technique of Fourier Transform Spectroscopy /8/ has become standard in many branches of physics and chemistry. A rapidly scanned Michelson interferometer offers considerable advantages for spectral analysis in terms of, for example, simplicity of frequency calibration, wide spectral range and large optical throughput giving a high signal-to-noise ratio. However, it is only possible to extract the exact spectrum of the radiation source under investigation in the case where the autocorrelation function is known, in the absence of noise, for all values of the path difference between the two interfering wave-trains. In a practical situation not only do we have to contend with noise in the data but also have no knowledge of the autocorrelation function beyond a certain maximum path difference, $N\delta$. Usually $N\delta$ is limited by the scanning range of the instrument.

The Fourier transform of such imperfectly sampled and truncated data gives a response function having undesirable and prominent sidelobes. Usually, in order to reduce these sidelobes, the autocorrelation data are weighted (or 'apodized') prior to transformation; however this also has the effect of degrading the maximum spectral resolution (by typically a factor of 2) /6/. So, in a given application, one is forced to choose a weighting function which provides a satisfactory compromise between spectral resolution and sidelobe structure.

One can, however, obviate the need to make such a compromise by adopting a totally different scheme of data analysis, based on information theory. The alternative method we present here is the Maximum Entropy Method (MEM); various authors have discussed and used it with regard to the transformation of imperfect autocorrelation data /1/, /2/, /5/, /6/, /11/, /12/. Because the data are not apodized, MEM provides a substantial increase in resolution yet at the same time the effects of noise are minimised. A further advantage of the method is that certain calibration parameters can be adjusted automatically; here, we pay particular attention to the problem of determining the zero path difference (ZPD) correction when this cannot be found easily in the experimental arrangement. The fact that it is possible to 'bootstrap' (iteratively) the calibration procedure using MEM, is a consequence of the fact that it is a non-linear algorithm yielding a positive spectrum.

We demonstrate the use of an efficient algorithm for MEM with this data autocalibration procedure and illustrate it with an example taken from experimental plasma physics.

2. Fourier Transform Spectroscopy

We are interested in the spectrum $S(\omega)$ of a radiation source. This is related to the autocorrelation function, $I(\tau)$, by

$$I(\tau) = I_0 + 4 \int_0^\infty S(\omega) \cos(\omega\tau)\, d\omega, \qquad (1)$$

where τ is the time delay between the two interfering wave-trains and I_0 is a constant background level on the whole inter-ferogram. The detector output from the Michelson interferometer, however, is sampled discretely. Thus an estimate of the spectrum is given by

$$S(\omega) = \frac{2}{\pi} \sum_{n=o}^{N} \left(I(n\Delta\tau - \tau_0) - I_0\right) \cos(\omega n\Delta\tau) \qquad (2)$$

where τ_0 is the (unknown) offset in time delay for zero path difference between the interferometer arms. The autocorrelation function (1) is sampled at the discrete delays τ_0, $\Delta\tau - \tau_0$, $2\Delta\tau - \tau_0$, ... , $N\Delta\tau - \tau_0$. In these circumstances, it is well known that the spectral resolution available is limited to

$$\Delta\omega = \frac{1}{2(N\Delta\tau - \tau_0)} \, . \qquad (3)$$

2.1 The Standard Inversion Technique

At this stage, one might estimate the most probable values for τ_0 and I_0 and, in the case of inversion by Fast Fourier Transform (FFT), interpolate the measured values of the autocorrelation function on to a grid. Some typical data taken from a Michelson interferometer operating at millimetre wavelengths are shown in Fig. 1. This interferometer was used to observe the electron cylotron emission (ECE) spectrum from the DITE tokamak plasma /9/. At this point an apodization function was applied to the data in order to improve the shape of the spectral response function. In this example we have applied a 'Cosine-squared' weighting function and used a 512-point computational mesh spanning a frequency range

$$-\frac{2}{\pi\Delta\tau\alpha} \leqslant \omega \leqslant \frac{2}{\pi\Delta\tau\alpha} \qquad (4)$$

where α is the 'oversampling' factor

$$\alpha = \frac{512}{2N} . \qquad (5)$$

If $\alpha \leqslant 1$, the data overflow the FFT grid and, to avoid this, we have chosen $\alpha = 4$ as a satisfactory value and set the autocorrelation function to zero for time delays beyond $\pm N\Delta\tau$. According to this prescription, we have transformed the data in Fig. 1 to yield the spectrum in Fig. 2. We also show the effect of omitting the apodization step in Fig. 3.

The results shown in Figs. 2 and 3 demonstrate clearly the problems associated with this method of analysis. Both spectra exhibit (unphysical) negative regions, although, as expected, negative sidelobes are minimised in the apodized case. However, in this case, the resolution has also been degraded. Inspection of Fig. 2 shows the loss in structure of the dominant peak at the second harmonic Electron Cyclotron frequency compared with that in Fig. 3 as well as artificially broadened peaks at the fundamental and third harmonic cyclotron frequencies. The Electron Cyclotron frequency was 61 GHz for these data.

2.2 The Maximum Entropy Method

The Maximum Entropy Method (MEM) offers an alternative approach to spectral analysis. Instead of trying to compute a spectrum from measured autocorrelation data directly, we examine instead which spectrum would be the most likely to have been produced given the observed autocorrelation data set. This way, problems associated with the incompleteness of the data set are almost entirely avoided.

The reconstructed spectrum is represented by its intensities, f_j, on a similar (say N = 512) grid as that described above. Then, to measure the misfit between the actual (noisy) data and the data which would be observed if the spectrum were correctly represented by any particular f_j, we use the statistical χ^2 test:

$$\chi^2 = \sum (\text{normalised residual})^2 . \qquad (6)$$

All spectra for which χ^2 is inadmissibly high are discarded; in MEM we seek the condition that $\chi^2/N = 1$. To distinguish between the remaining spectra, all of which are consistent with the data, we choose the spectrum having the maximum configurational entropy /10/.

$$S = - \sum p_j \log p_j \quad \text{where} \quad p_j = \frac{f_j}{\sum f_j} . \qquad (7)$$

In this way we select a unique spectrum which is consistent with the observed data. The entropy, S, is a measure of the missing information content of the spectrum, consequently its unconstrained maximization yields a flat, featureless spectrum. Thus we see that its constrained maximization gives us a spectrum which is as featureless as possible but is in agreement with the known autocorrelation data. This spectrum is therefore uniquely safe; it contains the minimum of false artefacts and noise and yet gives the maximum resolution and dynamic range attainable consistent with the quality of the data.

Implementation of this method requires the calculation of the autocorrelation data which would be observed from any arbitrary (positive) spectrum. We do this using an FFT and subsequently interpolate to obtain data at the exact delay values for which we possess measured data. We then compute the value of

$$\chi^2 = \sum_{n=1}^{N} \frac{\left[I(n\Delta\tau) - I'(n\Delta\tau - \tau_0)\right]^2}{\sigma_n^2}, \qquad (8)$$

where I' refers to the 'mock' data that would be produced by the trial spectrum, f_j, and τ_0 is a trial value (in this example it is estimated by inspection of the maximum in the measured auto-correlation data, Fig. (1). The value, σ_n, is the rms error on the nth datum. Starting with the flat spectrum, we iterate on f_j in such a way as to minimise χ^2/N but simultaneously maximise S. Details of the algorithm are given in ref. /11/. Fig. 4 shows the resultant spectrum and the corresponding value of χ^2/N after various iterations.

After 12 iterations $\chi^2/N = 3.0$ but reduces to 2.7 after 20 iterations. However, because τ_0 was held constant during the MEM iteration, χ^2/N was not necessarily fully minimised with respect to changes in τ_0, i.e. variations in τ_0 will produce variations in $I'(\tau_0)$ after any given MEM iteration. Using the spectra obtained after 15 iterations we have minimised χ^2/N with respect to τ_0, obtaining an optimal value of 4.6, as shown in Fig. 5.

This value was then used to calculate χ^2/N and the MEM iteration procedure repeated. Further optimisations gave $\chi^2/N = 1$ for $\tau_0 = 4.68$ after 16 iterations. The ultimate minimisation of χ^2/N with respect to τ_0 is shown in Fig. 6. In principle this recalibration procedure could be applied iteratively but, unless the initial estimate of τ_0 is far from optimum, one or two recalibrations will suffice.

The resulting optimum MEM spectrum is shown in Fig. 7. The structure in the second harmonic peak is clearly consistent with the measured autocorrelation data and is not an artifact of the Fourier transform. Accurate measurements of the profile of the second harmonic Electron Cyclotron Emission with good frequency resolution are necessary for detailed studies of the electron temperature profile in a tokamak /4/. In this example, the structure is believed to be due to a resonance in the interferometer frequency response and can be compensated for independently by comparing the spectrum of the plasma with that of a black body /3/.

3. Conclusion

In summary, we have used both the Maximum Entropy Method and the standard Fourier Transform Method to invert noisy autocorrelation data gathered using a millimetre-wave Michelson interferometer. Comparison of the results shows the MEM technique to have three clear advantages:

(1) For given autocorrelation data (with inevitable incompleteness and measurement errors), MEM gives an enhancement in the spectral resolution which can be obtained.

(2) The MEM technique eliminates negative sidelobe structures on the inverted spectrum. This is a consequence of the definition of configurational entropy (Eq. (7)) in which p_j must always be greater than zero.

(3) The use of MEM enables the zero-path-difference correction to be determined accurately.

Although we have concentrated here on an example in millimetre-wave plasma interferometry, it is clear that the MEM technique could be used to advantage in other regions of the electromagnetic spectrum.

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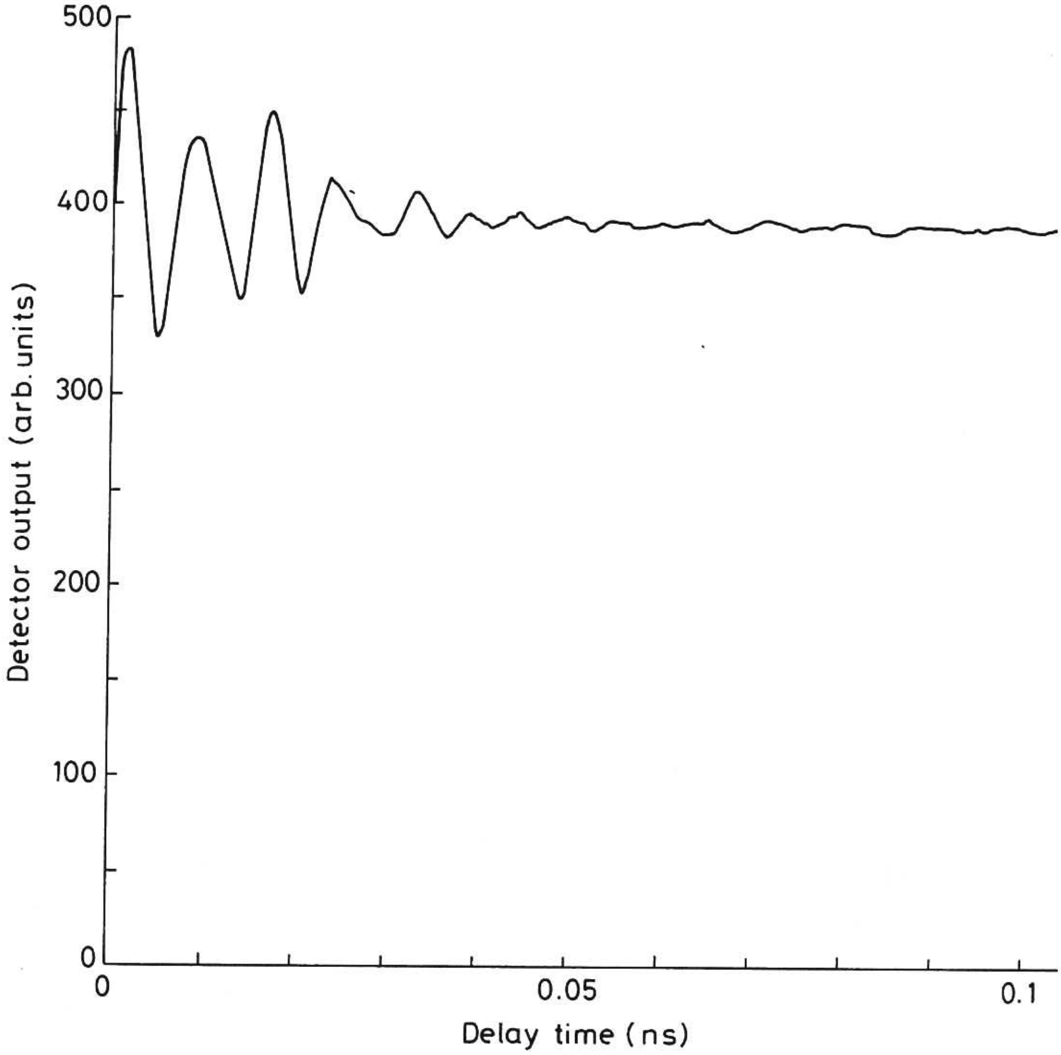

Fig.1 Experimental autocorrelation data, N = 194 points. Electron Cyclotron Emission intensity for magnetic field of 2.2 T in DITE tokamak.

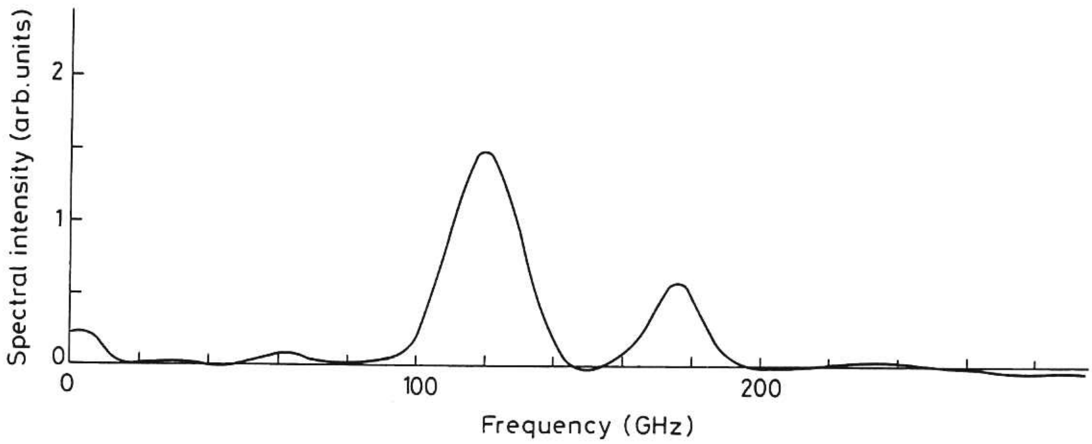

Fig.2 Inverted Spectrum with Cosine-squared weighting of autocorrelation data.

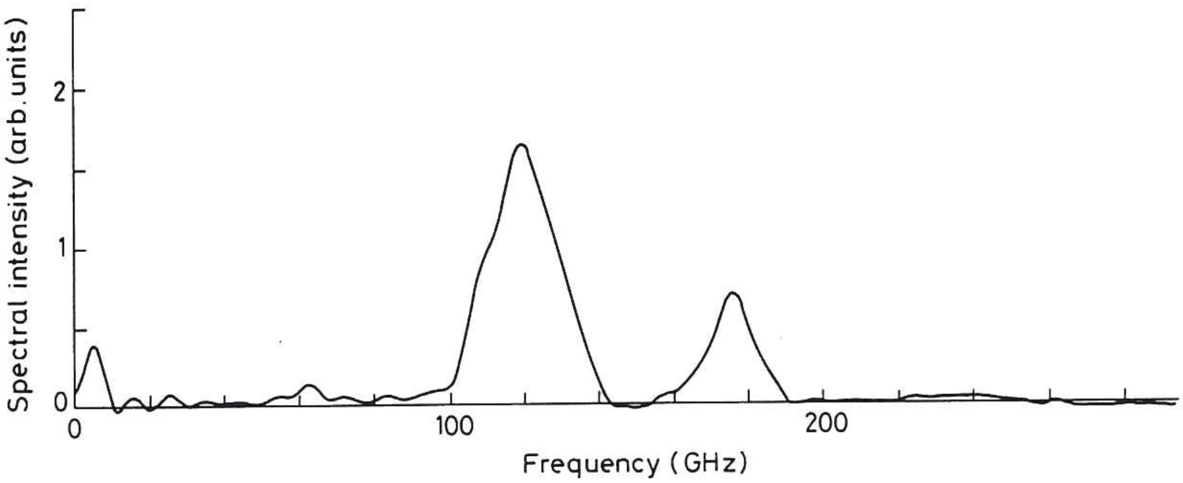

Fig.3 Inverted Spectrum with unweighted autocorrelation data.

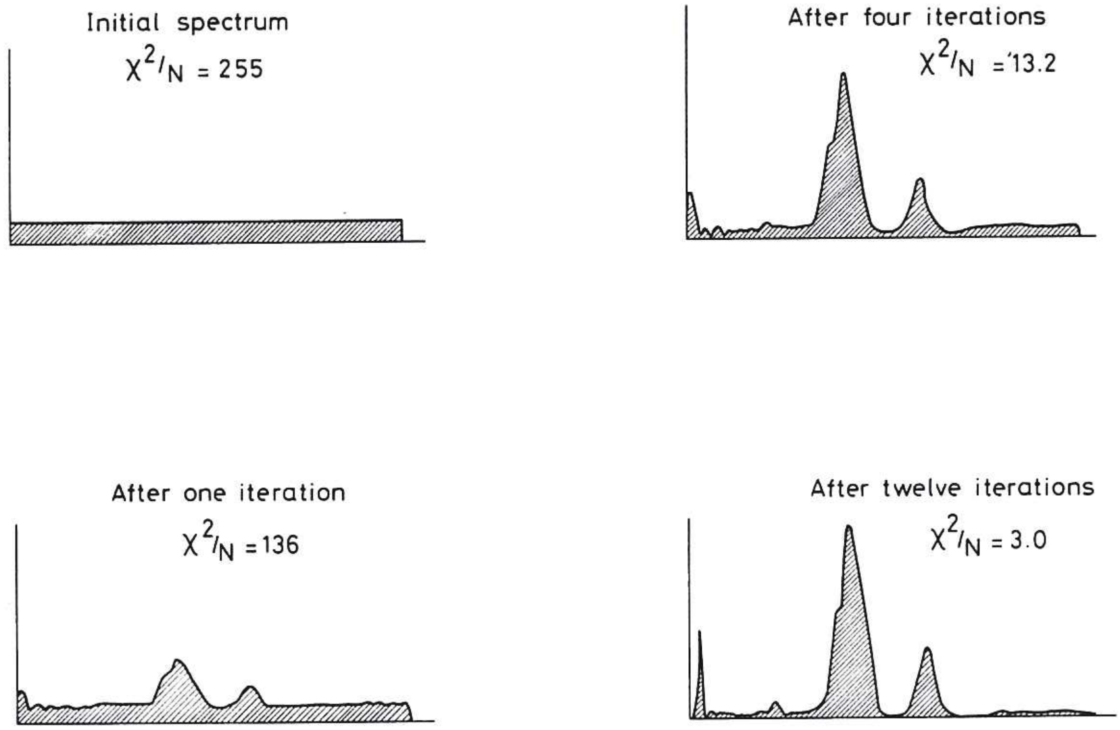

Fig.4 Development of trial spectrum during Maximum Entropy Iteration. Oversampling $\alpha = 4$, zero-path-difference correction, $\tau_0 = 4.5$ grid points.

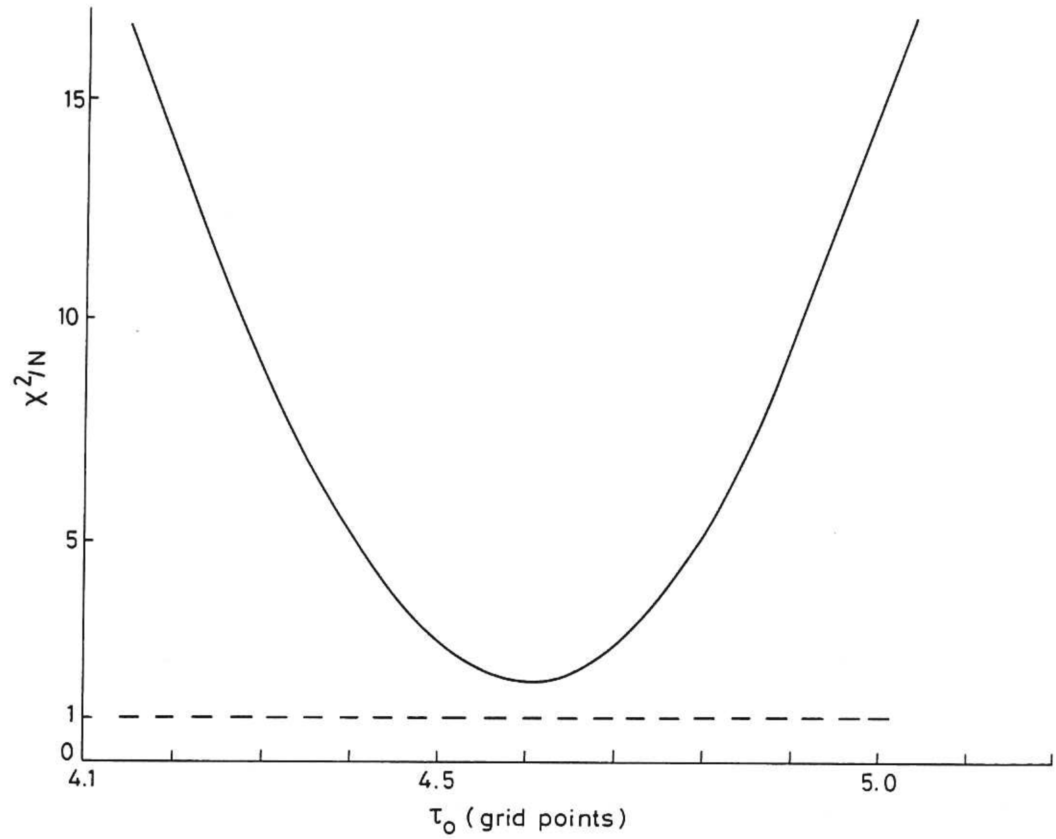

Fig.5 Minimisation of χ^2/N during Maximum Entropy Iteration assuming τ_o = 4.5 grid points.

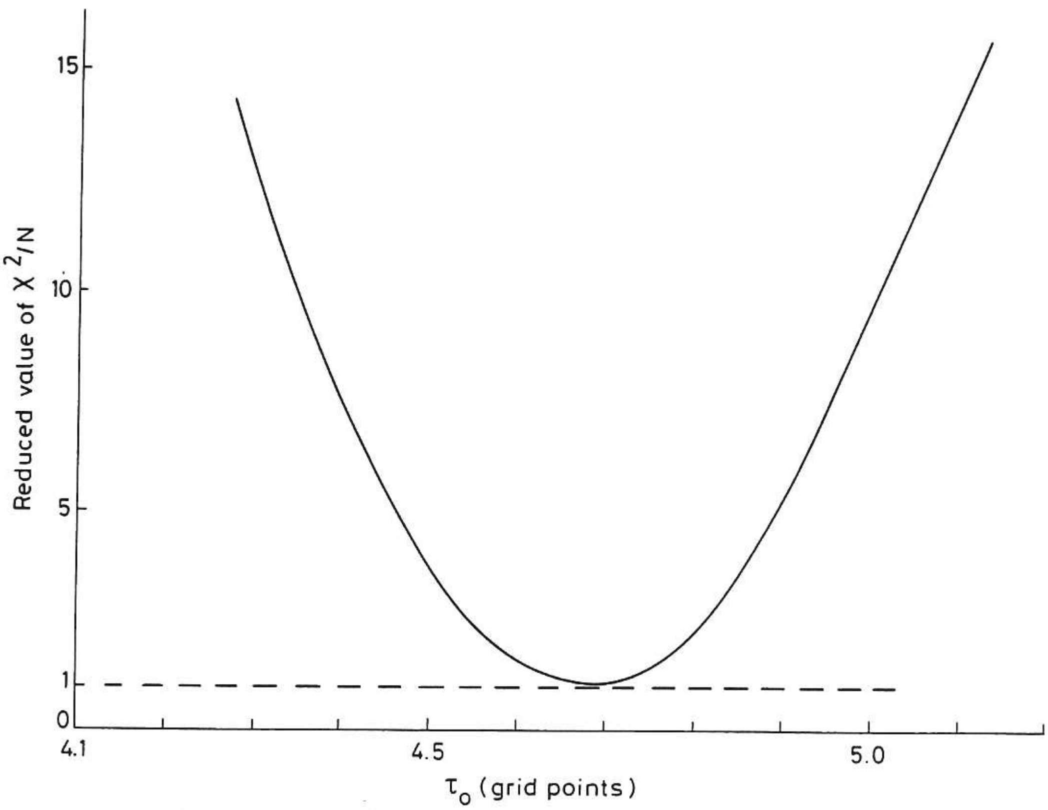

Fig.6 Minimisation of χ^2/N during Maximum Entropy Iteration assuming τ_o = 4.68 grid points.

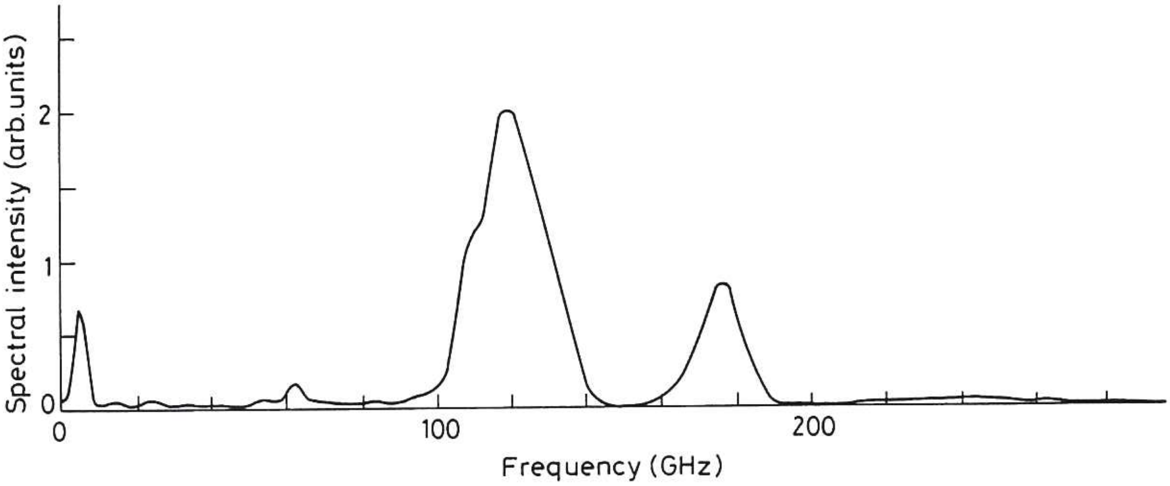

Fig.7 Optimised Maximum Entropy Spectrum. Oversampling $\alpha = 4$, zero-path-difference correction $\tau_0 = 4.68$, $\chi^2/N = 1.0$.